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SEPARATING PARTIAL NORMALITY CLASSES WITH WEIGHTED COMPOSITION OPERATORS ON L^2

H. EMAMALIPOUR, M. R. JABBARZADEH* AND Z. MOAYYERIZADEH

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Abstract. In this article, we discuss measure theoretic characterizations for weighted composition operators in some operator classes on $L^2(\Sigma)$ such as, n-power normal, n-power quasi-normal, k-quasi-paranormal and quasi-class$A$. Then we show that weighted composition operators can separate these classes.

Keywords: Conditional expectation, weighted composition operator, n-power normal, k-quasi-paranormal.
MSC(2010): Primary: 47B20; Secondary: 47B38.

1. **Introduction and preliminaries**

Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a sub-σ-finite algebra of Σ. We use the notation $L^2(\mathcal{A})$ for $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ and henceforth we write μ in place of $\mu_{|\mathcal{A}}$. If B is any subset of X, then we define $\mathcal{A}_B = \{B \cap A : A \in \mathcal{A}\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\varphi : X \to X$ be a measurable transformation such that $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and $\mu \circ \varphi^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite.

For a sub-σ-algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}}f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}}f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \qquad \text{for all} \quad A \in \mathcal{A}.$$

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As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is an orthogonal projection and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. We write $\mathcal{D}(E)$ for the domain of E. It is known that every non-negative measurable function and every $L^p(\Sigma)$-function with $1 \leq p \leq \infty$ is conditionable (see [12]). Put $\Sigma_n = \varphi^{-n}(\Sigma)$ and $E_n = E^{\varphi^{-n}(\Sigma)}$. Let $f \in \mathcal{D}(E) \subset L^0(\Sigma)$. Since $E_n(f)$ is a Σ_n-measurable function, there is a $g \in L^0(\Sigma)$ such that $E_n(f) = g \circ \varphi^n$. In general g is not unique. This deficiency can be solved by assuming $\sigma(g) \subseteq \sigma(h_n)$ (see [4]), because for each $g_1, g_2 \in L^0(\Sigma)$,

$$g_1 \circ \varphi^n = g_2 \circ \varphi^n \text{ if and only if } h_n g_1 = h_n g_2. \tag{1.1}$$

As a symbol, we then write $g = E_n(f) \circ \varphi^{-n}$ though we make no assumptions regarding the invertibility of φ^{-n}. With this setting, by the change of variables formula, we have

$$\int_X f d\mu = \int_X h_n E_n(f) \circ \varphi^{-n} d\mu, \qquad f \in L^0(\Sigma),$$

in the sense that if one of the integrals exists then so does the other, and they have the same value. For more details on the properties $E^{\mathcal{A}}$ see [12, 15, 18]. Let $u \in L^0(\Sigma)$. The weighted composition operator W on $L^2(\Sigma)$ induced by the pair (u, φ) is given by $W = M_u \circ C_\varphi$, where M_u is a multiplication operator and C_φ is a composition operator defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively. It is a classical fact that W is a bounded linear operator on $L^2(\Sigma)$ if and only if $J := hE(|u|^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$ (see [13]). It follows that $W^n = M_{u_n} \circ C_{\varphi^n}$ is a bounded operator on $L^2(\Sigma)$ precisely when $J_n := h_n E_n(|u_n|^2) \circ \varphi^{-n} \in L^\infty(\Sigma)$, where $n \geq 0$, $h_n = d\mu \circ \varphi^{-n}/d\mu$, $u_n = u(u \circ \varphi)(u \circ \varphi^2) \cdots (u \circ \varphi^{n-1})$. Put $\varphi_0 = I$, $h_0 = 1$, $J_1 = J$, $h_1 = h$ and $E_1 = E$. Throughout this paper we assume that $W : L^2(\Sigma) \to L^2(\Sigma)$ is a w.c.o. with non-negative weight function u and we also assume that $J \in L^\infty(\Sigma)$. Also we suppose $(X, \varphi^{-n}(\Sigma), \mu)$ is a σ-finite space. The results can be easily extended to the case of a complex-valued u.

The fundamental properties of weighted composition operators on measurable function spaces are studied by Harrington and Whitley [11], Singh [20], Lambert [13, 15], Campbell [4–6], Burnap and Jung [2, 3], Stochel [8], Takagi [21], Kumar and Maligranda [7] and many other mathematicians.

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $T = U|T|$ be the canonical polar decomposition for $T \in \mathcal{L}(\mathcal{H})$. An operator T is quasi-class$A$ if $T^*|T^2|T \geq T^*|T|^2 T$. For each $x \in \mathcal{H}$ and $k \in \mathbb{N}$, if $\|T^{k+2}x\|\|T^k x\| \geq \|T^{k+1}x\|^2$, then T is called k-quasi-paranormal operator. An operator T is p-$*$-paranormal if $\||T|^p U|T|^p x\|\|x\| \geq \||T|^p U^* x\|^2$ for all vectors x in $\mathcal{H}$. If $\|T^3 x\|\|Tx\| \geq \|T^*Tx\|^2$ for all vectors x in $\mathcal{H}$, then T is called quasi-$*$-paranormal. For some integer n, an operator T is n-power quasi-normal if $T^n T^* T = T^* T T^n$, and T is n-power normal operator if $T^nT^* = T^*T^n$. There are several well-known relationships among these classes (see [10, 16, 17, 19]). The hierarchical relationship between the classes is as follows: hyponormal$\Rightarrow$ quasi-class$A$ $\Rightarrow$ k-quasi-paranormal($k \in \mathbb{N}$); n-power normal$\Rightarrow$ n-power quasi-normal; 1-$*$-paranormal$\Rightarrow$ quasi-$*$-paranormal. In general, these inclusion relations are all proper. To study these classes, weighted composition operators, as an extension of weighted shift operators, are very useful tools. Weak hyponormal classes of composition operators are studied in [2, 3]. They have shown that composition operators can separate some weak hyponormal classes. The goal of this paper is to distinguish some partial normality classes of weighted composition operators. In Section 3, some examples are presented which show that weighted composition operators can separate these classes.

2. Characterizations

In this section we determine necessary and sufficient conditions for a weighted composition operator to be k-quasi-paranormal, quasi-class$A$ operator, n-power normal, n-power quasi-normal, quasi-$*$-paranormal and p-$*$-paranormal.

We shall make use of the following general properties of E and W (see [11, 13, 15, 18]):

- For $f, g \in \mathcal{D}(E)$, if g is Σ_1-measurable, then $E(fg) = E(f)g$.
- If $f \geq 0$, then $E(f) \geq 0$ and $\sigma(f) \subseteq \sigma(E(f))$.
- $\sigma(h_n \circ \varphi^n) = X$ and $\sigma(J \circ \varphi) = \sigma(E(u))$.
- $W^*f = hE(uf) \circ \varphi^{-1}$.
- $W^*Wf = hE(u^2) \circ \varphi^{-1} f$.
- $WW^*f = u(h \circ \varphi)E(uf)$.

Let $U|W|$ be the polar decomposition of W. It is easy to check that $|W| = M_{\sqrt{J}}$ and $U = M_{\frac{\chi_{\sigma(E(u))}}{\sqrt{J\circ\varphi}}} W$ (see [6]). Also, since for each $f, g \in L^2(\Sigma)$,

$$\begin{aligned}\langle U^*f, g\rangle &= \int_X \frac{(uf)\overline{g} \circ \varphi}{\sqrt{h \circ \varphi E(u^2)}} d\mu \\ &= \int_X h^{\frac{1}{2}}([E(u^2)]^{-\frac{1}{2}} E(uf)) \circ \varphi^{-1} \overline{g} d\mu \\ &= \langle h^{\frac{1}{2}}([E(u^2)]^{-\frac{1}{2}} E(uf)) \circ \varphi^{-1}, g\rangle,\end{aligned}$$

we have $U^*f = h^{\frac{1}{2}}([E(u^2)]^{-\frac{1}{2}} E(uf)) \circ \varphi^{-1}$.

Lemma 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and $k \in \mathbb{N}$. Then we have the following assertations:*

(i) [17, Proposition 2.1] *T is k-quasi-paranormal if and only if*

$$T^{*^{k+2}}T^{k+2} - 2\lambda T^{*^{k+1}}T^{k+1} + \lambda^2 T^{*^k}T^k \geq 0, \quad \textit{for each} \quad \lambda \geq 0. \tag{2.1a}$$

(ii) [16, p. 562] *T is quasi-$*$-paranormal if and only if*

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \geq 0, \quad \textit{for each} \quad \lambda \geq 0. \tag{2.1b}$$

(iii) [16, p. 561] *T is p-$*$-paranormal if and only if*

$$|T|^pU^*|T|^{2p}U|T|^p + 2\lambda|T|^{2p} + \lambda^2 \geq 0, \quad \textit{for each} \quad \lambda \geq 0. \tag{2.1c}$$

Theorem 2.2. *Let W be a weighted composition operator on $L^2(\Sigma)$. Then W is k-quasi-paranormal if and only if $J_{k+2}J_k \geq (J_{k+1})^2$ for each $k \in \mathbb{N}$.*

Proof. First notice that for each $f, g \in L^2(\Sigma)$, we have

$$\begin{aligned}\langle W^{*^k}W^kf, g\rangle &= \int_X u_kf\circ\varphi^k u_k\overline{g}\circ\varphi^k d\mu\\ &= \int_X h_kE_k(u_k^2)\circ\varphi^{-k}f\overline{g}.\end{aligned}$$

Hence we conclude that for each $k \in \mathbb{N}$, $W^{*^k}W^kf = h_kE_k(u_k^2)\circ\varphi^{-k}f$. By using (2.1a), W is k-quasi-paranormal if and only if for every $\lambda \geq 0$,

$$\begin{aligned}H(\lambda) := {}& h_{k+2}E_{k+2}(u_{k+2}^2)\circ\varphi^{-(k+2)}\\ &- 2\lambda h_{k+1}E_{k+1}(u_{k+1}^2)\circ\varphi^{-(k+1)} + \lambda^2h_kE_k(u_k^2)\circ\varphi^{-k} \geq 0.\end{aligned}$$

Note that $H(\lambda)$ is of the form $a - 2\lambda b + \lambda^2c$ with $a, b, c \geq 0$. Hence

$$H(\lambda) \geq 0 \text{ if and only if } ac \geq b^2, \text{ which is equivalent to } J_{k+2}J_k \geq (J_{k+1})^2.$$

Therefore the theorem is proved. □

Theorem 2.3. *The operator W is a quasi-classA operator if and only if*

$$h_3E_3\left(\sqrt{E_2\left(\frac{u_2^2(u\circ\varphi^2)^4}{h_2\circ\varphi^2}\right)}\right)\circ\varphi^{-3} \geq h_2E_2(u_2^2)\circ\varphi^{-2}.$$

Proof. Let $f, g \in L^2(\Sigma)$. Then we have

$$\begin{aligned}\langle W^*|W^2|Wf, g\rangle &= \langle (W^{*2}W^2)^{\frac{1}{2}}Wf, Wg\rangle\\ &= \int_X \sqrt{h_2}\sqrt{E_2(u_2^2)}\circ\varphi^{-2}(uf\circ\varphi)(u\overline{g}\circ\varphi)d\mu\\ &= \int_X \sqrt{h_2}\chi_{\sigma(h_2)}\sqrt{E_2(u_2^2)}\circ\varphi^{-2}(uf\circ\varphi)(u\overline{g}\circ\varphi)d\mu\\ &= \int_{\sigma(h_2)} \sqrt{E_2(u_2^2)}\circ\varphi^{-2}(uf\circ\varphi)(u\overline{g}\circ\varphi)\frac{d\mu\circ\varphi^{-2}}{\sqrt{h_2}}\\ &= \int_X \sqrt{E_2(\frac{u_2^2(u\circ\varphi^2)^4}{h_2\circ\varphi^2})}(f\circ\varphi^3)(\overline{g}\circ\varphi^3)d\mu\\ &= \int_X h_3E_3(\sqrt{E_2(\frac{u_2^2(u\circ\varphi^2)^4}{h_2\circ\varphi^2})})\circ\varphi^{-3}f\overline{g}d\mu.\end{aligned}$$

Hence

$$W^*|W^2|Wf = h_3E_3(\sqrt{E_2(\frac{u_2^2(u\circ\varphi^2)^4}{h_2\circ\varphi^2})})\circ\varphi^{-3}f.$$

On the other hand we have

$$\begin{aligned}\langle W^*|W|^2Wf, g\rangle &= \langle W^{*2}W^2f, g\rangle \\ &= \int_X u_2 f\circ\varphi^2 u_2\overline{g}\circ\varphi^2 d\mu \\ &= \int_X h_2E_2(u_2^2)\circ\varphi^{-2}f\overline{g}d\mu.\end{aligned}$$

Then $W^*|W|^2Wf = h_2E_2(u_2^2)\circ\varphi^{-2}f$. Since $W^*|W^2|Wf$ and $W^*|W|^2Wf$ are multiplication operators, so we get that $W^*|W^2|Wf \geq W^*|W|^2Wf$ if and only if $h_3E_3(\sqrt{E_2(\frac{u_2^2(u\circ\varphi^2)^4}{h_2\circ\varphi^2})})\circ\varphi^{-3} \geq h_2E_2(u_2^2)\circ\varphi^{-2}$. □

Lemma 2.4. [14] *Let $f \in L^2(\Sigma)$ and $Af := u(h\circ\varphi)E(uf)$. Then for all $0 < p < \infty$,*

$$A^pf = u(h^p\circ\varphi)[E(u^2)]^{p-1}E(uf).$$

Theorem 2.5. *The following assertions hold.*

(i) *W is quasi-$*$-paranormal if and only if $J_3 \geq J^3$.*
(ii) *W is p-$*$-paranormal if and only if p-paranormal.*
(iii) *W is n-power quasi-normal if and only if $(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} = hE(u^2)\circ\varphi^{-1}$, on $\sigma(u_n)$.*

Proof. (i) It is easy to verify that for all $f \in L^2(\Sigma)$, $W^*W^{*2}W^2Wf = h_3E_3(u_3^2)\circ\varphi^{-3}f$, and $W^*WW^*Wf = h^2(E(u^2))^2\circ\varphi^{-1}f$. Therefore by (2.1b), W is quasi-$*$-paranormal if and only if for all $\lambda \geq 0$,

$$\phi(\lambda) := h_3E_3(u_3^2)\circ\varphi^{-3} - 2\lambda h^2(E(u)^2)^2\circ\varphi^{-1} + \lambda^2hE(u)^2\circ\varphi^{-1} \geq 0.$$

By a similar argument as in Theorem 2.2, W is quasi-$*$-paranormal if and only if $h_3E_3(u_3^2)\circ\varphi^{-3} \geq h^3(E(u^2))^3\circ\varphi^{-1}$.

(ii) By using (2.1c), W is p-$*$-paranormal if and only if

$$|W|^pU^*|W|^{2p}U|W|^p + 2\lambda|W|^{2p} + \lambda^2 \geq 0.$$

This condition is equivalent to $|W|^pU^*|W|^{2p}U|W|^p \geq (|W|^{2p})^2$. It is easy to verify that for every $f \in L^2(\Sigma)$, $|W|^{2p}f = (W^*W)^pf = h^pE(u^2)^p\circ\varphi^{-1}f$ and also by Lemma 2.4 we get that

$$\begin{aligned}U|W|^pf &= M_{\frac{1}{\sqrt{J\circ\varphi}}}WM_{J^{\frac{p}{2}}}f, \\ |W|^{2p}U|W|^pf &= J^p(J^{\frac{p-1}{2}}\circ\varphi)uf\circ\varphi, \\ U^*|W|^{2p}U|W|^pf &= h^{\frac{1}{2}}[E(u^2)]^{-\frac{1}{2}}\circ\varphi^{-1}J^{\frac{p-1}{2}}E(u^2J^p)\circ\varphi^{-1}f, \\ |W|^pU^*|W|^{2p}U|W|^pf &= h^p[E(u^2)]^{p-1}E(u^2J^p)]\circ\varphi^{-1}f.\end{aligned}$$

Then W is p-$*$-paranormal if and only if $E(u^2J^p) \geq h^p \circ \varphi[E(u^2)]^{p+1}$. Now, by [14, Theorem 2.2], this condition is equivalent to W is p-paranormal.

(iii) For each $f \in L^2(\Sigma)$, we have $W^nW^*Wf = u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1}f\circ\varphi^n$ and $W^*WW^nf = hu_nE(u^2)\circ\varphi^{-1}f\circ\varphi^n$. Hence $W^nW^*W = W^*WW^n$ if and only if $(u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} - hu_nE(u^2)\circ\varphi^{-1})f\circ\varphi^n = 0$, for each $f \in L^2(\Sigma)$ and this is equivalent to $|u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} - hu_nE(u^2)\circ\varphi^{-1}||f|\circ\varphi^n = 0$. Thus

$$\int_X h_nE_n(|u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} - hu_nE(u^2)\circ\varphi^{-1}|)\circ\varphi^{-n}|f|d\mu = 0,$$

for each $f \in L^2(\Sigma)$. It follows that

$$h_nE_n(|u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} - hu_nE(u^2)\circ\varphi^{-1}|)\circ\varphi^{-n} = 0, \text{ on } X.$$

Thus by the change of variables formula we can deduce that

$$|u_n(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} - hu_nE(u^2)\circ\varphi^{-1}| = 0, \text{ on } X.$$

Consequently, W is n-power quasi-normal if and only if $(h\circ\varphi^n)E(u^2)\circ\varphi^{n-1} = hE(u^2)\circ\varphi^{-1}$ on $\sigma(u_n)$. $\square$

Lemma 2.6. *Let $f \in L^2(\Sigma)$. Then for each $n \in \mathbb{N}$ the following assertions hold.*

(i) $W^{*^n}f = h_nE_n(u_nf)\circ\varphi^{-n}$.
(ii) $W^nW^{*^n}f = u_n(h_n\circ\varphi^n)E_n(u_nf)$.

Proof. (i) Let $f, g \in L^2(\Sigma)$. Then we get that

$$\langle W^{*^n}f, g\rangle = \langle f, W^ng\rangle = \int_X E_n(u_nf)\,\bar{g}\circ\varphi^n d\mu = \int_X h_nE_n(u_nf)\circ\varphi^{-n}\bar{g}d\mu.$$

Hence $W^{*^n}f = h_nE(u_nf)\circ\varphi^{-n}$.
(ii) For each $f, g \in L^2(\Sigma)$, we have

$$\begin{aligned}\langle W^nW^{*^n}f, g\rangle &= \int_X h_nE_n(u_nf)\circ\varphi^{-n}h_nE_n(u_n\bar{g})\circ\varphi^{-n}d\mu \\ &= \int_X (h_n\circ\varphi^n)E_n(u_n\bar{g}E_n(u_nf))\,d\mu \\ &= \int_X (h_n\circ\varphi^n)E_n(u_nf)\,u_n\bar{g}d\mu\end{aligned}$$

Consequently $W^nW^{*^n}f = u_n(h_n\circ\varphi^n)E_n(u_nf)$. $\square$

In the following theorem we characterize those pair (u,φ), for which W is n-power normal.

Theorem 2.7. *W is n-power normal if and only if*

(i) $u_n(h_n\circ\varphi^n)E_n(u_n) = h_nE_n(u_n^2)\circ\varphi^{-n}$, *and*

(ii) $\varphi^{-n}(\Sigma) \cap \sigma(u_n) = \Sigma \cap \sigma(u_n)$.

Proof. Suppose W is n-power normal. Then by [1, Proposition2.2], W^n is normal, and so $W^n W^{*^n} = W^{*^n} W^n$. This is equivalent to

$$u_n(h_n \circ \varphi^n)E_n(u_n f) = h_n E_n(u_n^2) \circ \varphi^{-n} f, \qquad f \in L^2(\Sigma). \tag{2.2}$$

It follows that

$$u_n(h_n \circ \varphi^n)E_n(u_n)f = h_n E_n(u_n^2) \circ \varphi^{-n} f, \qquad f \in L^2(\varphi^{-1}(\Sigma)),$$

and so (i) is true. Now by (2.2) and (i) we obtain

$$u_n(h_n \circ \varphi^n)E_n(u_n f) = u_n(h_n \circ \varphi^n)E_n(u_n)f, \qquad f \in L^2(\Sigma).$$

Since $\sigma(h \circ \varphi^n) = X$, so

$$u_n E_n(u_n f) = u_n E_n(u_n)f, \text{ for each } f \in L^2(\Sigma). \tag{2.3}$$

Let $\sigma(u_n) = X$. Then $E_n(u_n f) = E_n(u_n)f$, for each $f \in L^2(\Sigma)$. Set $f = \chi_C$, with $\mu(C) < \infty$, then we get that

$$\int_X u_n \chi_C d\mu = \int_X E_n(u_n \chi_C) d\mu = \int_X E_n(u_n)\chi_C d\mu.$$

Thus for each $C \in \Sigma$, with finite measure, we have

$$\int_C u_n d\mu = \int_C E_n(u_n) d\mu.$$

Consequently, $E_n(u_n) = u_n$ and so by equation (2.3) it is easy to see that $E_n f = f$ for each $f \in L^2(\Sigma)$, and this is equivalent to $\varphi^{-n}(\Sigma) = \Sigma$. Now assume that $\sigma(u_n) \neq X$. Since W^n is normal, it is easy to see that $\overline{ran(W^n)}$ is a reducing subspace for W^n. Equation (2.3) implies that for each $f \in L^2(\Sigma)$,

$$W^n W^{*^n} f = u_n(h_n \circ \varphi^n)E_n(u_n f) = u_n(h_n \circ \varphi^n)E_n(u_n)f.$$

Hence we can conclude that

$$\overline{ran(W^n)} = \overline{ran(W^n W^{*n})} = L^2(\sigma(u_n E_n(u_n)), \Sigma, \mu) = L^2(\sigma(u_n), \Sigma, \mu).$$

Thus $L^2(\sigma(u_n), \Sigma, \mu)$ is a reducing subspace for W^n. On the other hand

$$\overline{ran(W^n)} = \overline{ran(W^n W^{*n})} = \overline{ran(W^{*n} W^n)} = L^2(h_n E_n(u_n^2) \circ \varphi^{-n}).$$

It follows that $\sigma(u_n) = \sigma(h_n E_n(u_n^2) \circ \varphi^{-n})$. Then if $u_n(\varphi^n(x)) = 0$, we conclude that $(h_n \circ \varphi^n)(x)E_n(u_n^2)(x) = 0$ and so $u_n(x) = 0$. Hence we obtain that $\varphi^n(\sigma(u_n)) \subseteq \sigma(u_n)$, and this shows that W^n is a normal operator on $L^2(\sigma(u_n), \Sigma, \mu)$. Note that for this space $u_n > 0$. We put $\Sigma_n = \Sigma \cap \sigma(u_n)$ and φ_n as the restriction φ to $\sigma(u_n)$. Thus by using the preceding paragraph we get that $\varphi_n^{-n}(\Sigma_n) = \Sigma_n$.

Conversely, suppose that (i) and (ii) are true. Now by the fact that $h_n \circ \varphi^n > 0$, we get that $\sigma(u_n E_n(u_n)) = \sigma(h_n E_n(u_n^2) \circ \varphi^{-n})$, On the other hand since $u_n \geq 0$ we have $\sigma(u_n) \subseteq \sigma(E_n(u_n))$. Hence we get that

$$\sigma(u_n) = \sigma(h_n E(u_n^2) \circ \varphi^{-n}). \tag{2.4}$$

Now let C be a $\varphi^{-n}(\Sigma)$-measurable set with finite measure. Then by (i) and equation (2.4), we obtain that

$$u_n(h_n \circ \varphi^n)E_n(u_n \chi_C) = u_n(h_n \circ \varphi^n)E_n(u_n)\chi_C = h_n E_n(u_n^2) \circ \varphi^{-n} \chi_C. \tag{2.5}$$

By using (ii), it is clear that (2.5) holds for any Σ-measurable subset C of $\sigma(u_n)$ with finite measure. Since $\sigma(u_n) = \sigma(h_n E(u_n^2) \circ \varphi^{-n})$. So both sides of (2.5) is 0 off $\sigma(u_n)$. Thus we can deduce that that (2.5) holds for any Σ-measurable subset C, with $\mu(C) < \infty$. Therefore $W^n {W^*}^n \chi_C = {W^*}^n W^n \chi_C$ for all $C \in \Sigma$. This implies that W^n is normal and, by [1, Proposition2.2], this is equivalent to W is n-power normal. $\square$

3. **Examples**

Example 3.1. Let $m := \{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider the space $\ell^2(m) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a measurable transformation. Direct computation shows that (see [15])

$$h(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j; \quad E(f)(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j},$$

and

$$h_n(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-n}(k)} m_j; \quad J_n(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-n}(k)} (u_n(j))^2 m_j,$$

for all non-negative sequence $f = \{f_n\}_{n=1}^{\infty} \in \ell^2(m)$ and $k \in \mathbb{N}$. By Theorem 2.2, W is k-quasi paranormal if and only if

$$\frac{1}{m_n} \sum_{\ell \in \varphi^{-(k+2)}(n)} (u_{k+2}(\ell))^2 m_\ell \frac{1}{m_n} \sum_{\ell \in \varphi^{-k}(n)} (u_k(\ell))^2 m_\ell$$
$$\geq \{\frac{1}{m_n} \sum_{\ell \in \varphi^{-(k+1)}(n)} (u_{k+1}(\ell))^2 m_\ell\}^2.$$

Moreover, by Theorem 2.3, W is quasi-class$A$ operator if and only if

$$\frac{1}{m_n}\sum_{\ell\in\varphi^{-3}(n)}\frac{(m_{\varphi^2(\ell)})^{\frac{1}{2}}}{\sum_{s\in\varphi^{-2}(\varphi^2(\ell))}m_s}\left(\sum_{k\in\varphi^{-2}(\varphi^2(\ell))}(u_2(k))^2 m_k (u\circ\varphi^2)^4(\ell)\right)^{\frac{1}{2}} m_\ell \geq \frac{1}{m_n}\sum_{\ell\in\varphi^{-2}(n)}(u_2(\ell))^2 m_\ell.$$

Also by Theorem 2.5(i), W is quasi-$*$-paranormal if and only if

$$\frac{1}{m_n}\sum_{\ell\in\varphi^{-3}(n)}(u_3(\ell))^2 m_\ell \geq \left(\frac{1}{m_n}\sum_{\ell\in\varphi^{-1}(n)}(u(\ell))^2 m_\ell\right)^3.$$

Example 3.2. Let X be the set of nonnegative integers and Σ be the σ-algebra of all subsets of X. Take μ to be the point mass measure determined by the

$$m = 1, 1, 1, a, b, a^2, b^2, a^3, b^3, \ldots;$$

where a and b are fixed positive real numbers. The powers of a occur for odd integers and those of b for even integers. Our point transformation φ is defined by

$$\varphi(n) = \begin{cases} 0 & n = 0, 1, 2, \\ n-2 & n \geq 3. \end{cases}$$

Note that this example was used in [2] and [3] to show that composition operators can separate almost all weak hyponormality classes. Define u by

$$u(n) = \begin{cases} 1 & n = 0, 1, 2, \\ n & n \geq 3. \end{cases}$$

It is easy to verify that

$$\varphi^k(n) = \begin{cases} 0 & n = 0, 1, 2, \ldots, 2k, \\ n-2k & n \geq 2k+1. \end{cases}$$

Since $u_2(n) = u(n)u(\varphi(n))$, we get that

$$u_2(n) = \begin{cases} 1 & n = 0, 1, 2, \\ n & n = 3, 4, \\ n(n-2) & n \geq 5. \end{cases}$$

It follows that

$$J_2(n) = \begin{cases} 3 + 9a + 16b & n = 0, \\ (2t+4)^2(2t+2)^2 b^2 & n = 2t, \\ (2t+5)^2(2t+3)^2 a^2 & n = 2t+1. \end{cases}$$

Also we have

$$u_3(n) = \begin{cases} 1 & n = 0, 1, 2, \\ n & n = 3, 4, \\ n(n-2) & n = 5, 6, \\ n(n-2)(n-4) & n \geq 7, \end{cases}$$

and hence we get that

$$J_3(n) = \begin{cases} 3 + 9a + 16b + (15a)^2 + (24b)^2 & n = 0, \\ (2t+6)^2(2t+4)^2(2t+2)^2 b^3 & n = 2t, \\ (2t+7)^2(2t+5)^2(2t+3)^2 a^3 & n = 2t+1. \end{cases}$$

Similar computations show that

$$u_4(n) = \begin{cases} 1 & n = 0, 1, 2, \\ n & n = 3, 4, \\ n(n-2) & n = 5, 6, \\ n(n-2)(n-4) & n = 7, 8, \\ n(n-2)(n-4)(n-6) & n \geq 9, \end{cases}$$

and

$$J_4(n) = \begin{cases} 3 + 9a + 16b + (15a)^2 + (24b)^2 + (105)^2 a^3 + (384)^2 b^3 & n = 0, \\ (2t+8)^2(2t+6)^2(2t+4)^2(2t+2)^2 b^4 & n = 2t, \\ (2t+9)^2(2t+7)^2(2t+5)^2(2t+3)^2 a^4 & n = 2t+1. \end{cases}$$

By Theorem 2.2, W is 2-quasi-paranormal if and only if $J_4(n)J_2(n) \geq (J_3)^2(n)$ for each $n \in \mathbb{N}_0$. It is easy to see that for $n = 2t$ and $n = 2t+1$ this inequality always holds. But for $n = 0$ this condition is equivalent to

$$\begin{aligned} (3 + 9a + 16b + (15a)^2 + (24b)^2 + (105)^2 a^3 + (384)^2 b^3)(3 + 9a + 16b) \\ \geq (3 + 9a + 16b + (15a)^2 + (24b)^2)^2. \end{aligned}$$

Namely put $a = 0.01$ and $b = 0.2$. Consequently W is 2-quasi-paranormal. Now, we will show that W is not quasi-class$A$ operator. Put

$$A(n) = \frac{1}{m_n} \sum_{\ell \in \varphi^{-3}(n)} \frac{(m_{\varphi^2(\ell)})^{\frac{1}{2}}}{\sum\limits_{s \in \varphi^{-2}(\varphi^2(\ell))} m_s} \left(\sum_{k \in \varphi^{-2}(\varphi^2(\ell))} (u_2(k))^2 m_k (u \circ \varphi^2)^4(\ell) \right)^{\frac{1}{2}} m_\ell.$$

Direct computations show that

$$A(n) = \begin{cases} \sqrt{3 + 9a + 16b} + 15a + 16b & n = 0, \\ (2t+6)(2t+4)(2t+2)^2 b^2 & n = 2t, \\ (2t+7)(2t+5)(2t+3)^2 a^2 & n = 2t+1. \end{cases}$$

By Theorem 2.3, W is a quasi class A operator if and only if $A(n) \geq J_2(n)$ for each $n \in \mathbb{N}_0$. With a and b given we have $A(0) = 5.85$ and $J_2(0) = 6.29$. Therefore W is not quasi class$A$ operator. Also since

$$J^3(n) = \begin{cases} 3^3 & n = 0, \\ ((2t+2)^2 b)^3 & n = 2t, \\ ((2t+3)^2 a)^3 & n = 2t+1, \end{cases}$$

and by Theorem 2.5(i), W is quasi-$*$-paranormal if and only if $J_3(n) \geq J^3(n)$ for each $n \in \mathbb{N}_0$. We get that for $n = 2t$ and $n = 2t+1$ this inequality always holds and for $n = 0$ is equivalent to $3 + 9a + 16b + (15a)^2 + (24b)^2 \geq 27$. With the same a and b, the value of the left side is 29.3525. Hence we conclude that W is quasi-$*$-paranormal.

Example 3.3. Let $X = (0, 1)$ equipped with the Lebesgue measure μ on the Lebesgue measurable subspaces and $\varphi : X \to X$ is defined by

$$\varphi(x) = \begin{cases} 2x & 0 < x < \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x < 1. \end{cases}$$

Direct computation shows that $h(x) = 1$ and for each $f \in L^2(\Sigma)$,

$$hE(f) \circ \varphi^{-1}(x) = \frac{1}{2} \left(f(\frac{x}{2}) + f(1 - \frac{x}{2}) \right).$$

Put $u(x) = \sqrt{x}$. Then for each $n \in \mathbb{N}$, we have $\sigma(u_n) = X$. Moreover $J(x) = \frac{1}{2} \left(u^2(\frac{x}{2}) + u^2(1 - \frac{x}{2}) \right) = \frac{1}{2}$, and so $J \circ \varphi^n(x) = \frac{1}{2}$. Hence by Theorem 2.5(iii), W is n-power quasi-normal. But W is not n-power normal because for example $(0, \frac{1}{2}) \in \Sigma$, but $(0, \frac{1}{2})$ does not essentially belong to $\varphi^{-1}(\Sigma)$ and so $(0, \frac{1}{2}) \notin \varphi^{-n}(\Sigma)$. Hence by Theorem 2.7, it is easy to see that for each $n \in \mathbb{N}$, W is not n-power normal.

Example 3.4. Let X be the set of positive integers, and Σ be the σ-algebra of all subsets of X. Also, let $m := \{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers and take μ to be the point mass measure on Σ defined by $\mu(\{n\}) = m_n$. Our point transformation φ is defined by

$$\varphi(x) = \begin{cases} x + 1 & \text{for odd } x, \\ x - 1 & \text{for even } x. \end{cases}$$

It is easy to verify that $\varphi^2 = \varphi^4 = ... = \varphi^{2n} = I$. Hence $J \circ \varphi^n = J$ for any even n. With the wight function u given by $u(x) = x + 1$. So $\sigma(u_n) = X$ for each $n \in \mathbb{N}$. Hence by Theorem 2.5(iii), $W = M_u \circ C_\varphi$ is n-power quasi-normal for each even n. On the other hand we have $\varphi^3 = \varphi^5 = ... = \varphi^{2n+1} = \varphi$, and this implies that $J \circ \varphi^n = J \circ \varphi$ for any odd n. Therefore with u and φ given in above $W = M_u \circ C_\varphi$ is not n-power quasi-normal for any odd n.

Example 3.5. Let X be the set of nonnegative integers, let Σ be the σ-algebra of all subsets of X, take μ to be the point mass measure determined by the

$$m = 1, 2, a, a^2, a^3, \cdots ;$$

where a is fixed positive real number. Our point transformation φ is defined by

$$\varphi(n) = \begin{cases} 0 & n = 0, \\ n-1 & n \geq 1. \end{cases}$$

A simple calculation by using of the given formula in Example 3.1 shows that, as a sequence, $h^3 = 3^3, (\frac{a}{2})^3, a^3, a^3, \cdots$. Also for every $n \in \mathbb{N}_0$, $h_3(n) = \frac{1}{m_n} \sum_{j \in \varphi^{-3}(n)} m_j$. Some direct computations show that $h_3 : 3+a+a^2, \frac{a^3}{2}, a^3, a^3, \cdots$.

By Theorem 2.5(i), C_φ is quasi-$*$-paranormal if and only if $h_3(n) \geq h^3(n)$ for each $n \in \mathbb{N}_0$, and this is equivalent to $3+a+a^2 \geq 27$. Hence this inequality holds if $a \in [4.44, \infty)$. We will show the corresponding composition operator is not 1-$*$-paranormal. It is easy to see that

$$h \circ \varphi(n) = \frac{1}{m_{\varphi(n)}} \sum_{j \in \varphi^{-1}(\varphi(n))} m_j;$$

hence, $h \circ \varphi : 3, 3, \frac{a}{2}, a, a, \cdots$. Furthermore,

$$E(f)(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j}.$$

Thus we get that $E(h) : \frac{3+a}{3}, \frac{3+a}{3}, a, a, \cdots$. By Theorem 2.5(ii), C_φ is 1-$*$-paranormal if and only if $E(h)(n) \geq h \circ \varphi(n)$ for each $n \in \mathbb{N}_0$. This condition is equivalent to $3+a \geq 9$, this inequality holds if and only if $a \geq 6$. Consequently for $a \in [4.44, 6)$, C_φ is not 1-$*$-paranormal.

Note that the following example was used in [9] to show that the p-hyponormal classes are distinct for p with $0 < p < \infty$. Now we will show that block matrix operators can separate $p - *-$paranormal classes.

Example 3.6. Let $M := [A_{ij}]_{0 \leq i,j \leq \infty}$ be a block matrix operator whose blocks are 6×3 matrices such that $A_{ij} = 0, i \neq j$, and $A \equiv A_0 = A_1 = A_2 = \cdots$, where $A_n = A_{nn}$ for each $n \in \mathbb{N}_0$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & \sqrt{b} \end{bmatrix}.$$

Note that a and b are fixed positive real numbers. Now, let $\ell^2(m)$ be the weighted Hilbert sequence space on $(\mathbb{N}_0, 2^{\mathbb{N}_0}, \mu)$. Also let μ be a measure on Σ defined by $\mu(\{n\}) = m_n$. Define $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\varphi(n) = \begin{cases} 3k & n = 6k, 6k+1, 6k+2, 6k+3; \\ 3k+1 & n = 6k+4; \\ 3k+2 & n = 6k+5. \end{cases}$$

Then by [8, Proposition 2.2], C_φ is unitarily equivalent to the block matrix M such that for every $k \in \mathbb{N}_0$

$$\sqrt{\frac{m_{6k+i-1}}{m_{3k}}} = 1, \quad \text{for} \quad 1 \leq i \leq 4, \quad k \in \mathbb{N}_0,$$

$$\sqrt{\frac{m_{6k+4}}{m_{3k+1}}} = \sqrt{a}, \quad \text{and} \quad \sqrt{\frac{m_{6k+5}}{m_{3k+2}}} = \sqrt{b}, \quad k \in \mathbb{N}_0.$$

Moreover by [8, Proposition 3.1] M is p-$*$-paranormal if and only if C_φ to be p-$*$-paranormal. By Theorem 2.5(ii), C_φ is p-$*$-paranormal if and only if $E(h^p) \geq h^p \circ \varphi$, this condition is equivalent to

$$\left(\frac{m(\varphi^{-1}(\varphi(n)))}{m_{\varphi(n)}}\right)^p \leq \frac{1}{m(\varphi^{-1}(\varphi(n)))} \sum_{j \in \varphi^{-1}(\varphi(n))} m_j \left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p. \tag{3.1}$$

By the same argument in the proof of [8, Theorem 3.3], we deduce that (3.1) is equivalent to

$$(\frac{a}{4})^p + (\frac{b}{4})^p \geq 2. \tag{3.2}$$

Let $0 < q < p$ and M be $p - *-$paranormal. Then by using (3.2), we can find a and b such that M is not $q - *-$paranormal. Namely for $a = 4.8$ and $b = 3$, by using (3.2), it is easy to see that M is 2-$*$-paranormal but it is not 1-$*$-paranormal.

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