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COMPARATIVE STUDY ON SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS VIA SHIFTED JACOBI COLLOCATION METHOD

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Abstract. In this paper, operational matrices of Riemann-Liouville fractional integration and Caputo fractional differentiation for shifted Jacobi polynomials are considered. Using the given initial conditions, we transform the fractional differential equation (FDE) into a modified fractional differential equation with zero initial conditions. Next, all the existing functions in modified differential equation are approximated by shifted Jacobi polynomials. Then, operational matrices and spectral collocation method are applied to obtain a linear or nonlinear system of algebraic equations. System of algebraic equations can be simultaneously solved (e.g. using $Mathematica^{TM}$). Main characteristic behind of the this technique is that only a small number of shifted Jacobi polynomials is needed to obtain a satisfactory result which demonstrates the validity and efficiency of the method. Comparison between this method and some other methods confirm the good performance of the presented method. Also, this method is generalized for the multi-point fractional differential equation.
Keywords: Fractional-order differential equation, Riemann-Liouville integral, Jacobi polynomial, collocation method.
MSC(2010): Primary: 65Pxx; Secondary: 34A08, 65L60, 33C45.

1. Introduction

Fractional calculus has been used to model the physical and engineering processes [31,33]. Differential equations with fractional order have recently proved to be valuable tools for the modeling of many physical phenomena [14, 34]. Recently, the applied scientists and the engineers find out that FDEs prepared a better approach to explain the complex phenomena in nature, for example, oscillation of earthquakes [17], fluid-dynamic traffic model [18], non-Brownian

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motion, signal processing, systems identification, control, viscoelastic materials and polymers (see [5,22,34] and references therein). Also, Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [19]. Due to its tremendous scope and applications in several disciplines, a considerable attention has been given to exact and numerical solutions of fractional differential equations which is an extremely difficult task. Even numerical approximation of fractional differentiation of rough functions is not easy as it is an ill-posed problem [16]. Analytic results on the existence and uniqueness of solutions to the fractional differential equations have been studied by many authors [22,34]. Finding approximate or exact solutions of FDEs is an important task. Except for a limited number of these equations, we have difficulty in finding their analytical solutions. Therefore, there have been attempts to develop new methods for obtaining numerical solutions which reasonably approximate the exact solutions. There are several approximate methods such as Adomian decomposition method [36], variational iteration method [43], homotopy perturbation method [32], homotopy analysis method [16], collocation method [35], Galerkin method [12], operational matrix method [38] and other methods [24, 28, 44]. Orthogonal functions have received special attention in dealing with different problems. The main excellence behind the approach using this method is that it reduces the solution of original problem to the solution of a system of algebraic equations which it leads to simplify the solution procedure. In these methods, a truncated orthogonal series is considered for numerical solution of differential equations, with the goal of obtaining efficient computational solutions for example shifted Legendre polynomials [6], hat basis functions [1], shifted Jacobi polynomials [9] and etc. Some wavelet basis approaches have also been successfully employed to solve the fractional order differential equations, for example Haar wavelet operational matrix [27], Chebyshev wavelets [25], block pulse operational matrix [26] and etc. Motivation of the present paper is to extend the application of Jacobi polynomials to provide an approximate algorithm for solving nonlinear and linear FDEs. In order to achieve this aim, we homogenize original problem to derive a problem with zero initial conditions by changing of variables. It is worthy to mention that, the method based on using the operational matrices of the orthogonal function for solving differential equations is computer oriented.

This manuscript is organized as follows: we introduce some necessary definitions and preliminaries for fractional calculus, Jacobi polynomials and their relevant properties needed here after in Section 2. In Section 4, the Jacobi operational matrices of fractional derivative and integration are introduced. Section 5 is dedicated to propose a technique in order to apply the shifted Jacobi fractional operational matrices for solving multi-order fractional differential equation. In Section 6 numerical results that is exhibiting the accuracy and efficiency of presented algorithm are showed. Section 7 is devoted to give a brief conclusion.

2. Preliminaries and notation

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

2.1. **Fractional integration in the Riemann-Liouville sense.** There are several definitions of a fractional integration of order $v \geq 0$, and not necessarily equivalent to each other [29, 33]. In this study work, we use the Riemann-Liouville fractional integral operator.

Definition 2.1. Riemann-Liouville fractional integral operator of order $v(v \geq 0)$ is defined as

$$\begin{aligned} J^v f(x) &= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt\,, \quad v>0, \quad x>0\,, \\ J^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

We mention only two properties of the operator J^v,

$$J^v x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+v)} x^{\beta+v}\,,$$

$$J^\beta J^v f(x) = J^{\beta+v} f(x) = J^v J^\beta f(x).$$

Similar to the integer-order integration, the Riemann-Liouvilles fractional integration is a linear operation

$$J^v(\lambda f(x) + \mu g(x)) = \lambda J^v f(x) + \mu J^v g(x),$$

where λ and μ are constants.

Definition 2.2. Caputo fractional derivatives of order v is defined as

$$D^v f(x) = J^{m-v} D^m f(x) = \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{m-v-1} \frac{d^m}{dt^m} f(t)dt\,, \tag{2.2}$$

for $m-1 < v \leq m$, $m \in \mathbb{N}$ and $x > 0$, where D^m is the classical differential operator of order m.

For the Caputo derivative we have

$$D^v x^\beta = \begin{cases} 0\,, & \text{for } \beta < v\,, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}\,, & \text{for } \beta \geq v. \end{cases}$$

Recall that for $v \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operation, i.e.

$$D^v(\lambda f(x) + \mu g(x)) = \lambda D^v f(x) + \mu D^v g(x)\,,$$

where μ and λ are constants.

Lemma 2.3. *If $m-1 < v \leq m$, $m \in \mathbb{N}$, then* [3]

$$D^v J^v f(x) = f(x)\,,$$

$$J^v D^v f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{x^i}{i!}, \quad x > 0. \tag{2.3}$$

2.2. **Jacobi polynomials.** Jacobi polynomials denote by $p_n^{(\alpha,\beta)}(t)$ are defined as the orthogonal polynomials with respect to the weight function $w^{(\alpha,\beta)}(t) = (1-t)^{\alpha}(1+t)^{\beta}(\alpha > -1, \beta > -1)$ which $t \in [-1,1]$ (see [4,10,39,41]), i.e.,

$$\int_{-1}^{1} p_n^{(\alpha,\beta)}(t) p_m^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t)dt = \gamma_n^{\alpha,\beta}\delta_{nm}\,,$$

where δ_{nm} is the Kronecker's delta and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}.$$

Jacobi polynomials can be created by means of the following recurrence formula

$$p_i^{(\alpha,\beta)}(t) = \frac{(\alpha+\beta+2i-1)\{\alpha^2-\beta^2+t(\alpha+\beta+2i)(\alpha+\beta+2i-2)\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} p_{i-1}^{(\alpha,\beta)}(t) - \frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} p_{i-2}^{(\alpha,\beta)}(t), \qquad i = 2,3,\ldots$$

where $p_0^{(\alpha,\beta)}(t) = 1$ and $p_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta+2}{2}t + \frac{\alpha-\beta}{2}$.
Jacobi polynomial $p_n^{(\alpha,\beta)}(t)$ of degree n given by

$$p_n^{(\alpha,\beta)}(t) = 2^{-n}\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}(t-1)^i(t+1)^{n-i}.$$

In order to use these polynomials on the interval $x \in [0,L]$ we defined the so-called shifted Jacobi polynomials by introducing the change of variable $t = \frac{2x}{L} - 1$. Let the shifted Jacobi polynomials $p_i^{(\alpha,\beta)}(\frac{2x}{L}-1)$ be denoted by $p_{L,i}^{(\alpha,\beta)}(x)$. Then $p_{L,i}^{(\alpha,\beta)}(x)$ can be generated from

$$p_{L,i}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+2i-1)\{\alpha^2-\beta^2+(\frac{2x}{L}-1)(\alpha+\beta+2i)(\alpha+\beta+2i-2)\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} p_{L,i-1}^{(\alpha,\beta)}(x) - \frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} p_{L,i-2}^{(\alpha,\beta)}(x)\,, \quad i = 2,3,\ldots$$

where $p_{L,0}^{(\alpha,\beta)}(x) = 1$ and $p_{L,1}^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}(\frac{2x}{L}-1) + \frac{\alpha-\beta}{2}$.

Analytic form of the shifted Jacobi polynomials $p_{L,i}^{(\alpha,\beta)}(x)$ of degree i is given by

$$p_{L,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!L^k} x^k.$$

Shifted Jacobi polynomials are orthogonal with respect to the weight function $W_L^{(\alpha,\beta)}(x) = x^\beta (L-x)^\alpha$ in the interval $[0, L]$ with the orthogonality property

$$\int_0^L p_{L,n}^{(\alpha,\beta)}(x) p_{L,m}^{(\alpha,\beta)}(x) W_L^{(\alpha,\beta)}(x) dx = h_k \delta_{nm}\,,$$

where

$$h_k = \frac{L^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}.$$

Some properties of the shifted Jacobi polynomials, which are needed here, are as follows [10, 21]

(1) $p_{L,n}^{(\alpha,\beta)}(0) = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)n!}$,

(2) $p_{L,n}^{(\alpha,\beta)}(L) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}$,

(3) $\frac{d^i}{dx^i} p_{L,n}^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+\beta+i+1)}{L^i \Gamma(n+\alpha+\beta+1)} p_{L,n-i}^{(\alpha+i,\beta+i)}(x)$.

For $\alpha = \beta$ one recovers the shifted ultra spherical polynomials (symmetric Jacobi polynomials). For $\alpha = \beta = -\frac{1}{2}$, $\alpha = \beta = \frac{1}{2}$ and $\alpha = \beta = 0$ the shifted Chebyshev of the first and second kinds and shifted Legendre polynomials, respectively, and for the nonsymmetric shifted Jacobi polynomials, the two important special cases $\alpha = -\beta = \frac{1}{2}$ and $\alpha = -\beta = -\frac{1}{2}$ (shifted Chebyshev polynomials of the third and fourth kinds, respectively) are also recovered [9].

3. Approximation of function

Suppose that the weighted space $L^2_{W_L^{(\alpha,\beta)}}[0, L]$ in the usual way, with the following inner product and norm [9]

$$(u, v)_{W_L^{(\alpha,\beta)}} = \int_0^L u(x)v(x)W_L^{(\alpha,\beta)}(x)dx, \qquad ||u||_{W_L^{(\alpha,\beta)}} = (u, u)^{\frac{1}{2}}_{W_L^{(\alpha,\beta)}}.$$

Let $P_{L,m}^{\alpha,\beta} = Span\{p_{L,0}^{(\alpha,\beta)}(x), p_{L,1}^{(\alpha,\beta)}(x), \ldots, p_{L,m}^{(\alpha,\beta)}(x)\}$. Since $P_{L,m}^{\alpha,\beta}$ is a finite dimensional vector space of $L^2_{W_L^{(\alpha,\beta)}}[0, L]$, for every $f \in L^2_{W_L^{(\alpha,\beta)}}[0, L]$ there exists a unique best approximation $f_m(x) \in P_{L,m}^{\alpha,\beta}$, such that for each $u \in P_{L,m}^{\alpha,\beta}$,

$$||f - f_m||_{W_L^{(\alpha,\beta)}} \leq ||f - u||_{W_L^{(\alpha,\beta)}} \text{ (see [24])}.$$

Since the function $f_m(x) \in P^{\alpha,\beta}_{L,m}$, it can be expressed in terms of shifted jacobi polynomials as

$$f_m(x) = \sum_{j=0}^{m} c_j p^{(\alpha,\beta)}_{L,j}(x) = C^T \Phi(x), \tag{3.1}$$

where the coefficient vector C and the vector of shifted Jacobi functions $\Phi(x)$ are given by

$$\begin{aligned} C^T &= [c_0, c_1, \ldots, c_m], \\ \Phi(x)^T &= [p^{(\alpha,\beta)}_{L,0}(x), p^{(\alpha,\beta)}_{L,1}(x), \ldots, p^{(\alpha,\beta)}_{L,m}(x)]. \end{aligned} \tag{3.2}$$

The set of shifted Jacobi polynomials forms a complete $L^2_{W_L^{(\alpha,\beta)}}[0, L]$-orthogonal system and so

$$c_j = \frac{1}{h_j} \int_0^L W_L^{(\alpha,\beta)}(x) u(x) p^{(\alpha,\beta)}_{L,j}(x) dx \qquad j = 0, 1, \ldots m.$$

In the following lemma, an upper bound of approximation error is presented by assuming that the function f is $m + 1$ times continuously differentiable, which is notated by $f \in C^{(m+1)}[0, L]$.

Lemma 3.1. *Let the function $f : [0, L] \to \mathbb{R}$ be in $C^{(m+1)}[0, L]$ and let f_m be the best approximation from $P^{\alpha,\beta}_{L,m}$. Then*

$$||f - f_m||_{W_L^{(\alpha,\beta)}} \leq \frac{M}{(m+1)!} \sqrt{\frac{L^{3+2m+\alpha+\beta}\ \Gamma(1+\alpha)\ \Gamma(3+2m+\beta)}{\Gamma(4+2m+\alpha+\beta)}},$$

where $M = \max_{x \in [0,L]} \left| f^{(m+1)}(x) \right|$.

Proof. Consider that $y_1(x) = f(0) + f'(0)x + \cdots + f^m(0)\dfrac{x^m}{m!}$ is the Taylor polynomial of f at zero which the upper bound of error of Taylor polynomial is given as

$$|f(x) - y_1(x)| \leq \frac{M x^{m+1}}{(m+1)!}, \qquad \forall x \in [0, L]$$

where $M = \max_{x \in [0,L]} \left| f^{(m+1)}(x) \right|$.
Since f_m is the best approximation to f from $P^{\alpha,\beta}_{L,m}$, y_1 and $f_m \in P^{\alpha,\beta}_{L,m}$, we

have

$$\begin{aligned}||f-f_m||^2_{W_L^{(\alpha,\beta)}} \leq ||f-y_1||^2_{W_L^{(\alpha,\beta)}} &= \int_0^L (f(x)-y_1(x))^2\, W_L^{(\alpha,\beta)}(x)dx \\ &\leq \int_0^L \left(\frac{Mx^{m+1}}{(m+1)!}\right)^2 W_L^{(\alpha,\beta)}(x)dx \\ &\leq \left(\frac{M}{(m+1)!}\right)^2 \int_0^L x^{2m+2+\beta}(L-x)^\alpha dx \\ &= \left(\frac{M}{(m+1)!}\right)^2 \frac{L^{3+2m+\alpha+\beta}\,\Gamma(1+\alpha)\,\Gamma(3+2m+\beta)}{\Gamma(4+2m+\alpha+\beta)}\end{aligned}$$

and by taking the square roots we have the above bound. □

Lemma 3.1 shows that, when $f^{(m+1)}$ (the $(m+1)$-th derivative of f) is continuous, then the best approximatin f_m converges to f, as $m \to \infty$. In the following we are going to show that, this best approximation converges to f, if f is continuous.

Definition 3.2. Modulus of continuity $\omega(f,\delta)$ of function f on $[t_0,t_f]$ is defined as

$$\omega(f,\delta) = \sup_{\substack{x,y\in[t_0,t_f]\\ |x-y|\leq\delta}} |f(x)-f(y)|.$$

Lemma 3.3. *$f(x)$ is uniformly continuous on $[t_0,t_f]$ if and only if* $\lim_{\delta\to 0}\omega(f,\delta)=0$.

Proof. [37]. □

Theorem 3.4. *If $f(x)$ be bounded on $[0,1]$, then*

$$||f-p(f,m)||_\infty \leq \frac{3}{2}\omega(f,\frac{1}{\sqrt{m}}),$$

where $p(f,m)=\sum_{k=0}^m f(\frac{k}{m})B_{k,m}$ *and* $||f||_\infty = \sup\{\,|f(x)| : x \in \text{domain of f}\,\}$. *Also, if f satisfies a Lipschitz condition of order α on $[0,1]$, then*

$$||f-p(f,m)||_\infty \leq \frac{3}{2}km^{-\frac{\alpha}{2}},$$

where k is Lipschitz constant [37].

Lemma 3.5. *If $f(x)$ be bounded on $[0,1]$ and*

$$P^{\alpha,\beta}_{L,m} = Span\{p^{(\alpha,\beta)}_{L,0}(x), p^{(\alpha,\beta)}_{L,1}(x),\ldots,p^{(\alpha,\beta)}_{L,m}(x)\},$$

then

$$||f-f_m||_2 \leq \frac{3}{2}\omega(f,\frac{1}{\sqrt{m}}),$$

where f_m is the best approximation f out of $P^{\alpha,\beta}_{L,m}$.

Proof. Since f_m is the best approximation for f out of $P^{\alpha,\beta}_{L,m}$, $p(f,m) \in P^{\alpha,\beta}_{L,m}$ and Bernstein ploynomials of degree m are a basis for $P^{\alpha,\beta}_{L,m}$, by using $||f||_2 \leq ||f||_\infty$ we have

$$||f - f_m||_2 \leq ||f - p(f,m)||_2 \leq ||f - p(f,m)||_\infty \leq \frac{3}{2}\omega(f, \frac{1}{\sqrt{m}}).$$

□

Note that if f is defined on $[t_0, t_f]$, we can transmit $[t_0, t_f]$ to $[0,1]$ and if f is uniformly continuous on $[0,1]$, by using Lemmas 3.3 and 3.5 and Theorem 3.4, we have $\lim_{m\to\infty} \omega(f, \frac{1}{\sqrt{m}}) = 0$.

4. **Operational matrices of the Jacobi polynomials**

In general, let $\Phi(t) = [\Phi_0(t), \Phi_1(t), \ldots, \Phi_{n-1}(t)]^T$, where the elements $\Phi_0(t)$, $\Phi_1(t), \ldots, \Phi_{n-1}(t)$ are independent functions on $[a,b]$. Operational matrices for the integration $I_{n\times n}$, differentiation $D_{n\times n}$, dual $Q_{n\times n}$ and product $G_{n\times n}$ are respectively introduced as [13]

$$\int_a^x \Phi(t)dt \simeq I\Phi(x)\,, \qquad \frac{d}{dt}\Phi(x) \simeq D\Phi(x)\,,$$
$$Q = \int_a^b \Phi(t)\Phi^T(t)dt\,, \qquad \Phi(x)\Phi(x)^T C \simeq G\Phi(x).$$

4.1. **Operational matrix of fractional integration.** Main objective of this subsection is to generalize the shifted Jacobi operation matrix of integration for fractional calculus.

Definition 4.1. For $v \geq 0$, we define the operational matrix of fractional integration of the vector $\Phi(x)$ by $I^{(v)}$, where [21]

$$J^v\Phi(x) \simeq I^{(v)}\Phi(x). \tag{4.1}$$

Theorem 4.2. *If $I^{(v)}$ is the $m \times m$ operational matrix of Riemann-Liouville fractional integral of order v, then the elements of this matrix are obtained as* [21]

$$\{I^{(v)}_{ij}\}_{ij=0}^{m-1} = \sum_{k=0}^{i}\sum_{l=0}^{j} p_k^{(i)} p_l^{(j)} \frac{\Gamma(k+1)B(k+l+v+\beta+1, \alpha+1)}{\Gamma(k+v+1)h_k}.$$

Remark 4.3. If $v \in \mathbb{N}$, the operational matrix of integral by definition of Eq. (2.1) gives the same result as above equation. Now the following Lemmas can be presented an upper bound for estimating the error of Riemann-Liouville

fractional integral operator. In first, we define the error vector E_v as:
$E_v = J^v\Phi - I^{(v)}\Phi = [E_{0,v}, E_{1,v}, \dots, E_{m-1,v}]^T$, where

$$E_{k,v} = J^v p_k^{\alpha,\beta}(x) - \sum_{j=0}^{m-1} I_{kj}^{(v)} p_j^{\alpha,\beta}(x)\,, \qquad k = 0, 1, \dots, m-1.$$

In order to propose the following Lemma, we define some preliminaries as mentioned in [21].
Let $\Omega = (0,1)$, and for any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_{w_s^{(\alpha,\beta)}}(\Omega)$ in the usual way, and denote its inner product, semi-norm and norm by $(u,v)_{r,w_s^{(\alpha,\beta)}}, |v|_{r,w_s^{(\alpha,\beta)}}$ and $\|v\|_{r,w_s^{(\alpha,\beta)}}$, respectively.
In particular, $L^2_{w_s}(\Omega) = H^0_{w_s^{(\alpha,\beta)}}(\Omega), (u,v)_{w_s} = (u,v)_{0,w_s^{(\alpha,\beta)}}$ and $\|v\|_{w_s} = \|v\|_{0,w_s^{(\alpha,\beta)}}$.

$$\begin{aligned} H^r_{w_s}(\Omega) &= \{f|f \ \ \textbf{is measurable} \ \&\, \|v\|_{r,w_s} < \infty\}\,, \\ \|f\|^2_{r,w_s^{(\alpha,\beta)}} &= \sum_{k=0}^{r} \left\|\partial_x^k f\right\|^2_{w_s^{(\alpha+k,\beta+k)}}\,, \quad |f|_{r,w_s^{(\alpha,\beta)}} = \|\partial_x^r f\|_{w_s^{(\alpha+r,\beta+r)}}\,. \end{aligned}$$

Lemma 4.4. (See [22]) *If* $E_{k,v} = J^v p_k^{\alpha,\beta}(x) - \sum_{j=0}^{m-1} I_{kj}^{(v)} p_j^{\alpha,\beta}(x) \in H^r_{w_s}(\Omega)$ *then an error bound of Riemann-Liouville fractional integral operator of order* $v > 0$ *of the vector* Φ *can be expressed by:*

$$\| E_{k,v} \|^2_{\mu,w_s^{(\alpha,\beta)}} \leq c^2((m-1)(m+\alpha+\beta-1))^{\mu-r} \sum_{ij=0}^{k} \rho_i^{(k)}\rho_j^{(k)} B(i+j+2v+\beta-r+1, \alpha+r+1),$$

where $r \in \mathbb{N}$, $0 \le \mu \le r$ *and* $B(x,y)$ *is the beta function , a generic positive constant* c *independent of any function and* $\rho_i^{(k)} = p_i^{(k)} i!/\Gamma(i+v-r+1)$.

The maximum norm of error vector E_v is achieved as [21]

$$\| E_v \|_\infty \leq \begin{cases} \frac{x_0^v}{m!|\Gamma(v-m+1)|}\left(\frac{S}{x_0}\right)^m \binom{m+\beta}{m} \sqrt{B(\alpha+1,\beta+1)}\,, & \beta \ge 0\,, \\ \frac{x_0^v}{m!|\Gamma(v-m+1)|}\left(\frac{S}{x_0}\right)^m \sqrt{B(\alpha+1,\beta+1)}\,, & \beta < 0, \end{cases}$$

which $S = max\{1-x_0, x_0\}$.

4.2. **Operational matrix of fractional derivatives.** In this subsection, we introduce the fractional differentiation operation matrix of the shifted Jacobi polynomials.

Theorem 4.5. *Let* $\Phi(x)$ *be shifted Jacobi vector defined in Eq.* (3.2) *and let also* $v > 0$. *Then* [10]

$$D^v\Phi(x) \simeq \mathbf{D}^{(v)}\Phi(x), \tag{4.2}$$

where $\mathbf{D}^{(v)}$ *is the* $(m+1)\times(m+1)$ *operational matrix of fractional derivative of order* v *in the Caputo sense and is defined by*

$$\mathbf{D}^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \Delta_v(\lceil v\rceil,0) & \Delta_v(\lceil v\rceil,1) & \Delta_v(\lceil v\rceil,2) & \cdots & \Delta_v(\lceil v\rceil,m) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta_v(i,0) & \Delta_v(i,1) & \Delta_v(i,2) & \cdots & \Delta_v(i,m) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta_v(m,0) & \Delta_v(m,1) & \Delta_v(m,2) & \cdots & \Delta_v(m,m) \end{pmatrix}$$

where

$$\Delta_v(i,j) = \sum_{k=\lceil v\rceil}^{i} \delta_{ijk}$$

and δ_{ijk} *is given by*

$$\delta_{ijk} = \frac{(-1)^{i-k} L^{\alpha+\beta-v+1}\Gamma(j+\beta+1)\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{h_j\Gamma(j+\alpha+\beta+1)\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(k-v+1)(i-k)!} \times \sum_{l=0}^{j} \frac{(-1)^{j-1}\Gamma(j+l+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(l+k+\beta-v+1)}{\Gamma(l+\beta+1)\Gamma(l+k+\alpha+\beta-v+2)(j-l)!\; l!}.$$

Note that in $\mathbf{D}^{(v)}$*, the first* $\lceil v\rceil$ *rows, are all zeros.*

5. **Applications of the fractional operational matrices**

In this section, in order to show the high importance of operational matrices of fractional integral and differentiation, we apply them to numerically solve multi-order fractional differential equation. Existence and uniqueness and continuous dependence of the solution to this problem are discussed in [38].

5.1. **Linear multi-order boundary value problems.** Consider the linear multi-order fractional differential equation:

$$D^v u(x) + \sum_{i=1}^{k} e_i D^{v_i} u(x) + u(x) = g(x)\,, \tag{5.1}$$

with initial conditions

$$u^{(i)}(0) = d_i\,, \quad i = 0,1,\ldots,n-1\,, \tag{5.2}$$

where e_i, for $i = 0,1,\ldots,k$ are real constant coefficients and also $n < v \le n+1$, $v_k < v_{k-1} < \ldots < v_1 < v$ and D^v denotes the Caputo fractional derivative of order v. To solve problem (5.1) at first we try to make changes in the problem to convert the original problem to a problem with zero initial conditions. Therefore we define

$$z(x) = u(x) - \hat{u}(x)\,, \tag{5.3}$$

where $\hat{u}(x)$ is a known function that implies the initial condition (5.2) and is determined using Taylor polynomial (i.e. $\hat{u}^{(i)}(0) = d_i$ for $i = 0, 1, \ldots, n-1$) and $z(x)$ is the new unknown function. It can be easily seen that $z^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n-1$. So by replacing Eq. (5.3) in Eqs. (5.1) and (5.2), the modified fractional differential equation is obtained as

$$D^v z(x) + \sum_{i=1}^{k} \hat{e}_i D^{v_i} z(x) + z(x) + \hat{u}(x) = \hat{g}(x)\,, \tag{5.4}$$

$$z^{(i)}(0) = 0\,, \quad i = 0, 1, \ldots, n-1\,, \tag{5.5}$$

Let $s(x) = \hat{g}(x) - \hat{u}(x)$. We can rewrite Eq. (5.4) as

$$D^v z(x) + \sum_{i=1}^{k} \hat{e}_i D^{v_i} z(x) + z(x) = s(x)\,, \tag{5.6}$$

In order to solve problem (5.6) we approximate $D^v z(x)$ and $s(x)$ by Eq. (3.1) in terms of the shifted Jacobi polynomials as

$$D^v z(x) \simeq \sum_{i=0}^{m} c_i p_{L,i}^{(\alpha,\beta)}(x) = C^T \Phi(x)\,, \tag{5.7}$$

$$s(x) \simeq \sum_{i=0}^{m} s_i p_{L,i}^{(\alpha,\beta)}(x) = S^T \Phi(x)\,, \tag{5.8}$$

where $S^T = [s_0, \ldots, s_m]$ and $C^T = [c_0, \ldots, c_m]$ are known and unknown vectors, respectively. From Eq. (2.3) and initial conditions (5.5) we have

$$z(x) = J^v D^v z(x) + \sum_{i=0}^{n-1} \frac{z^{(i)}(0)}{i!} x^i = J^v D^v z(x)\,. \tag{5.9}$$

Applying Eqs. (4.1), (5.7) and (5.9) we obtain

$$z(x) = J^v D^v z(x) \simeq C^T J^v \Phi(x) \simeq C^T I^{(v)} \Phi(x)\,. \tag{5.10}$$

By Eqs. (5.10) and (4.2) we have

$$D^{v_i} z(x) \simeq C^T I^{(v-v_i)} \Phi(x)\,, \quad i = 0, 1, \ldots, k. \tag{5.11}$$

Substituting Eqs. (5.7), (5.8), (5.10) and (5.11) in (5.6), the residual function $R_m(x)$ can be written as

$$R_m(x) = (C^T + \sum_{i=1}^{k} \hat{e}_i C^T I^{(v-v_i)} + C^T I^{(v)} - S^T)\Phi(x).$$

Then, we generate $(m+1)$ linear algebraic equations, by applying the collocation method. Unknown vector C can be found by simultaneously solving the linear algebraic equation system (e.g. using $Mathematica^{TM}$). Consequently, we have from Eq. (5.10)

$$z(x) = C^T I^{(v)} \Phi(x).$$

From Eq. (5.3) the approximated solution $u(x)$ is obtained as

$$u(x) = \hat{u}(x) + C^T I^{(v)} \Phi(x).$$

5.2. **Nonlinear multi-order boundary value problems.** In this section, we consider a nonlinear multi-order fractional differential equation

$$D^v u(x) = F\left(x, u(x), D^{\beta_1} u(x), \ldots, D^{\beta_k} u(x)\right), \tag{5.12}$$

with initial conditions

$$u^{(i)}(0) = d_i \quad i = 0, 1, \ldots, n-1\,,$$

where $n-1 < v \leq n$, $0 < \beta_1 < \beta_2 < \ldots, \beta_k < v$, and D^v denotes the Caputo fractional derivative of order v. It should be noted that F can be a nonlinear function in general. In order to numerically solve problem (5.12), we homogenize the problem (5.12) by using the presented method in Section 5.1 and Eq. (5.3) as

$$\begin{aligned} D^v z(x) &= \hat{F}(x, z(x) + \hat{u}(x), D^{\beta_1} z(x), \ldots, D^{\beta_k} z(x)), \\ z^{(i)}(0) &= 0\,, \quad i = 0, 1, \ldots, n-1. \end{aligned} \tag{5.13}$$

In order to use shifted Jacobi polynomials for this problem, we first approximate $z(x), D^v z(x)$ and $D^{\beta_j} z(x) (j = 1, \ldots, k)$ by Eqs. (5.10) , (5.7) and (5.11), respectively. By substituting these equations in Eq. (5.13), we write the residual function $R(x)$ as

$$\begin{aligned} R(x) = {} & -C^T \Phi(x) + \\ & \hat{F}(x, C^T I^{(v)} \Phi(x) + \hat{u}(x), C^T I^{(v-\beta_1)} \Phi(x), \ldots, C^T I^{(v-\beta_k)} \Phi(x)). \end{aligned} \tag{5.14}$$

To find the unknown vector C, we collocate Eq. (5.14) in $(m+1)$ points and the obtained nonlinear system of algebraic equations can be simultaneously solved (e.g. using $Mathematica^{TM}$). Consequently, we have

$$z(x) = C^T I^{(v)} \Phi(x), \qquad u(x) = \hat{u}(x) + C^T I^{(v)} \Phi(x).$$

Presented method is generalized for the nonlinear multi-point fractional differential equation in Example 6.2.

6. **Illustrative examples**

To illustrate the effectiveness of the proposed methods in the present paper, several test examples are carried out in this section. Presented method is stated in general case for arbitrary α, β in Jacobi polynomials and wherever not stated $\alpha = \beta = 0$ are considered.

Example 6.1. In this example, we consider the following initial value problem in the case of the inhomogeneous Bagely- Torvik equation [7, 10, 38]

$$D^2u(x) + D^{\frac{3}{2}}u(x) + u(x) = g(x)\,, \quad u(0) = 1, \quad u'(0) = 1, \quad x \in [0,1]\,, \tag{6.1}$$

where $g(x) = 1 + x$. Exact solution of this problem is $u(x) = 1 + x$.
We consider

$$u(x) = z(x) + 1 + x. \tag{6.2}$$

Then Eq. (6.1) is converted to

$$D^2z(x) + D^{\frac{3}{2}}z(x) + z(x) = 0\,. \quad z(0) = z'(0) = 0. \tag{6.3}$$

Now, the unknown function $D^2z(x)$ is approximated as

$$D^2z(x) = C^T\Phi(x)\,,$$

and by Eq.s (2.2) and (2.3) the following equations are obtained

$$D^{\frac{3}{2}}z(x) = C^TI^{(\frac{1}{2})}\Phi(x)\,, \quad z(x) = C^TI^{(2)}\Phi(x). \tag{6.4}$$

Using above equations, we rewrite Eq. (6.3) as

$$C^T[I + I^{(\frac{1}{2})} + I^{(2)}]\Phi(x) = 0,$$

Using the presented method with $m = 2$, the solution of above system is $C^T = 0$, which by using Eqs. (6.4) and (6.2) the exact solution is achieved.

Example 6.2. Consider the following nonlinear multi-point boundary Bagely-Torvik equation [10, 30]

$$D^2u(x) + D^{\frac{3}{2}}u(x) + u(x) = g(x) \quad u(0) = 0, \quad u(1) = 1, \quad x \in [0,1]\,, \tag{6.5}$$

where $g(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}}$, and the exact solution is $u(x) = x^2$.
At first, we can convert Eq. (6.5) to

$$D^2z(x) + D^{\frac{3}{2}}z(x) + z(x) = \hat{g}(x)\,, \quad z(0) = 0, \quad z(1) = 0, \quad x \in [0,1]\,, \tag{6.6}$$

where $z(x) = u(x) - x$, and $\hat{g}(x) = x^2 - x + 2 + 4\sqrt{\frac{x}{\pi}}$. Now, we approximate solution as

$$D^1 z(x) = C^T \Phi(x)\,, \qquad z(x) = C^T I^{(1)} \Phi(x)\,,$$
$$D^2 z(x) = C^T \mathbf{D}^{(1)} \Phi(x)\,, \qquad D^{\frac{3}{2}} z(x) = C^T \mathbf{D}^{(\frac{1}{2})} \Phi(x)\,,$$
$$\hat{g}(x) = G^T \Phi(x)\,,$$

where $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(\frac{1}{2})}$ are the operational matrix of derivative in the Caputo sense. The residual function for Eq. (6.6) is written as

$$R(x) = [C^T \mathbf{D}^{(1)} + C^T \mathbf{D}^{(\frac{1}{2})} + C^T I^{(1)} - G^T]\Phi(x).$$

Using the collocation method together with implying the condition

$$z(1) = C^T I^{(1)} \Phi(1) = 0,$$

the system of algebraic equations are achieved. Solution of system for $m = 4$ is

$$C^T = [0, 1, -1.16209 \times 10^{-17}, 1.59492 \times 10^{-18}, 5.48351 \times 10^{-18}],$$

so $z(x) = C^T I^{(1)} \Phi(x)$ and by using definition $z(x)$, $u(x) = z(x) + x$. In Figure 1, absolute error with $m = 4$ is proposed which shows the maximum absolute error of the presented method is a multiple of 10^{-18}. Also, the maximum absolute error of the method in [30] is a multiple of 10^{-14} which shows the good performance and preciseness of the presented technique.

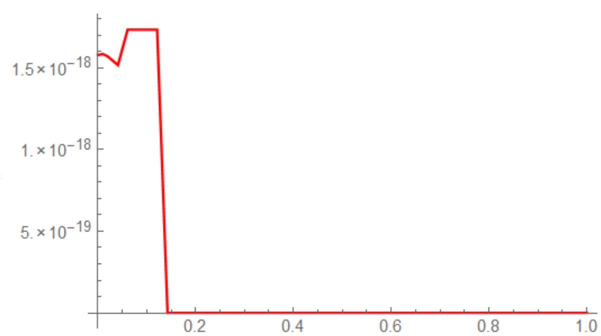

FIGURE 1. Absolute error with $m = 4$ for Example 2.

Example 6.3. Consider the following linear initial value problem with fractional-order [10, 16, 21, 38]

$$D^v u(x) + u(x) = 0\,, \quad 0 < v < 2\,, \quad I = (0, 1),$$
$$u(0) = 1\,, u'(0) = 0\,,$$

the second initial solution is for $v > 1$ only. Exact solution is given as [8,21,38],

$$u(x) = E_{v,1}(-x^v)\,,$$

where

$$E_{\delta,\epsilon}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\delta r + \epsilon)}$$

is the generalized Mittag-Leffler function.
We homogenize the main problem as

$$D^v z(x) + z(x) + 1 = 0\,, \quad 0 < v < 2\,, \quad I = (0,1),$$
$$z(0) = 0,\ z'(0) = 0\,,$$

Now, we approximate the solution as

(1) For $v < 1$:
$D^1 z(x) = C^T \Phi(x)\,, \qquad z(x) = C^T I^{(1)} \Phi(x)\,,$
$D^v z(x) = C^T I^{(1-v)} \Phi(x)\,, \quad 1 = G^T \Phi(x),$
and

(2) For $v > 1$:
$D^2 z(x) = C^T \Phi(x)\,, \qquad z(x) = C^T I^{(2)} \Phi(x)\,,$
$D^v z(x) = C^T I^{(2-v)} \Phi(x)\,, \quad 1 = G^T \Phi(x).$

Absolute error for $v = 0.85$ and $m = 2, 5, 8$ and 9 are shown in Table 1. From Table 1, it can be seen that a good approximation can be achieved by using a few terms of shifted Jacobi polynomials. Also, Figure 2 presents the numerical results for $u(x)$ with $m = 9$ and $v = 0.5,\ 0.75,\ 0.95$, and 1, where for $v = 1$, the exact solution is $u(x) = e^{-x}$. It is can realized from Figure 2 that the approximated solutions for different values v approach uniformly to $u(x) = e^{-x}$, by increasing v.
Figure 3 shows the numerical results for $u(x)$ for $m = 9$ and $v = 1.5,\ 1.75,\ 1.95$, and 2. For $v = 2$, the exact solution is $u(x) = cos(x)$. From Figure 3, we see that when the values v approach 2, the approximated solutions converge to $u(x) = cos(x)$. Absolute error for different values of v and $m = 9$ are shown in Table 2. From Table 2, we see that when v approaches to integer values ($v = 1,\ 2$) the error is reduced, which is expected. In order to make a comparison, we show the maximum absolute error of [10] in Table 3.

Table 1. Absolute error for different values of m for $v = 0.85$ for Example 3.

m	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
2	3.0×10^{-3}	7.8×10^{-3}	3.0×10^{-3}	7.7×10^{-3}	3.5×10^{-3}
5	7.8×10^{-4}	3.1×10^{-4}	4.8×10^{-4}	5.3×10^{-4}	6.1×10^{-4}
8	3.6×10^{-4}	1.0×10^{-6}	8.9×10^{-5}	1.3×10^{-4}	4.5×10^{-5}
9	2.2×10^{-4}	1.2×10^{-4}	8.2×10^{-5}	1.2×10^{-5}	8.5×10^{-5}

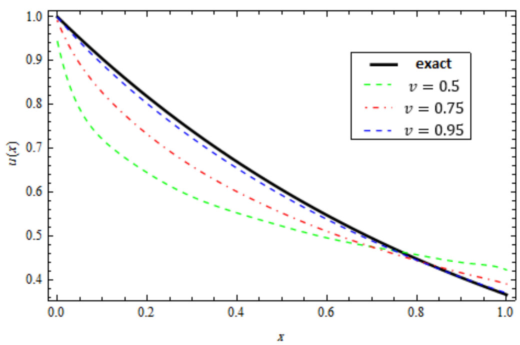

FIGURE 2. Comparison of $u(x)$ and $u_m(x)$ with $m = 9$ and $v = 0.5, 0.75, 0.95, 1$ for Example 3.

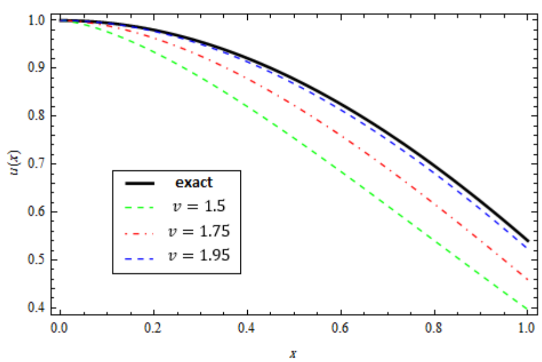

FIGURE 3. Comparison of $u(x)$ and $u_m(x)$ with $m = 9$ and $v = 1.5, 1.75, 1.95, 2$ for Example 3.

Example 6.4. Let the following nonlinear initial value problem with fractional-order [10, 20, 38, 40]

$$D^3u(x) + D^{\frac{5}{2}}u(x) + u^2(x) = x^4 \,,$$
$$u(0) = u'(0) = 0 \,, \quad u''(0) = 2.$$

Table 2. Absolute error for different values of v and for $m = 9$ for Example 3.

v	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.2	2.8×10^{-3}	1.8×10^{-3}	1.0×10^{-3}	2.6×10^{-4}	1.1×10^{-3}
0.4	3.8×10^{-2}	8.6×10^{-3}	4.7×10^{-3}	3.8×10^{-3}	3.6×10^{-3}
0.6	1.3×10^{-3}	7.8×10^{-4}	4.9×10^{-4}	1.0×10^{-4}	5.2×10^{-4}
0.8	3.6×10^{-4}	2.0×10^{-4}	1.3×10^{-4}	2.0×10^{-5}	1.4×10^{-4}
1	2.7×10^{-13}	9.8×10^{-14}	2.2×10^{-13}	7.9×10^{-14}	2.8×10^{-13}
1.2	6.6×10^{-5}	3.4×10^{-5}	2.4×10^{-5}	1.4×10^{-6}	2.4×10^{-5}
1.4	4.7×10^{-5}	2.5×10^{-5}	1.8×10^{-5}	1.8×10^{-6}	1.9×10^{-5}
1.8	5.9×10^{-6}	3.4×10^{-6}	2.4×10^{-6}	4.0×10^{-7}	2.6×10^{-6}
2	4.0×10^{-13}	1.3×10^{-13}	3.2×10^{-13}	1.2×10^{-13}	3.9×10^{-13}

Table 3. Maximum absolute error for $m = 10$ and different values of v with $\alpha = \beta = 0$ in [10] for Example 3.

v	0.2	0.4	0.6	0.8	1
Error	0.1684	0.0363	0.0100	0.0018	10^{-14}
v	1.2	1.4	1.6	1.8	2
Error	0.0046	0.0014	3.8×10^{-4}	7.3×10^{-5}	1.9×10^{-14}

The exact solution of this problem is $u(x) = x^2$. Main problem is converted to

$$D^3 z(x) + D^{\frac{5}{2}} z(x) + z^2(x) + 2x^2 z(x) = 0\,,$$
$$z(0) = z'(0) = z''(0) = 0\,,$$

where $z(x) = u(x) - x^2$. By applying the described technique as $z(x) = C^T I^{(3)}\Phi(x)$ with $m = 4$, the unknown coefficient is achieved $C^T = 0$, so the exact solution is obtained from $u(x) = z(x) + x^2 = x^2$.

Example 6.5. Consider the following initial value problem of multi-term nonlinear FDE [10]

$$D^\zeta u(x) + D^\eta u(x).D^\theta u(x) + u^2(x) = f(x),$$
$$f(x) = x^6 + \frac{6x^{3-\zeta}}{\Gamma(4-\zeta)} + \frac{36x^{6-\eta-\theta}}{\Gamma(4-\eta)\Gamma(4-\theta)}\,,$$
$$u(0) = u'(0) = u''(0) = 0,$$
$$\zeta \in (2,3)\,, \quad \eta \in (1,2)\,, \quad \theta \in (0,1)\,.$$

Exact solution of this problem is $u(x) = x^3$. We applied the presented method for this problem as

$$D^\zeta z(x) = C^T\Phi(x)\,, \qquad z(x) = C^T I^{(\zeta)}\Phi(x)\,,$$
$$D^v z(x) = C^T I^{(\zeta-v)}\Phi(x)\,, \qquad f(x) = G^T\Phi(x).$$

Figure 4 shows the analytical and numerical results for $m = 4, 8, 12$ and $\zeta = 2.5, \eta = 1.5, \theta = 0.9$. Also, absolute error for different values of ζ, η, θ and for $m = 5$, 6 are shown in Tables 4 and 5, respectively. From Tables 4 and 5, we see that the method gives the accurate results using a low degree of Jacobi polynomial. We compare our method with the results presented in [10] and present maximum absolute error of [10] for different values of ζ, η, θ with $m = 24$ for $\alpha = \beta = 1.5$ are shown in Table 6. From comparison between the results of our method and method of [10], it can be stated that our method gives acceptable solutions with a low degree of m which it shows that in view of computational this method is advantageous for solving fractional differential equations.

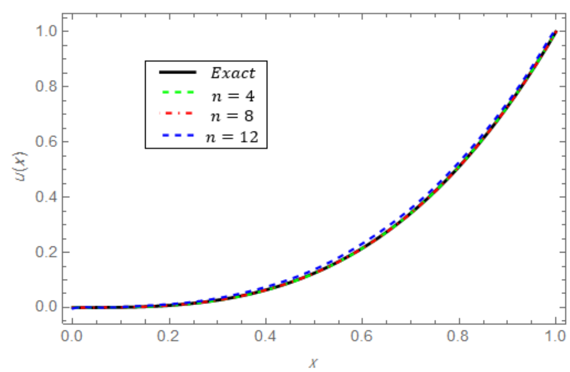

FIGURE 4. Comparison of $u(x)$ and $u_m(x)$ with $m = 4, 8, 12$ and $\zeta = 2.5, \eta = 1.5, \theta = 0.9$ for Example 5.

TABLE 4. Absolute error for different values of ζ, η, θ and $\alpha = \beta = 0$ for $m = 5$ for Example 5.

ζ, η, θ	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
2.5, 1.5, 0.9	6.9×10^{-6}	5.3×10^{-5}	6.2×10^{-5}	1.1×10^{-4}	1.7×10^{-4}
2.75, 1.75, 0.75	6.2×10^{-6}	5.1×10^{-5}	5.7×10^{-5}	9.9×10^{-5}	1.4×10^{-4}
2.99, 1.99, 0.99	1.5×10^{-6}	3.1×10^{-6}	1.3×10^{-5}	2.5×10^{-5}	4.3×10^{-5}

Example 6.6. Consider the following nonlinear initial value problem [2, 8, 38]

$$D^v u(x) = g(x) - [u(x)]^{3/2} ,$$
$$u(0) = 0, \quad u'(0) = 0, \quad 0 < v < 2 ,$$

Table 5. Absolute errors by $\zeta = 2.000001,\ \eta = 1.000001,\ \theta = 0.000001$ and $\alpha = \beta = 0$ for $m = 6$ for Example 5.

x	0.1	0.3	0.5	0.7	0.9
Error	5.0×10^{-11}	7.3×10^{-11}	1.5×10^{-10}	1.9×10^{-10}	2.1×10^{-10}

Table 6. Maximum absolute error for different values of ζ, η, θ with $m = 24$ and $\alpha = \beta = 1.5$ in [10] for Example 5.

ζ	η	θ	Error
2.000001	1.000001	0.000001	6.29×10^{-12}
2.5	1.5	0.9	3.15×10^{-5}
2.75	1.75	0.75	1.06×10^{-4}
2.99	1.99	0.99	1.95×10^{-5}

where

$$g(x) = \frac{40320}{\Gamma(9-v)}x^{8-v} - 3\frac{\Gamma(5+v/2)}{\Gamma(5-v/2)}x^{4-v/2} + \frac{9}{4}\Gamma(1+v) + (\frac{3}{2}x^{v/2} - x^4)^3.$$

As before, the second initial condition is for $v > 1$ only. Exact solution of this problem is given as

$$u(x) = x^8 - 3x^{4+v/2} + \frac{9}{4}x^v.$$

Using the described technique, we can write the residual function as

$$R(x) = C^T\Phi(x) + [C^T I^{(v)}\Phi(x)]^{\frac{3}{2}} - G^T\Phi(x),$$

which $D^v u(x) = C^T\Phi(x)$ and $g(x) = G^T\Phi(x)$. Figure 5 shows the analytical and numerical results for $m = 9$ and $v = 0.5,\ 0.75,\ 0.95$ and 1. Furthermore, the numerical results for $v = 1.5$ and $m = 5,\ 7,\ 9$ are plotted in Figure 6. Figure 7 shows the analytical and numerical results for $m = 9$ and $v = 1.5,\ 1.75,\ 1.95$ and 2. Figures 5 and 7 show that when v limits to 1 and 2, the approximated solutions $u_m(x)$ uniformly approach to the exact $u(x)$, respectively. Also the absolute error for different values of v and $m = 9$ are shown in Table 7. Absolute error of the presented method in [38] for different values of v and for $m = 10$ are shown in Table 8.

Example 6.7. Consider the following fractional differential equation with variable coefficients [11, 42]

$$aD^2u(x) + b(x)Du(x) + c(x)D^{\alpha_2}u(x) + e(x)D^{\alpha_1}u(x) + k(x)u(x) = f(x)\,,$$

where, $0 \le t \le 1$, $0 < \alpha_1 < \alpha_2 < 1$, and

$$f(x) = -a - b(x)x - \frac{c(x)}{\Gamma(3-\alpha_2)}x^{2-\alpha_2} - \frac{e(x)}{\Gamma(3-\alpha_1)}x^{2-\alpha_1} + k(x)(2 - \frac{1}{2}x^2)$$

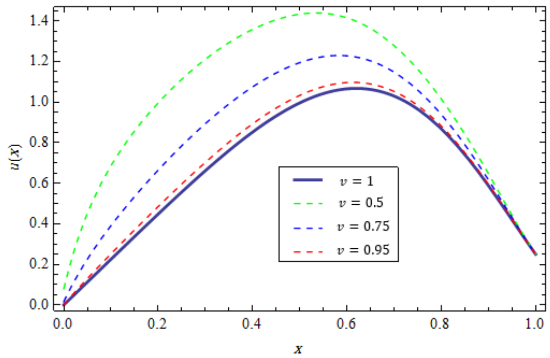

FIGURE 5. Comparison of $u(x)$ and $u_m(x)$ with $m = 9$ and $v = 0.5, 0.75, 0.95, 1$ for Example 6.

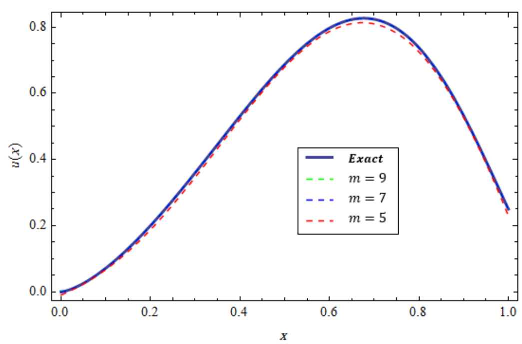

FIGURE 6. Comparison of $u(x)$ and $u_m(x)$ with $m = 5, 7, 9$ and $v = 1.5$ for Example 6.

subject to initial conditions $u(0) = 2\,, u'(0) = 0$.
Exact solution of this problem is $u(x) = 2 - \frac{1}{2}x^2$. Homogenized problem is

$$aD^2 z(x) + b(x)Dz(x) + c(x)D^{\alpha_2} z(x) + e(x)D^{\alpha_1} z(x) +$$
$$k(x)z(x) + 2k(x) = f(x),$$
$$z(0) = z'(0) = 0$$

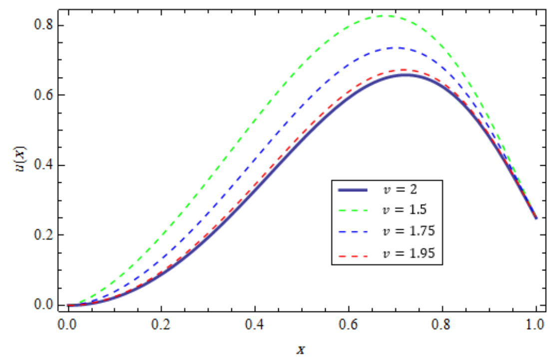

FIGURE 7. Comparison of $u(x)$ and $u_m(x)$ with $m = 9$ and $v = 1.5, 1.75, 1.95, 2$ for Example 6.

TABLE 7. Absolute error for different values of v and for $m = 9$ for Example 6.

v	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.2	2.7×10^{-2}	5.4×10^{-3}	8.6×10^{-4}	4.5×10^{-4}	1.6×10^{-3}
0.4	5.7×10^{-2}	1.6×10^{-2}	4.0×10^{-3}	2.6×10^{-3}	4.0×10^{-3}
0.6	2.3×10^{-2}	1.3×10^{-2}	4.5×10^{-3}	2.9×10^{-3}	3.2×10^{-3}
0.8	4.3×10^{-3}	4.2×10^{-3}	2.0×10^{-3}	1.4×10^{-3}	1.3×10^{-3}
1.2	1.5×10^{-3}	1.4×10^{-3}	1.5×10^{-3}	1.3×10^{-3}	8.5×10^{-4}
1.4	4.3×10^{-4}	6.2×10^{-4}	8.3×10^{-4}	8.1×10^{-4}	6.0×10^{-4}
1.6	2.0×10^{-5}	9.0×10^{-5}	2.0×10^{-4}	2.0×10^{-4}	1.5×10^{-4}
1.8	2.5×10^{-5}	1.7×10^{-5}	1.4×10^{-7}	1.1×10^{-5}	2.2×10^{-5}

where $z(x) = u(x) - 2$.
Problem is solved for $a = 0.1$, $b(x) = x$, $c(x) = x + 1$, $e(x) = x^2$, $k(x) = (x+1)^2$, $\alpha_1 = 0.781$ and $\alpha_2 = 0.891$. Absolute error for different values of x for $m = 8$ are shown in Table 9.
Problem has been also tested for solution when $a = 5$, $b(x) = \sqrt{x}$, $c(x) = x^2 - x$, $e(x) = 3x$, $k(x) = x^3 - x$, $\alpha_1 = \frac{\sqrt{7}}{70}$ and $\alpha_2 = \frac{\sqrt{13}}{13}$ and the absolute errors are shown in Table 10. Maximum absolute errors of the methods in [11] and [42] are a multiple of 10^{-2} and 10^{-3}, respectively. Tables 9 and 10 demonstrate that the presented method provides better results than the given methods in [11,42].

TABLE 8. Absolute error are presented in [38] for different values of v and for $m = 10$ for Example 6.

v	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.2	2.2×10^{-1}	2.3×10^{-1}	3.6×10^{-2}	5.3×10^{-1}	1.7×10^{-0}
0.4	6.3×10^{-2}	6.0×10^{-2}	2.4×10^{-2}	1.2×10^{-1}	3.0×10^{-1}
0.6	1.5×10^{-2}	1.3×10^{-2}	9.6×10^{-3}	2.1×10^{-2}	3.7×10^{-2}
0.8	2.9×10^{-3}	2.1×10^{-3}	2.3×10^{-3}	2.5×10^{-3}	2.1×10^{-3}
1.2	1.9×10^{-3}	1.6×10^{-3}	2.8×10^{-2}	2.9×10^{-3}	1.6×10^{-2}
1.4	2.0×10^{-4}	1.6×10^{-3}	7.6×10^{-3}	4.9×10^{-3}	3.3×10^{-2}
1.6	6.3×10^{-5}	7.3×10^{-4}	1.7×10^{-3}	2.3×10^{-3}	1.3×10^{-2}
1.8	3.8×10^{-5}	2.0×10^{-4}	2.6×10^{-4}	5.9×10^{-4}	2.8×10^{-3}

TABLE 9. Absolute error for $\alpha_1 = 0.781, \alpha_2 = 0.891$ with $m = 8$ for Example 7.

x	0.1	0.3	0.5	0.7	0.9
Error	2.3×10^{-5}	3.3×10^{-5}	2.5×10^{-5}	7.0×10^{-6}	8.2×10^{-6}

TABLE 10. Absolute error for $\alpha_1 = \frac{\sqrt{7}}{70}, \alpha_2 = \frac{\sqrt{13}}{13}$ with $m = 8$ for Example 7.

x	0.1	0.3	0.5	0.7	0.9
Error	1.1×10^{-6}	5.9×10^{-6}	1.1×10^{-5}	1.6×10^{-5}	2.1×10^{-5}

Example 6.8. Consider the nonlinear fractional differential equation [11, 27, 42]

$$aD^{\alpha}u(x) + bD^{\alpha_2}u(x) + cD^{\alpha_1}u(x) + e(u(x))^3 = f(x)\,,$$

where $0 \leq t \leq 1$, $2 < \alpha \leq 3$, $0 < \alpha_1 \leq 1$, $1 < \alpha_2 \leq 2$ and

$$f(x) = \frac{2a}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{2b}{\Gamma(4-\alpha_2)}x^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)}x^{3-\alpha_1} + e(\frac{x^3}{3})^3$$

subject to initial conditions

$$u(0) = u'(0) = u''(0) = 0.$$

Exact solution of this problem is $u(x) = \frac{x^3}{3}$. We let $\alpha = 2.2,\ \alpha_1 = 0.75,\ \alpha_2 = 1.25,\ a = b = c = e = 1$ and applied the presented method for this example. Numerical results for $m = 4, 8, 10$ are shown in Table 11. In [27], absolute errors with $m = 64,\ 128$ for this numerical test are a multiple of 10^{-4} and 10^{-5}, respectively. Aalso, in [42], absolute errors with $m = 32,\ 64$ are a multiple of 10^{-4} and 10^{-5}, respectively, and in [11] absolute errors is a multiple of 10^{-3}.

In our method, preciseness 10^{-5} is obtained by $m = 4$. Figure 8 is presented to show that the approximation solution with $m = 10$ globally converges to the exact solution.

TABLE 11. Absolute error with $m = 4, 8, 10$ for Example 8.

x	$m = 4$	$m = 8$	$m = 10$
0	1.7×10^{-5}	1.2×10^{-6}	5.5×10^{-7}
0.1	2.4×10^{-6}	2.4×10^{-7}	1.2×10^{-7}
0.2	1.6×10^{-5}	3.6×10^{-7}	4.8×10^{-8}
0.3	2.4×10^{-5}	5.2×10^{-8}	6.8×10^{-8}
0.4	2.8×10^{-5}	2.9×10^{-7}	1.2×10^{-7}
0.5	3.0×10^{-5}	5.1×10^{-8}	2.7×10^{-8}
0.6	3.0×10^{-5}	2.9×10^{-7}	8.4×10^{-8}
0.7	3.0×10^{-5}	4.6×10^{-9}	8.2×10^{-8}
0.8	3.1×10^{-5}	2.5×10^{-7}	5.7×10^{-9}
0.9	3.7×10^{-5}	2.1×10^{-7}	3.9×10^{-8}
1	4.9×10^{-5}	6.2×10^{-7}	2.1×10^{-7}

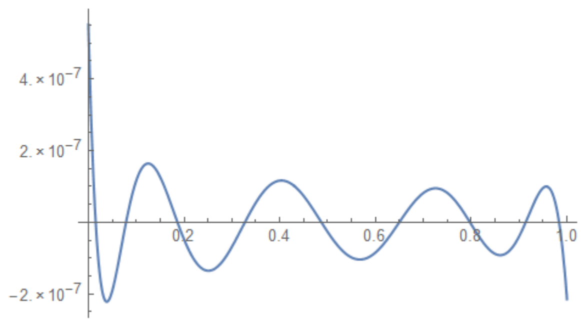

FIGURE 8. Error with $m = 10$ for Example 8.

7. Conclusion

Our aim in this paper is to present a simple method in viewpoint of application which has the better performance and preciseness than the other existing method. For this reason, fractional integration and derivative operational matrices of shifted Jacobi polynomials was employed to prepare a new technique for numerically solving nonlinear multi-order fractional differential equation. In order to gain this aim, original problem was homogenized by changing variable. Fractional derivative in the Caputo sense and fractional integration in the Riemann-Liouville sense were introduced and those operational matrices for shifted Jacobi polynomials in general case were stated. Using these matrices and collocation method, a new method for obtaining a numerical solution for multi-order fractional differential equation was discussed. Main advantages of the developed method is that, high accurate solutions were achieved by a few number of the Jacobi polynomials. Numerical results were compared favorably with the analytical solutions. Comparison between the mentioned method and other existing methods showed that this method gives the same or the better solution. Also, it is shown in Example 6.2 that this method can be generalized to the multi-point fractional differential equation. It can be highlited that our method gives acceptable solutions with a low degree of m which it shows that in view of computational this method is advantageous for solving fractional differential equations (see Example 6.5).

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