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PULLBACK $\mathcal{D}$-ATTRACTORS FOR NON-AUTONOMOUS PARTLY DISSIPATIVE REACTION-DIFFUSION EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. At present paper, we establish the existence of pullback $\mathcal{D}$-attractor for the process associated with non-autonomous partly dissipative reaction-diffusion equation in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In order to do this, by energy equation method we show that the process, which possesses a pullback $\mathcal{D}$-absorbing set, is pullback $\widehat{D}_0$-asymptotically compact.
Keywords: Pullback attractors, partly dissipative, reaction-diffusion equations.
MSC(2010): Primary: 37B55; Secondary: 37L05, 37L30, 35B40, 35B41, 35K57.

1. **Introduction**

Consider the following non-autonomous partly dissipative reaction-diffusion equation

(1.1)
$$\frac{\partial u}{\partial t} - \nu \Delta u + \lambda u + h(u) + \alpha v = f(x,t) \qquad \text{in } [\tau, +\infty) \times \mathbb{R}^n,$$
(1.2)
$$\frac{\partial v}{\partial t} + \delta v - \beta u = g(x,t) \qquad \text{in } [\tau, +\infty) \times \mathbb{R}^n,$$

with the initial data

$$u(\tau, x) = u_\tau(x), v(\tau, x) = v_\tau(x) \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

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where $\nu, \lambda, \delta > 0$ with $\lambda < \delta$, $f, g \in L_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta > 0$. Suppose that the nonlinear function h satisfies

$$h(s)s \geq -C_0 s^2,\ h(0) = 0,\ h'(s) \geq -C_1, \quad s \in \mathbb{R}, \tag{1.4}$$

$$|h'(s)| \leq C_2(1 + |s|^r), \qquad s \in \mathbb{R}, \tag{1.5}$$

with $r \geq 0$ for $n \leq 2$ and $r \leq \min\{\frac{4}{n}, \frac{2}{n-2}\}$ for $n \geq 3$, where C_i, $i = 0, 1, 2$, are positive constants and $0 \leq C_0 < \lambda$.

The long-time behavior of the solutions of non-autonomous evolution partial differential equations can be characterized in terms of uniform attractors, that is, the compact sets that uniformly (with respect to time symbol) attract every bounded subset of the phase space and are minimal in the sense of attracting property. By constructing the skew-product flow, one can reduce the non-autonomous system to a autonomous one in an extended phase space. In this situation, the structure of the uniform attractor can be described as the union of kernel sections. See details in [6]. At the same time, pullback (or cocycle) attractors have been introduced to non-autonomous systems to investigate the dynamics of them, see, e.g. [5, 7]. By definition, pullback attractor is a parameterized family of compact sets which attracts every bounded set of phase space from $-\infty$. It is noticed that in contrast to the existence of uniform attractors, the existence of pullback attractors can be obtained under the weak assumptions on the external forces. Thus, the development of the theory on pullback attractors plays an important role in understanding the dynamics of non-autonomous systems.

Recently, authors in [4, 9, 12, 17] develop the theory of pullback attractors in the classical sense and study the pullback $\mathcal{D}$-attractors for non-autonomous systems under the consideration of universes of initial data changing in time. In this case, the pullback attracting property of pullback $\mathcal{D}$-attractors is about the families of sets depending on time which are not necessary to be bounded.

The purpose of this paper is to investigate the existence of pullback $\mathcal{D}$-attractors for system (1.1)-(1.3). For the evolution equations in unbounded domains, since the Sobolev embedding is not compact, there are some difficulties in obtaining the compact attracting sets for the semigroup (autonomous case) or process (non-autonomous case) corresponding to them. One way to overcome such difficulties is to investigate the problem in weighted spaces, see, e.g. [2, 8]. By using the method of tail estimates of the solutions, authors in [15] obtained that the semigroup associated with autonomous system (1.1)-(1.3) (f and g are independent of t) is asymptotically compact. Motivated by the energy equation method, see, e.g. [3,13,16,19–21], we show that the process associated with (1.1)-(1.3) is pullback $\widehat{D}_0$-asymptotically compact, and obtain the existence of pullback $\mathcal{D}$-attractor.

For convenience, hereafter, let $H^s(\mathbb{R}^n)$ be the standard Sobolev spaces. Denote by $H = L^2(\mathbb{R}^n)$ with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$, respectively. For any $1 < p \le \infty$, let $\|\cdot\|_p$ be the norm of $L^p(\mathbb{R}^n)$. Let X be a Banach space with distance $d(\cdot,\cdot)$, C be generic positive constant which may vary from line to line or even in the same line.

This paper is organized as follows: in the next section, we give some definitions and recall some results which will be used in the following sections; in Section 3, we prove the existence of pullback $\mathcal{D}$-attractor for the process associated with (1.1)-(1.3) in $H \times H$.

2. **Preliminaries**

We first recall some basic definitions and abstract results on the existence of pullback $\mathcal{D}$-attractors. Let U be a process acting in a Banach space X, i.e., a family $\{U(t,\tau) : -\infty < \tau \le t < \infty\}$ of mappings $U(t,\tau) : X \to X$ satisfying
(1) $U(\tau,\tau) = Id$ (identity),
(2) $U(t,\tau) = U(t,r)U(r,\tau)$ for all $\tau \le r \le t$.

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 2.1. It is said that $\widehat{D}_0 \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $U(t,\tau)$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \le t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t), \text{ for all } \tau \le \tau_0(t,\widehat{D}).$$

Definition 2.2. A family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for the process $U(t,\tau)$ on X if
(1) $\mathcal{A}_{\mathcal{D}}(t)$ is compact for every $t \in \mathbb{R}$,
(2) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0, \qquad \text{for all } \widehat{D} \in \mathcal{D}, \text{and all } t \in \mathbb{R},$$

(3) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for $-\infty < \tau \le t < +\infty$, where $\operatorname{dist}(A,B)$ is the Hausdorff semi-distance between A and B, defined by

$$\operatorname{dist}(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y), \text{ for } A, B \subset X.$$

In order to illuminate the invariance of pullback $\mathcal{D}$-attractor, the following property is needed.

Definition 2.3. A process $U(t,\tau)$ on X is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y$, then $U(t,\tau)x = y$.

It is clear that if the process is closed, then it is norm-to-weak continuous, and if it is continuous or weak continuous, then it is norm-to-weak continuous.

Definition 2.4. It is said that a process $U(t,\tau)$ on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$, any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

The family $\mathcal{D}$ is said to be *inclusion-closed* if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$. Recall the results in [9].

Theorem 2.5. *Consider a closed process $U(t,\tau)$, $t \geq \tau$, $t,\tau \in \mathbb{R}$, on X, a universe $\mathcal{D}$ in $\mathcal{P}(X)$ which is inclusion-closed, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ which is closed and pullback $\mathcal{D}$-absorbing for $U(t,\tau)$, and assume also that $U(t,\tau)$ is pullback $\widehat{D}_0$-asymptotically compact. Then there exists a minimal and unique pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau) D_0(\tau)}^{X}, \quad \forall\, t \in \mathbb{R}.$$

3. Pullback $\mathcal{D}$–attractor of system (1.1)-(1.3)

We notice that if (u,υ) is a solution of (1.1)-(1.2) with the data (α,β,f,g), then $(u,-\upsilon)$ is a solution of (1.1)-(1.2) with the data $(-\alpha,-\beta,f,-g)$. Since $\alpha\beta > 0$, we assume without loss of generality in this paper that α and β are both positive. By the standard Fatou-Galerkin methods (see, e.g. [1,10,14,18]), we can get the well-posedness of (1.1)-(1.3).

Theorem 3.1. *Assume that h satisfies (1.4)-(1.5) and $f, g \in L_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$. Then for any $\tau \in \mathbb{R}$, any initial data $(u_\tau, \upsilon_\tau) \in H \times H$ and any $T > \tau$, there exists a unique solution $(u,\upsilon) = (u(t;\tau,u_\tau,\upsilon_\tau), \upsilon(t;\tau,u_\tau,\upsilon_\tau))$ for problem (1.1)-(1.3) satisfying*

$$(u,\upsilon) \in C([\tau,T]; H \times H), \qquad u \in L^2(\tau,T;H^1(\mathbb{R}^n)),$$

and the mapping $(u_\tau,\upsilon_\tau) \to (u(t),\upsilon(t))$ is continuous in $H \times H$.

By Theorem3.1, we can define a continuous process $\{U(t,\tau)\}$ on $H \times H$ by

$$U(t,\tau)(u_\tau,\upsilon_\tau) = (u(t),\upsilon(t)) := (u(t;\tau,u_\tau,\upsilon_\tau), \upsilon(t;\tau,u_\tau,\upsilon_\tau)) \quad \text{for all } t \geq \tau.$$

Let $\mathcal{D}$ be the class of all families $\{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $H \times H$ such that

$$\lim_{t \to -\infty} e^{\sigma t}[D(t)]^+ = 0, \tag{3.1}$$

where

$$0 < \sigma < \min\{\lambda - C_0, \delta, \frac{2\lambda}{r+1}\}$$

and

$$[D(t)]^+ = \sup\{\| u \|^2 + \| \upsilon \|^2 \colon (u,\upsilon) \in D(t)\}.$$

For the external terms, we suppose that

$$\int_{-\infty}^{t} e^{\sigma s} \parallel f(s) \parallel^2 ds < +\infty, \qquad \int_{-\infty}^{t} e^{\sigma s} \parallel g(s) \parallel^2 ds < +\infty, \qquad \forall\, t \in \mathbb{R}. \tag{3.2}$$

Lemma 3.2. *Assume that (1.4)-(1.5) hold and $f, g \in L_{loc}(\mathbb{R}, H)$ satisfy (3.2). Then there exists a pullback $\mathcal{D}$-absorbing set in $H \times H$.*

Proof. Taking the inner product of (1.1) with βu in H, we have

$$\frac{\beta}{2}\frac{d}{dt} \parallel u \parallel^2 + \beta\nu \parallel \nabla u \parallel^2 + \beta\lambda \parallel u \parallel^2 + \beta \int_{\mathbb{R}^n} h(u)u + \beta\alpha \int_{\mathbb{R}^n} uv = \beta \int_{\mathbb{R}^n} f(x,t)u. \tag{3.3}$$

Similarly, taking the inner product of (1.2) with αv in H, we have

$$\frac{\alpha}{2}\frac{d}{dt} \parallel v \parallel^2 + \alpha\delta \parallel v \parallel^2 - \beta\alpha \int_{\mathbb{R}^n} uv = \alpha \int_{\mathbb{R}^n} g(x,t)v. \tag{3.4}$$

Summing up (3.3) and (3.4), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\beta \parallel u \parallel^2 + \alpha \parallel v \parallel^2) + \beta\nu \parallel \nabla u \parallel^2 + \beta\lambda \parallel u \parallel^2 + \alpha\delta \parallel v \parallel^2 + \beta \int_{\mathbb{R}^n} h(u)u \\ &= \beta \int_{\mathbb{R}^n} f(x,t)u + \alpha \int_{\mathbb{R}^n} g(x,t)v. \end{aligned} \tag{3.5}$$

Note that the terms on the right-hand side of (3.5) can be estimated by

$$\mid \beta \int_{\mathbb{R}^n} f(x,t)u \mid \leq \beta \parallel f(x,t) \parallel \parallel u \parallel \leq \frac{\beta(\lambda - C_0)}{2} \parallel u \parallel^2 + \frac{\beta}{2(\lambda - C_0)} \parallel f(x,t) \parallel^2, \tag{3.6}$$

and

$$\mid \alpha \int_{\mathbb{R}^n} g(x,t)v \mid \leq \alpha \parallel g(x,t) \parallel \parallel v \parallel \leq \frac{1}{2}\alpha\delta \parallel v \parallel^2 + \frac{\alpha}{2\delta} \parallel g(x,t) \parallel^2 . \tag{3.7}$$

By (3.6)-(3.7) and (1.4), we obtain from (3.5) that

$$\begin{aligned} &\frac{d}{dt}(\beta \parallel u \parallel^2 + \alpha \parallel v \parallel^2) + 2\beta\nu \parallel \nabla u \parallel^2 + \beta(\lambda - C_0) \parallel u \parallel^2 + \alpha\delta \parallel v \parallel^2 \\ &\leq \frac{\beta}{\lambda - C_0} \parallel f(x,t) \parallel^2 + \frac{\alpha}{\delta} \parallel g(x,t) \parallel^2, \end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned} &\frac{d}{dt}(\beta \parallel u \parallel^2 + \alpha \parallel v \parallel^2) + 2\beta\nu \parallel \nabla u \parallel^2 + \sigma(\beta \parallel u \parallel^2 + \alpha \parallel v \parallel^2) \\ &\leq \frac{\beta}{\lambda - C_0} \parallel f(x,t) \parallel^2 + \frac{\alpha}{\delta} \parallel g(x,t) \parallel^2 . \end{aligned} \tag{3.9}$$

Neglecting $2\beta\nu \parallel \nabla u \parallel^2$ and utilizing Gronwall lemma, we get that

(3.10)

$$\begin{aligned}
&\beta \parallel u(t) \parallel^2 + \alpha \parallel \upsilon(t) \parallel^2 \\
&\quad \leq e^{-\sigma(t-\tau)}(\beta \parallel u(\tau) \parallel^2 + \alpha \parallel \upsilon(\tau) \parallel^2) + \int_\tau^t e^{-\sigma(t-s)}(\frac{\beta}{\lambda - C_0} \parallel f(x,s) \parallel^2 \\
&\qquad + \frac{\alpha}{\delta} \parallel g(x,s) \parallel^2) ds \\
&\quad \leq e^{-\sigma(t-\tau)}(\beta \parallel u(\tau) \parallel^2 + \alpha \parallel \upsilon(\tau) \parallel^2) + \frac{\beta e^{-\sigma t}}{\lambda - C_0} \int_{-\infty}^t e^{\sigma s} \parallel f(x,s) \parallel^2 ds \\
&\qquad + \frac{\alpha e^{-\sigma t}}{\delta} \int_{-\infty}^t e^{\sigma s} \parallel g(x,s) \parallel^2 ds.
\end{aligned}$$

By (3.1) we know that for any $(u(\tau), \upsilon(\tau)) \in D(\tau)$, there exists a $\tau_0(t, \widehat{D})$ such that

$$\parallel u(t) \parallel^2 + \parallel \upsilon(t) \parallel^2 \leq \frac{2e^{-\sigma t}}{\gamma}(\frac{\beta}{\lambda - C_0} \int_{-\infty}^t e^{\sigma s} \parallel f(x,s) \parallel^2 ds + \frac{\alpha}{\delta} \int_{-\infty}^t e^{\sigma s} \parallel g(x,s) \parallel^2 ds)$$

$$\triangleq R(t)^2, \qquad \forall \tau \leq \tau_0(t, \widehat{D}), \tag{3.11}$$

where $\gamma = \min\{\alpha, \beta\}$. Note that

$$\lim_{t \to -\infty} e^{\sigma t}(R(t))^2 = 0, \tag{3.12}$$

we get that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$, defined by

$$D_0(t) = \{(u(t), \upsilon(t)) \in H \times H : \parallel u(t) \parallel^2 + \parallel \upsilon(t) \parallel^2 \leq R(t)^2\}, \tag{3.13}$$

is a pullback $\mathcal{D}$-absorbing set. □

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. Then for any $(u(\tau), \upsilon(\tau)) \in D(\tau)$ and any $t \in \mathbb{R}$, there exists $\tau_1(t, \widehat{D}) \leq t$ such that for all $\tau \leq \tau_1(t, \widehat{D})$, the following inequality holds*

$$\parallel \nabla u(t) \parallel^2 \leq Ce^{-\sigma t}(\int_{-\infty}^t e^{\sigma s}(\parallel f(x,s) \parallel^2 + \parallel g(x,s) \parallel^2) ds, \tag{3.14}$$

where the positive constant C is independent of $D(\tau)$ and t.

Proof. Taking the inner product of (1.1) with $-\Delta u$ in H, we get that

(3.15)

$$\frac{1}{2}\frac{d}{dt}||\nabla u||^2 + \nu||\triangle u||^2 + \lambda||\nabla u||^2 = \int_{\mathbb{R}^n} h(u)\Delta u + \alpha \int_{\mathbb{R}^n} \upsilon \triangle u - \int_{\mathbb{R}^n} f(t) \triangle u.$$

We now estimate every term on the right-hand side of (3.15). By (1.4), we have

$$\int_{\mathbb{R}^n} h(u)\Delta u = -\int_{\mathbb{R}^n} h'(u) \mid \nabla u \mid^2 \leq C_1 \parallel \nabla u \parallel^2 . \tag{3.16}$$

Also, we find that

$$|\int_{\mathbb{R}^n} f(t)\triangle u| \leq \| f \| \| \Delta u \| \leq \frac{1}{2}\nu \| \Delta u \|^2 + \frac{1}{2\nu} \| f \|^2, \tag{3.17}$$

and

$$| \alpha \int_{\mathbb{R}^n} \upsilon\triangle u | \leq \alpha \| \upsilon \| \| \triangle u \| \leq \frac{1}{2}\nu \| \Delta u \|^2 + \frac{\alpha^2}{2\nu} \| \upsilon \|^2 . \tag{3.18}$$

It follows from (3.15)-(3.18) that

$$\frac{d}{dt} \| \nabla u \|^2 \leq 2C_1 \| \nabla u \|^2 + \frac{1}{\nu} \| f \|^2 + \frac{\alpha^2}{\nu} \| \upsilon \|^2 . \tag{3.19}$$

Integrating (3.9) with respect to t and considering (3.10), we get that

$$\begin{aligned}
&\int_t^{t+1} \| \nabla u(s) \|^2 ds + \frac{\alpha^2}{\nu} \int_t^{t+1} \| \upsilon(s) \|^2 ds \\
&\leq C(\beta \| u(t) \|^2 + \alpha \| \upsilon(t) \|^2) + C\int_t^{t+1} (\| f(x,s) \|^2 + \| g(x,s) \|^2)ds \\
&\leq C(\beta \| u(t) \|^2 + \alpha \| \upsilon(t) \|^2) + Ce^{-\sigma t}\int_{-\infty}^{t+1} e^{\sigma s}(\| f(x,s) \|^2 + \| g(x,s) \|^2)ds \\
&\leq Ce^{-\sigma(t-\tau)}(\beta \| u(\tau) \|^2 + \alpha \| \upsilon(\tau) \|^2) + Ce^{-\sigma t}\int_{-\infty}^{t+1} e^{\sigma s}(\| f(x,s) \|^2 + \| g(x,s) \|^2)ds.
\end{aligned} \tag{3.20}$$

By (3.20) and using uniform Gronwall lemma (see, e.g. [18]), we obtain from (3.19) that

$$\begin{aligned}
\| \nabla u(t+1) \|^2 \leq\ & Ce^{-\sigma(t-\tau)}(\beta \| u(\tau) \|^2 + \alpha \| \upsilon(\tau) \|^2) \\
&+ Ce^{-\sigma t}\int_{-\infty}^{t+1} e^{\sigma s}(\| f(x,s) \|^2 + \| g(x,s) \|^2)ds,
\end{aligned} \tag{3.21}$$

where the constant C is independent of $\| u(\tau) \|, \| \upsilon(\tau) \|$ and t. The estimate (3.21) implies the desired estimate (3.14). □

In order to get the pullback $\widehat{D}_0$-asymptotical compactness for the process, we endow external terms with the additional assumptions:

$$(A) \quad \lim_{k\to\infty} \int_{-\infty}^{t} e^{\sigma s} \int_{|x|\geq k} | f(x,s) |^2 dxds = 0,$$
$$\lim_{k\to\infty} \int_{-\infty}^{t} e^{\sigma s} \int_{|x|\geq k} | g(x,s) |^2 dxds = 0, \qquad \forall t \in \mathbb{R}.$$

Lemma 3.4. *Let the assumptions of Lemma 3.2 and (A) hold, and $\widehat{D}_0$ be defined by (3.13). Then for every $\varepsilon > 0$, any $t \in \mathbb{R}$, there exists $\tilde{k} = \tilde{k}(\varepsilon, t) \geq 0$*

such that

(3.22)

$$\lim_{\tau\to-\infty}\sup\int_{\tau}^{t}e^{-2\lambda(t-s)}\int_{|x|\geq k}|\,U(s,\tau)(u_\tau,\upsilon_\tau)\,|^2\,dxds<\varepsilon, k\geq\tilde{k}, (u_\tau,\upsilon_\tau)\in D_0(\tau).$$

Proof. Let θ be a smooth function satisfying $0\leq\theta(s)\leq 1$ for $s\in\mathbb{R}^+$, and

$$\theta(s)=0 \quad\text{for}\quad 0\leq s\leq 1 \quad\text{and}\quad \theta(s)=1 \quad\text{for}\quad s\geq 2.$$

Then there exists a constant C such that $|\theta'(s)|\leq C$ for $s\in\mathbb{R}^+$.

Multiplying (1.1) by $\beta\theta(|x|^2/k^2)u$ and integrating in $\mathbb{R}^n$, we get that

$$\begin{aligned}&\frac{1}{2}\beta\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2-\beta\nu\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})u\triangle u+\beta\lambda\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2\\ &=-\beta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})h(u)u-\beta\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})u\upsilon+\beta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})fu.\end{aligned}\tag{3.23}$$

Similarly, multiplying (1.2) by $\alpha\theta(|x|^2/k^2)\upsilon$ and integrating in $\mathbb{R}^n$, we get that

(3.24)

$$\frac{\alpha}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\upsilon|^2+\alpha\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\upsilon|^2=\beta\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})u\upsilon+\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})g\upsilon.$$

Summing up (3.23) and (3.24), we have

(3.25)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u|^2+\alpha|\upsilon|^2)+\beta\lambda\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2+\alpha\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\upsilon|^2\\ &=\beta\nu\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})u\triangle u-\beta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})h(u)u+\beta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})fu+\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})g\upsilon.\end{aligned}$$

We now estimate every term on the right-hand side of (3.25). First,

(3.26)

$$\beta\nu\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})u\triangle u=-\beta\nu\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2-\beta\nu\int_{\mathbb{R}^n}\frac{2x}{k^2}\cdot\theta'(\frac{|x|^2}{k^2})u\nabla u.$$

Note that

$$
\begin{aligned}
&| -\beta\nu \int_{\mathbb{R}^n} \frac{2x}{k^2} \cdot \theta'(\frac{|x|^2}{k^2}) u \nabla u \, | \\
&\qquad \leq C \int_{k \leq |x| \leq \sqrt{2}k} \frac{|x|}{k^2} |u| |\nabla u| \\
&\qquad \leq \frac{C}{k} \int_{k \leq |x| \leq \sqrt{2}k} |u| |\nabla u| \\
&\qquad \leq \frac{C}{k} \int_{\mathbb{R}^n} |u| |\nabla u| \\
&\qquad \leq \frac{C}{k} \parallel u \parallel \parallel \nabla u \parallel . \qquad (3.27)
\end{aligned}
$$

For the second term on the right-hand side of (3.25), by (1.4), we have

$$
-\beta \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) h(u) u \leq \beta C_0 \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) \mid u \mid^2 . \qquad (3.28)
$$

For the third term on the right-hand side of (3.25), we have

$$
\begin{aligned}
\beta \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) f u &\leq \mid \beta \int_{|x| \geq k} \theta(\frac{|x|^2}{k^2}) f u \mid \\
&\leq \beta (\int_{|x| \geq k} |f|^2)^{\frac{1}{2}} (\int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2)^{\frac{1}{2}} \qquad (0 \leq \theta \leq 1) \\
&\leq \frac{\beta}{2(\lambda - C_0)} \int_{|x| \geq k} |f|^2 + \frac{\beta(\lambda - C_0)}{2} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2. \qquad (3.29)
\end{aligned}
$$

Similarly, we can obtain

$$
\mid \alpha \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) g \upsilon \mid \leq \frac{\alpha}{2\delta} \int_{|x| \geq k} |g|^2 + \frac{\alpha\delta}{2} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |\upsilon|^2. \qquad (3.30)
$$

From (3.25)-(3.30), we get that

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) (\beta |u|^2 + \alpha |\upsilon|^2) + \beta(\lambda - C_0) \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |u|^2 + \alpha\delta \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2}) |\upsilon|^2 \\
&\leq \frac{C}{k} \parallel u \parallel \parallel \nabla u \parallel + \frac{\beta}{\lambda - C_0} \int_{|x| \geq k} |f|^2 + \frac{\alpha}{\delta} \int_{|x| \geq k} |g|^2. \qquad (3.31)
\end{aligned}
$$

Since $0 < \sigma < \min\{\lambda - C_0, \delta\}$, we can find $\varepsilon_0 > 0$ such that $0 < \sigma + \varepsilon_0 < \min\{\lambda - C_0, \delta\}$. It follows from (3.31) that

$$\frac{d}{dr}(e^{(\sigma+\varepsilon_0)r}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u|^2+\alpha|\upsilon|^2))$$
$$=(\sigma+\varepsilon_0)e^{(\sigma+\varepsilon_0)r}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u|^2+\alpha|\upsilon|^2)+e^{(\sigma+\varepsilon_0)r}\frac{d}{dr}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u|^2+\alpha|\upsilon|^2)$$
(3.32)
$$\le \frac{C}{k}e^{(\sigma+\varepsilon_0)r}\parallel u\parallel\parallel\nabla u\parallel+\frac{\beta e^{(\sigma+\varepsilon_0)r}}{\lambda-C_0}\int_{|x|\ge k}|f|^2+\frac{\alpha e^{(\sigma+\varepsilon_0)r}}{\delta}\int_{|x|\ge k}|g|^2.$$

Integrating (3.32) on $[\tau, s]$ with $\tau \le s \le t$, we get that

(3.33)
$$\begin{aligned}&e^{(\sigma+\varepsilon_0)s}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u(s)|^2+\alpha|\upsilon(s)|^2)\\&\le e^{(\sigma+\varepsilon_0)\tau}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u_\tau|^2+\alpha|\upsilon_\tau|^2)+\frac{C}{k}\int_\tau^s e^{(\sigma+\varepsilon_0)r}\parallel u\parallel\parallel\nabla u\parallel dr\\&\quad+\frac{\beta}{\lambda-C_0}\int_\tau^s e^{(\sigma+\varepsilon_0)r}\int_{|x|\ge k}|f|^2dr+\frac{\alpha}{\delta}\int_\tau^s e^{(\sigma+\varepsilon_0)r}\int_{|x|\ge k}|g|^2dr.\end{aligned}$$

Multiplying (3.33) with $e^{(2\lambda-\sigma-\varepsilon_0)s-2\lambda t}$, we have

$$\begin{aligned}&e^{-2\lambda(t-s)}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u(s)|^2+\alpha|\upsilon(s)|^2)\\&\le e^{-2\lambda t}e^{(2\lambda-\sigma-\varepsilon_0)s}e^{(\sigma+\varepsilon_0)\tau}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u_\tau|^2+\alpha|\upsilon_\tau|^2)+\frac{Ce^{-2\lambda t}e^{(2\lambda-\sigma-\varepsilon_0)s}}{k}\\&\quad\times\int_\tau^t e^{(\sigma+\varepsilon_0)r}\parallel u\parallel\parallel\nabla u\parallel dr+\frac{\beta e^{-2\lambda t}e^{(2\lambda-\sigma-\varepsilon_0)s}}{\lambda-C_0}\int_\tau^t e^{(\sigma+\varepsilon_0)r}\int_{|x|\ge k}|f|^2dr\\&\quad+\frac{\alpha e^{-2\lambda t}e^{(2\lambda-\sigma-\varepsilon_0)s}}{\delta}\int_\tau^t e^{(\sigma+\varepsilon_0)r}\int_{|x|\ge k}|g|^2dr.\end{aligned}$$

Integrating the above inequality with respect to s over $[\tau, t]$, we obtain

(3.34)
$$\begin{aligned}&\int_\tau^t e^{-2\lambda(t-s)}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(\beta|u(s)|^2+\alpha|\upsilon(s)|^2)ds\\&\le\frac{e^{-(\sigma+\varepsilon_0)(t-\tau)}}{2\lambda-\sigma-\varepsilon_0}(\beta\parallel u_\tau\parallel^2+\alpha\parallel\upsilon_\tau\parallel^2)+\frac{Ce^{-(\sigma+\varepsilon_0)t}}{(2\lambda-\sigma-\varepsilon_0)k}\int_\tau^t e^{(\sigma+\varepsilon_0)s}\parallel u\parallel\parallel\nabla u\parallel ds\\&\quad+\frac{\beta e^{-(\sigma+\varepsilon_0)t}}{(2\lambda-\sigma-\varepsilon_0)(\lambda-C_0)}\int_\tau^t e^{(\sigma+\varepsilon_0)s}\int_{|x|\ge k}|f(s)|^2ds\\&\quad+\frac{\alpha e^{-(\sigma+\varepsilon_0)t}}{(2\lambda-\sigma-\varepsilon_0)\delta}\int_\tau^t e^{(\sigma+\varepsilon_0)s}\int_{|x|\ge k}|g(s)|^2ds.\end{aligned}$$

Now we bound each term on the right-hand side of (3.34). First, since $(u_\tau, \upsilon_\tau) \in D_0(\tau)$, by (3.12), for any $t \in \mathbb{R}$ we have

$$\frac{e^{-(\sigma+\varepsilon_0)(t-\tau)}}{2\lambda-\sigma-\varepsilon_0}(\beta \parallel u_\tau \parallel^2 + \alpha \parallel \upsilon_\tau \parallel^2) \to 0,\ as\ \tau \to -\infty. \tag{3.35}$$

Note that

$$\int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel u \parallel\parallel \nabla u \parallel ds \le \frac{1}{2}\int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel u \parallel^2 ds + \frac{1}{2}\int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel \nabla u \parallel^2 ds. \tag{3.36}$$

By (3.10), we get that

$$\begin{aligned} \parallel u(s) \parallel^2 \le & \frac{1}{\beta}e^{-\sigma(s-\tau)}(\beta \parallel u_\tau \parallel^2 + \alpha \parallel \upsilon_\tau \parallel^2) + \frac{e^{-\sigma s}}{\lambda - C_0}\int_{-\infty}^s e^{\sigma\nu} \parallel f(\nu) \parallel^2 d\nu \\ & + \frac{\alpha e^{-\sigma s}}{\beta\delta}\int_{-\infty}^s e^{\sigma\nu} \parallel g(\nu) \parallel^2 d\nu, \quad s \in [\tau, t]. \end{aligned} \tag{3.37}$$

Then, we have

$$\begin{aligned} & \int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel u(s) \parallel^2 ds \\ & \le \frac{1}{\beta}(\beta \parallel u_\tau \parallel^2 + \alpha \parallel \upsilon_\tau \parallel^2)e^{\sigma\tau}\int_\tau^t e^{\varepsilon_0 s}ds + \frac{1}{\lambda - C_0}\int_\tau^t e^{\varepsilon_0 s}ds\int_{-\infty}^t e^{\sigma\nu} \parallel f(\nu) \parallel^2 d\nu \\ & \quad + \frac{\alpha}{\beta\delta}\int_\tau^t e^{\varepsilon_0 s}ds\int_{-\infty}^t e^{\sigma\nu} \parallel g(\nu) \parallel^2 d\nu \\ & \le \frac{e^{\varepsilon_0 t}}{\beta\varepsilon_0}e^{\sigma\tau}(\beta \parallel u_\tau \parallel^2 + \alpha \parallel \upsilon_\tau \parallel^2) + \frac{e^{\varepsilon_0 t}}{\varepsilon_0(\lambda - C_0)}\int_{-\infty}^t e^{\sigma\nu} \parallel f(\nu) \parallel^2 d\nu \\ & \quad + \frac{\alpha e^{\varepsilon_0 t}}{\beta\delta\varepsilon_0}\int_{-\infty}^t e^{\sigma\nu} \parallel g(\nu) \parallel^2 d\nu \\ & < \infty, \qquad \forall \tau \le t, \end{aligned} \tag{3.38}$$

which along with (3.12), for any $t \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$\lim_{\tau\to-\infty}\sup\left(\frac{Ce^{-(\sigma+\varepsilon_0)t}}{2(2\lambda-\sigma-\varepsilon_0)k}\int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel u(s) \parallel^2 ds\right) < \varepsilon, \quad k \ge k_1,\ (u_\tau, \upsilon_\tau) \in D_0(\tau). \tag{3.39}$$

Reasoning as above, from (3.14), there exists $k_2 \in \mathbb{N}$ such that

$$\lim_{\tau\to-\infty}\sup\left(\frac{Ce^{-(\sigma+\varepsilon_0)t}}{2(2\lambda-\sigma-\varepsilon_0)k}\int_\tau^t e^{(\sigma+\varepsilon_0)s} \parallel \nabla u(s) \parallel^2 ds\right) < \varepsilon,\ k \ge k_2,\ (u_\tau, \upsilon_\tau) \in D_0(\tau). \tag{3.40}$$

For the third term and the last term on the right-hand side of (3.4), it follows from assumption (A) that there exists $k_3 \in \mathbb{N}$ such that

$$\int_{-\infty}^{t} e^{\sigma s} \int_{|x|\geq k} |\, f(s)\,|^2 \,ds < \varepsilon, \;\; \int_{-\infty}^{t} e^{\sigma s} \int_{|x|\geq k} |\, g(s)\,|^2 \,ds < \varepsilon, \;\; k \geq k_3. \tag{3.41}$$

Combining (3.34)-(3.41), letting $k > \max\{k_1, k_2, k_3\}$, and taking into account that

$$\int_{\tau}^{t} e^{-2\lambda(t-s)} \int_{|x|\geq 2k} (|u(s)^2 + |\upsilon(s)|^2) ds \leq \int_{\tau}^{t} e^{-2\lambda(t-s)} \int_{\mathbb{R}^n} \theta(\frac{|x|^2}{k^2})(|u(s)|^2 + |\upsilon(s)|^2) ds,$$

we get the result. □

As the proof of Lemma 4 in [19], we have the following result.

Lemma 3.5. *Assume that* (1.4)-(1.5) *hold and* $f, g \in L_{loc}(\mathbb{R}, H)$. *If* $(u_{\tau_n}, \upsilon_{\tau_n}) \to (u_\tau, \upsilon_\tau)$ *weakly in* $H \times H$, *then there exist subsequences* $\{u_{\tau_{n_j}}\}$ *of* $\{u_{\tau_n}\}$ *and* $\{\upsilon_{\tau_{n_j}}\}$ *of* $\{\upsilon_{\tau_n}\}$ *such that*

$$U(t,\tau)(u_{\tau_{n_j}}, \upsilon_{\tau_{n_j}}) \to U(t,\tau)(u_\tau, \upsilon_\tau) \qquad \text{weakly in } \; H \times H, \; \text{for all } \tau \leq t.$$

And for all $\tau \leq T$,

$$\begin{aligned} &U(\cdot,\tau)(u_{\tau_{n_j}}, \upsilon_{\tau_{n_j}}) \to U(\cdot,\tau)(u_\tau, \upsilon_\tau) && \text{weakly in } L^2(\tau, T; H \times H), \\ &P_1 U(\cdot,\tau) u_{\tau_{n_j}} \to P_1 U(\cdot,\tau) u_\tau && \text{weakly in } L^2(\tau, T; H^1(\mathbb{R}^n)), \\ &\frac{\partial}{\partial \cdot} P_1 U(\cdot,\tau) u_{\tau_{n_j}} \to \frac{\partial}{\partial \cdot} P_1 U(\cdot,\tau) u_\tau && \text{weakly in } L^2(\tau, T; H^{-1}), \end{aligned}$$

where P_1 *is the canonical projection from Hilbert space* $E \times F$ *to* E.

Theorem 3.6. *Assume that* (1.4)-(1.5) *hold and* $f, g \in L_{loc}(\mathbb{R}, H)$ *satisfy* (3.2) *with* $0 < \sigma < \min\{\lambda - C_0, \delta, \frac{2\lambda}{r+1}\}$ *and assumption* (A). *Then the process* $U(t,\tau)$ *corresponding to problem* (1.1)-(1.3) *possesses a unique pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ *in* $H \times H$.

Proof. By the proof of Lemma 3.2, we obtain that $\widehat{D}_0 = \{D_0(t)\}_{t\in\mathbb{R}}$ defined by (3.13) is a pullback $\mathcal{D}$-absorbing set for the process $U(t,\tau)$. To prove the result, by Theorem 2.5, we only need to prove that for any $t \in \mathbb{R}$, any sequences $\tau_n \to -\infty$, and all $(u_{\tau_n}, \upsilon_{\tau_n}) \in D_0(\tau_n)$, the sequence $\{U(t,\tau)(u_{\tau_n}, \upsilon_{\tau_n})\}$ is precompact in $H \times H$.

From (3.13) we know that for any $t \in \mathbb{R}$, there exists $\tau_{\widehat{D}_0}(t) \leq t$ such that

$$U(t,\tau) D_0(\tau) \subset D_0(t), \qquad \forall\, \tau \leq \tau_{\widehat{D}_0}(t).$$

Then for every $k \in \mathbb{Z}_+$, there exists $\tau_{\widehat{D}_0}(k,t) \leq t - k$ such that

$$U(t-k,\tau) D_0(\tau) \subset D_0(t-k), \qquad \forall\, \tau \leq \tau_{\widehat{D}_0}(k,t). \tag{3.42}$$

Since for any $t \in \mathbb{R}$, $k \geq 0$, $D_0(t-k)$ is a bounded subset in $H \times H$, by a diagonal procedure, we can select $\{\tau_{n'}, (u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\} \subset \{\tau_n, (u_{\tau_n}, \upsilon_{\tau_n})\}$, which $\tau_{n'}$ is a decreasing sequence, such that for every $k \geq 0$, there exists a sequence $\omega_k = (u_k, \upsilon_k) \subset H \times H$ such that

$$U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to \omega_k \qquad \text{weakly in } H \times H. \tag{3.43}$$

Thus, it follows from Lemma 3.5 that

$$\begin{aligned} \omega_0 = (u_0, \upsilon_0) &= \lim_{n'} {}_{(H\times H)_\omega} U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \\ &= \lim_{n'} {}_{(H\times H)_\omega} U(t, t-k) U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \\ &= U(t, t-k) \lim_{n'} {}_{(H\times H)_\omega} U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \\ &= U(t, t-k)\omega_k, \text{ for all } k \geq 0, \end{aligned} \tag{3.44}$$

where $\lim_{(H\times H)_\omega}$ denotes the weak limit in $H \times H$. The equality (3.44) also implies that

$$(\beta^{\frac{1}{2}} P_1 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}), \alpha^{\frac{1}{2}} P_2 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})) \to (\beta^{\frac{1}{2}} u_0, \alpha^{\frac{1}{2}} \upsilon_0) \text{ weakly in } H \times H,$$

where P_i $(i = 1, 2)$ is the canonical projection. By the lower semi-continuity of the norm, we get that

$$\begin{aligned} &\|(\beta^{\frac{1}{2}} u_0, \alpha^{\frac{1}{2}} \upsilon_0)\|_{H\times H} \\ &\leq \liminf_{n'\to\infty} \| (\beta^{\frac{1}{2}} P_1 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}), \alpha^{\frac{1}{2}} P_2 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})) \|_{H\times H} . \end{aligned} \tag{3.45}$$

If we can also prove

$$\begin{aligned} &\limsup_{n'\to\infty} \| (\beta^{\frac{1}{2}} P_1 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}), \alpha^{\frac{1}{2}} P_2 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})) \|_{H\times H} \\ &\qquad \leq \| (\beta^{\frac{1}{2}} u_0, \alpha^{\frac{1}{2}} \upsilon_0) \|_{H\times H}, \end{aligned} \tag{3.46}$$

then we can get that

$$(\beta^{\frac{1}{2}} P_1 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}), \alpha^{\frac{1}{2}} P_2 U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})) \to (\beta^{\frac{1}{2}} u_0, \alpha^{\frac{1}{2}} \upsilon_0) \text{ strongly in } H \times H,$$

which implies that

$$U(t, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to (u_0, \upsilon_0) \text{ strongly in } H \times H,$$

and the result will be proved.

By (3.5), for all $\tau \leq t$ and all $(u_\tau, \upsilon_\tau) \in H \times H$, we have

(3.47)
$$\begin{aligned}
&\beta \parallel P_1U(t,\tau)(u_\tau,\upsilon_\tau) \parallel^2 + \alpha \parallel P_2U(t,\tau)(u_\tau,\upsilon_\tau) \parallel^2 \\
&= e^{-2\lambda(t-\tau)}(\beta \parallel u_\tau \parallel^2 + \alpha \parallel \upsilon_\tau \parallel^2) + 2\alpha(\lambda-\delta)\int_\tau^t e^{-2\lambda(t-s)} \parallel P_2U(s,\tau)(u_\tau,\upsilon_\tau) \parallel^2 ds \\
&\quad -2\beta\nu \int_\tau^t e^{-2\lambda(t-s)} \|\nabla P_1U(s,\tau)(u_\tau,\upsilon_\tau)\|^2 ds - 2\beta \int_\tau^t e^{-2\lambda(t-s)} \langle h(P_1U(s,\tau)(u_\tau,\upsilon_\tau)), \\
&\qquad P_1U(s,\tau)(u_\tau,\upsilon_\tau)\rangle ds + 2\beta \int_\tau^t e^{-2\lambda(t-s)} \langle f, P_1U(s,\tau)(u_\tau,\upsilon_\tau)\rangle ds \\
&\quad +2\alpha \int_\tau^t e^{-2\lambda(t-s)} \langle g, P_2U(s,\tau)(u_\tau,\upsilon_\tau)\rangle ds.
\end{aligned}$$

Then for all $k \geq 0$, $\tau_{n'} \leq t-k$, we obtain

(3.48)
$$\begin{aligned}
&\beta \parallel P_1U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 + \alpha \parallel P_2U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 \\
&= \beta||P_1U(t,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})||^2 + \alpha||P_2U(t,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})||^2 \\
&= e^{-2\lambda k}(\beta \parallel P_1U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 + \alpha \parallel P_2U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \|)^2 \\
&\quad + 2\alpha(\lambda-\delta)\int_{t-k}^t e^{-2\lambda(t-s)} \parallel P_2U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 ds \\
&\quad - 2\beta\nu \int_{t-k}^t e^{-2\lambda(t-s)} \parallel \nabla P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 ds \\
&\quad - 2\beta \int_{t-k}^t e^{-2\lambda(t-s)} \langle h(P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})), P_1U(s,t-k)U(t-k, \\
&\qquad \tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\rangle ds \\
&\quad + 2\beta \int_{t-k}^t e^{-2\lambda(t-s)} \langle f, P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\rangle ds \\
&\quad + 2\alpha \int_{t-k}^t e^{-2\lambda(t-s)} \langle g, P_2U(s,t-k)U(t-k,\tau)(u_\tau,\upsilon_\tau)\rangle ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

We now estimate $I_1 - I_6$ one by one. First, from (3.42), we have

(3.49) $$U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \in D_0(t-k) \quad \text{for any } \tau_{n'} \leq \tau_{\widehat{D}_0(k,t)},\ k \geq 0.$$

Then we have

(3.50)
$$\begin{aligned}
&\lim_{n'\to\infty} \sup I_1 \\
&= \lim_{n'\to\infty} (e^{-2\lambda k}(\beta||P_1U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})||^2 + \alpha||P_2U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})||^2)) \\
&\leq e^{-2\lambda k}\delta_1(R(t-k))^2,\ k \geq 0,
\end{aligned}$$

where $\delta_1 = \max\{\alpha, \beta\}$. For I_2, since $U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})$ $\to \omega_k$ weakly in $H \times H$, and it follows, from Lemma 3.5, that $P_2U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to P_2U(s, t-k)\omega_k$ weakly in $L^2(t-k, t; H)$. Then we can deduce that

$$\int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_2U(s, t-k)\omega_k \parallel^2 ds$$
$$\leq \lim_{n'\to\infty} \inf \int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_2U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \parallel^2 ds.$$

Thus, by the fact that $\lambda < \delta$ we have

$$\lim_{n'\to\infty} \sup I_2 \leq 2\alpha(\lambda - \delta) \int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_2U(s, t-k)\omega_k \parallel^2 ds. \tag{3.51}$$

Similarly, reasoning as above, we have $\nabla P_1U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to \nabla P_1U(s, t-k)\omega_k$ weakly in $L^2(t-k, t; H)$. Then,

$$\int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel \nabla P_1U(s, t-k)\omega_k \parallel^2 ds$$
$$\leq \lim_{n'\to\infty} \inf \int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel \nabla P_1U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \parallel^2 ds.$$

Thus,

(3.52)

$$\lim_{n'\to\infty} \sup I_3 \leq -2\beta\nu \int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel \nabla P_1U(s, t-k)\omega_k \parallel^2 ds.$$

Now, we estimate I_4 by decomposing $\mathbb{R}^n$ into a bounded domain and its complement to overcome the lack of compactness of Sobolev imbeddings. Note that

$$\begin{aligned}
&-2\beta \int_{t-k}^{t} e^{-2\lambda(t-s)} \langle h(P_1U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})), P_1U(s, t-k) \\
&\quad \circ U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\rangle ds \\
&= -2\beta \int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\geq m} h(P_1U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}))P_1U(s, t-k) \\
&\quad \circ U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})dxds \\
&\quad -2\beta \int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\leq m} h(P_1U(s, t-k)U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}))P_1U(s, t-k) \\
&\quad \circ U(t-k, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})dxds.
\end{aligned}$$

By Lemma 3.4, for any $\varepsilon \geq 0$, there exists $\tilde{\tau} \leq t$, $\tilde{m} \in \mathbb{N}$ and $\tilde{n} \in \mathbb{N}$ such that

$$\int_{\tau_{n'}}^{t} e^{-2\lambda(t-s)} \int_{|x|\geq m} | P_1U(s, \tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) |^2 dxds < \varepsilon,$$

for $\tau_{n'} \le \min\{\tilde{\tau}, t-k\}$, $m \ge \tilde{m}$, $n' > \tilde{n}$ and $(u_{0_{n_j}}, \upsilon_{0_{n_j}}) \in D_0(\tau_{n'})$. Without loss of generality, we can choose $\tilde{n}$ large enough such that $\tau_{n'} \le \min\{\tilde{\tau}, t-k\}$ for all $n' \ge \tilde{n}$.

Combining (1.4) and (1.5), we have $|h(u)| \le C(|u| + |u|^{r+1})$, and then we can deduce that for $n' \ge \tilde{n}$, $m \ge \tilde{m}$, and $(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \in D_0(\tau_{n'})$, there holds

$$\begin{aligned}
&\int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\ge m} h(P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}))P_1U(s,t-k) \\
&\qquad \circ U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})ds \\
&\le C\int_{\tau_{n'}}^{t} e^{-2\lambda(t-s)} \int_{|x|\ge m} |\ P_1U(s,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |^2\ dxds + C\int_{t-k}^{t} e^{-2\lambda(t-s)} \\
&\quad \times \int_{|x|\ge m} |\ P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |^{r+1} |\ P_1U(s,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |\ dxds \\
&\le \varepsilon C + C(\int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\ge m} |\ P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |^{2(r+1)}\ dxds)^{\frac{1}{2}} \\
&\qquad \times (\int_{\tau_{n'}}^{t} e^{-2\lambda(t-s)} \int_{|x|\ge m} |\ P_1U(s,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |^2\ dxds)^{\frac{1}{2}} \\
&\le \varepsilon C + \varepsilon C(\int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\ge m} |\ P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ |^{2(r+1)}\ dxds)^{\frac{1}{2}} \\
&\le \varepsilon C + \varepsilon C(\int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})\ \parallel_{H^1}^{2(r+1)}\ ds)^{\frac{1}{2}}.
\end{aligned}$$

Let $\Omega_m = \{x \in \mathbb{R}^n |\ |x| \le m\}$, and note that

$$U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to \omega_k \qquad \text{weakly in } H \times H.$$

From Lemma 3.5, we get that

$$P_1U(\cdot,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to P_1U(\cdot,t-k)\omega_k \text{weakly in } L^2(t-k,t;H^1(\Omega_m)),$$

$$\frac{\partial}{\partial \cdot}P_1U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \to \frac{\partial}{\partial \cdot}P_1U(\cdot,t-k)\omega_k \text{ weakly in } L^2(t-k,t;H^{-1}(\Omega_m)).$$

By the compactness result in [11], we have

$$P_1U(\cdot,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \to P_1U(\cdot,t-k)\omega_k \text{ strongly in } L^2(t-k,t;L^2(\Omega_m).$$

Thus, for $m \ge \tilde{m}$ we have

(3.53)

$$\begin{aligned}
\lim_{n'\to\infty} \sup I_4 &\le -2\beta \int_{t-k}^{t} e^{-2\lambda(t-s)} \int_{|x|\le m} h(P_1U(s,t-k)\omega_k)P_1U(s,t-k)\omega_k ds \\
&\quad + \varepsilon \lim_{n'\to\infty} \sup(\int_{t-k}^{t} e^{-2\lambda(t-s)} ||P_1U(s,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}})||_{H^1}^{2(r+1)} ds)^{\frac{1}{2}} + \varepsilon.
\end{aligned}$$

By Lemma 3.5 and (3.43), we have

(3.54)
$$
\begin{aligned}
&2\beta\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle f(s),P_1U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\rangle ds\\
&\to 2\beta\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle f(s),P_1U(s,t-k)\omega_k\rangle ds,\ n'\to\infty,
\end{aligned}
$$

and

(3.55)
$$
\begin{aligned}
&2\alpha\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle g(s),P_2U(s,t-k)U(t-k,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\rangle ds\\
&\to 2\alpha\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle g(s),P_2U(s,t-k)\omega_k\rangle ds,\ n'\to\infty.
\end{aligned}
$$

Considering (3.50)-(3.55) and letting $m\to\infty$, we get from (3.48) that

(3.56)
$$
\begin{aligned}
&\lim_{n'\to\infty}\sup(\beta\parallel P_1U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\parallel^2+\alpha\parallel P_2U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\parallel^2)\\
&\le e^{-2\lambda k}\delta_1(R(t-k))^2+2\alpha(\lambda-\delta)\int_{t-k}^{t}e^{-2\lambda(t-s)}\parallel P_2U(s,t-k)\omega_k\parallel^2 ds\\
&\quad-2\beta\nu\int_{t-k}^{t}e^{-2\lambda(t-s)}\parallel\nabla P_1U(s,t-k)\omega_k\parallel^2 ds\\
&\quad-2\beta\int_{t-k}^{t}e^{-2\lambda(t-s)}\int_{\mathbb{R}^n}h(P_1U(s,t-k)\omega_k)P_1U(s,t-k)\omega_k ds\\
&\quad+2\beta\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle f(s),P_1U(s,t-k)\omega_k)\rangle ds\\
&\quad+2\alpha\int_{t-k}^{t}e^{-2\lambda(t-s)}\langle g(s),P_2U(s,t-k)\omega_k)\rangle ds\\
&\quad+\varepsilon\lim_{n'\to\infty}\sup(\int_{t-k}^{t}e^{-2\lambda(t-s)}\parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}})\parallel_{H^1}^{2(r+1)}ds)^{\frac{1}{2}}+\varepsilon.
\end{aligned}
$$

Combining (3.44) and (3.47), we can obtain

(3.57)
$$
\begin{aligned}
&\parallel(\beta^{\frac{1}{2}}u_0,\alpha^{\frac{1}{2}}\upsilon_0)\parallel_{H\times H}^2=\parallel(\beta^{\frac{1}{2}}P_1U(t,t-k)\omega_k,\alpha^{\frac{1}{2}}P_2U(t,t-k)\omega_k)\parallel^2\\
&=e^{-2\lambda k}(\beta\parallel u_k\parallel^2+\alpha\parallel\upsilon_k\parallel^2)+2\alpha(\lambda-\delta)\int_{t-k}^{t}e^{-2\lambda(t-s)}\parallel P_2U(s,t-k)\omega_k\parallel^2 ds\\
&\quad-2\beta\nu\int_{t-k}^{t}e^{-2\lambda(t-s)}\parallel\nabla P_1U(s,t-k)\omega_k\parallel^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&- 2\beta\int_{t-k}^{t} e^{-2\lambda(t-s)}\langle h(P_1U(s,t-k)\omega_k), P_1U(s,t-k)\omega_k)\rangle ds\\
&+ 2\beta\int_{t-k}^{t} e^{-2\lambda(t-s)}\langle f(s), P_1U(s,t-k)\omega_k)\rangle ds\\
&+ 2\alpha\int_{t-k}^{t} e^{-2\lambda(t-s)}\langle g(s), P_2U(s,t-k)\omega_k)\rangle ds.
\end{aligned}
$$

By (3.56) and (3.57), we can deduce that

(3.58)

$$
\begin{aligned}
&\lim_{n'\to\infty}\sup(\beta \parallel P_1U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2 +\alpha \parallel P_2U(t,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel^2)\\
&\leq e^{-2\lambda k}\delta_1(R(t-k))^2 + (\beta \parallel u_0 \parallel^2 +\alpha \parallel \upsilon_0 \parallel^2) - e^{-2\lambda k}(\beta \parallel\ u_k \parallel^2 +\alpha \parallel \upsilon_k \parallel^2)\\
&+\varepsilon \lim_{n'\to\infty}\sup(\int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel_{H^1}^{2(r+1)} ds)^{\frac{1}{2}} + \varepsilon.
\end{aligned}
$$

Finally, we only need to prove that for any $t \in \mathbb{R}$,

$$
\lim_{n'\to\infty}\sup(\int_{t-k}^{t} e^{-2\lambda(t-s)} \parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel_{H^1}^{2(r+1)} ds)^{\frac{1}{2}} < +\infty.
$$

From (3.21) we get that

$$
\begin{aligned}
&\parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel_{H^1}^{2(r+1)}\\
&\leq Ce^{-\sigma(r+1)s}(e^{\sigma\tau_{n'}}(\beta \parallel u_{\tau_{n'}} \parallel^2 +\alpha \parallel \upsilon_{\tau_{n'}} \parallel^2))^{r+1}\\
&\quad + Ce^{-\sigma(r+1)s}(\int_{-\infty}^{t} e^{\sigma s}(\parallel f(s) \parallel^2 + \parallel g(s) \parallel^2)ds)^{r+1}.
\end{aligned}
$$

Then we can easily obtain that for $0<\sigma<\min\{\lambda, \frac{2\lambda}{r+1}\}$,

$$
\begin{aligned}
&\int_{\tau_{n'}}^{t} e^{-2\lambda(t-s)} \parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}},\upsilon_{\tau_{n'}}) \parallel_{H^1}^{2(r+1)} ds\\
&\leq C[(e^{\sigma\tau_{n'}}(\beta||u_{\tau_{n'}}||^2 + \alpha||\upsilon_{\tau_{n'}}||^2))^{r+1} + (\int_{-\infty}^{t} e^{\sigma s}(\parallel f(s) \parallel^2 + \parallel g(s) \parallel^2)ds)^{r+1}]\\
&\qquad \times \int_{-\infty}^{t} e^{-2\lambda(t-s)}e^{-\sigma(r+1)s}ds\\
&\leq \frac{Ce^{-\sigma(r+1)t}}{2\lambda-\sigma(r+1)}[(e^{\sigma\tau_{n'}}(\beta \parallel u_{\tau_{n'}} \parallel^2 +\alpha \parallel \upsilon_{\tau_{n'}} \parallel^2))^{r+1} + (\int_{-\infty}^{t} e^{\sigma s}(\parallel f(s) \parallel^2\\
&\qquad + \parallel g(s) \parallel^2)ds)^{r+1}].
\end{aligned}
$$

Since $(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \in D_0(\tau_{n'})$, by (3.13), we have

(3.59)

$$
\begin{aligned}
&\lim_{n'\to\infty} \sup\Big(\int_{\tau_{n'}}^{t} e^{-2\lambda(t-s)} \parallel P_1U(s,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \parallel_{H^1}^{2(r+1)} ds\Big)^{\frac{1}{2}} \\
&\le \frac{2Ce^{-\frac{\sigma(r+1)t}{2}}}{\sqrt{2\lambda-\sigma(r+1)}}\Big(\int_{-\infty}^{t} e^{\sigma s}(\parallel f(s) \parallel^2 + \parallel g(s) \parallel^2)ds\Big)^{\frac{r+1}{2}} \\
&< \infty.
\end{aligned}
$$

By (3.58) and (3.59), we can get

(3.60)

$$
\begin{aligned}
&\lim_{n'\to\infty} \sup(\beta \parallel P_1U(t,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \parallel^2 + \alpha \parallel P_2U(t,\tau_{n'})(u_{\tau_{n'}}, \upsilon_{\tau_{n'}}) \parallel^2) \\
&\le e^{-2\lambda k}\delta_1(R(t-k))^2 + (\beta \parallel u_0 \parallel^2 + \alpha \parallel \upsilon_0 \parallel^2) - e^{-2\lambda k}(\beta \parallel u_k \parallel^2 + \alpha \parallel \upsilon_k \parallel^2) \\
&\quad + \varepsilon \frac{2Ce^{-\frac{\sigma(r+1)t}{2}}}{\sqrt{2\lambda-\sigma(r+1)}}\Big(\int_{-\infty}^{t} e^{\sigma s}(\parallel f(s) \parallel^2 + \parallel g(s) \parallel^2)ds\Big)^{\frac{r+1}{2}} + \varepsilon.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&e^{-2\lambda k}(R(t-k))^2 \\
&= \frac{2e^{-\sigma t}e^{-(2\lambda-\sigma)k}}{\gamma}\Big(\frac{\beta}{\lambda - C_0}\int_{-\infty}^{t-k} e^{\sigma s} \parallel f(x,s) \parallel^2 ds + \frac{\alpha}{\delta}\int_{-\infty}^{t-k} e^{\sigma s} \parallel g(x,s) \parallel^2 ds\Big) \\
&\to 0, \quad as\ k \to \infty.
\end{aligned}
$$

Letting $\varepsilon \to 0$ and $k \to \infty$ in (3.60) and considering (3.1), we get (3.46). This completes the proof. □

Remark 3.7. In this paper, we only get the existence of pullback $\mathcal{D}$-attractor for system (1.1)-(1.3). For more information on pullback $\mathcal{D}$-attractor obtained in Theorem 3.6 such as dimension, regularity and inner structure, we leave them for future study.

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