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SOLVABILITY OF AN IMPULSIVE BOUNDARY VALUE PROBLEM ON THE HALF-LINE VIA CRITICAL POINT THEORY

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Abstract. In this paper, an impulsive boundary value problem on the half-line is considered and existence of solutions is proved using Minimization Principal and Mountain Pass Theorem.

Keywords: Impulsive BVPs, unbounded interval, critical point, minimization principal, Mountain Pass Theorem.
MSC(2010): Primary: 34B37; Secondary: 34B40, 35A15, 58E30.

1. Introduction

This paper is concerned with the following impulsive boundary value problem (BVP for short) set on the positive half-line

$$(1.1)\qquad \begin{cases} -(p(t)u'(t))' &= f(t,u(t)), \qquad \text{a.e. } t \geq 0,\ t \neq t_j \\ u(0)=u(+\infty) &= 0, \\ \triangle(p(t_j)u'(t_j)) &= h(t_j)I_j(u(t_j)), \quad j \in \{1,2,\ldots\}, \end{cases}$$

where $f : [0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e.

(i) $f(.,u)$ is measurable, for each $u \in \mathbb{R}$,
(ii) $f(t,.)$ is continuous, for a.e. $t \in [0,+\infty)$.

The coefficient $p : [0,+\infty) \longrightarrow (0,+\infty)$ satisfies $\frac{1}{p} \in L^1[0,+\infty)$, and

$$M = \int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

As an example for p, one may take the exponential function. Here $t_0 = 0 < t_1 < t_2 < \ldots < t_j < \ldots < t_m \to +\infty$, as $m \to \infty$, are the impulse points, while the impulsive functions $I_j : \mathbb{R} \longrightarrow \mathbb{R}$ are assumed continuous. Finally

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

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where $u'(t_j^+) = \lim\limits_{t \to t_j^+} u'(t)$ and $u'(t_j^-) = \lim\limits_{t \to t_j^-} u'(t)$ stand for the right and the left limits of u' at t_j, respectively. As for $h : [0, +\infty) \longrightarrow [0, +\infty)$, it is a continuous function that satisfies $\sum_{j=1}^{+\infty} h(t_j) < +\infty$.

Many phenomena in nonlinear dynamics and natural sciences may be subject to jump discontinuities in velocity or short-term perturbations that can be seen as impulses, for instance the administration of a drug in the periodic treatment of some diseases. We refer the reader to [4] and the references therein for more details on the derivation of such models and some of their qualitative aspects. In the recent literature, we can find a lot of mathematical results of stability and existence of solutions for BVPs set on bounded intervals of the real line and associated to impulsive equations. Most of these mathematical results use topological methods (fixed point theorems, Leray-Schauder degree, ...). We cite the papers [5, 7, 10, 12, 17] where boundary value problems associated to second-order differential operators are investigated. Generally, Mountain Pass Theorem and Ekeland's Variational Principle are sufficient to get existence of single or even multiple solutions; see, e.g., [2, 11, 14] where polynomial type growth conditions are assumed for the nonlinear source term. However, only few papers employ variational approaches to deal with such problems on unbounded domains. For example, in the paper [7], the following boundary value problem is posed on the positive half-line:

$$\left\{ \begin{array}{rcll} -u''(t) + u(t) & = & \lambda f(t, u(t)), & \text{a.e. } t \geq 0,\ t \neq t_j \\ u'(0) & = & g(u(0)), & u'(+\infty) = 0, \\ \triangle u'(t_j)) & = & I_j(u(t_j)), & j \in \{1, 2, \ldots, l\} \end{array} \right.$$

where $f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$, $g, I_j \in C(\mathbb{R})$ $(1 \leq j \leq l)$, and $0 = t_0 < t_1 < \ldots < t_l < +\infty$ are finite impulse points. Then existence of solutions are obtained under some restrictions upon the positive parameter λ.

The aim of this work is to consider the more general differential operator $-(p(.)u'(.))'$ and a Carathéodory nonlinearity f satisfying sub-linear, linear or super-linear growth condition at positive infinity or at the origin. The impulse point are infinite and the jump conditions concern the derivatives $p(t_j)u'(t_j)\,(j = 1, 2, \ldots)$. The solutions are proved to exist in a weighted Sobolev space. For this, some continuous and compact embeddings are first established in Section 2 and three existence theorems are demonstrated in Section 3, two of them use Minimization Principal and one employs Mountain Pass Theorem. Examples of applications illustrate each of the obtained result.

2. Preliminaries

2.1. **The functional framework.** Define the space

$H^1_{0,p}(0,+\infty) = \{u \in AC[0,+\infty),\mathbb{R}) \mid u(0) = u(+\infty) = 0, \sqrt{p}u' \in L^2[0,+\infty)\}.$

Lemma 2.1. *$H^1_{0,p}(0,+\infty)$ embeds in $L^2(0,+\infty)$.*

Proof. For $u \in H^1_{0,p}(0,+\infty)$, we have

$$|u(t)| = \left|\int_t^{+\infty} u'(s)ds\right| = \left|\int_t^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}ds\right|.$$

Then, by the Cauchy-Schwartz inequality

$$\begin{aligned} |u(t)|^2 &\leq \left(\int_t^{+\infty} p(s)u'^2(s)ds\right)\left(\int_t^{+\infty} \tfrac{1}{p(s)}ds\right) \\ &\leq \left(\int_0^{+\infty} p(s)u'^2(s)ds\right)\left(\int_t^{+\infty} \tfrac{1}{p(s)}ds\right). \end{aligned}$$

Hence

$$\int_0^{+\infty} |u(t)|^2dt \leq \left(\int_0^{+\infty}(\int_t^{+\infty} \tfrac{1}{p(s)}ds)dt\right)\left(\int_0^{+\infty} p(s)|u'(s)|^2ds\right),$$

that is

$$\|u\|_{L^2} \leq \sqrt{M}\|\sqrt{p}u'\|_{L^2}.$$

□

Notice that $H^1_{0,p}(0,+\infty)$ is a Banach space equipped with the norm

$$\|u\|_{0,p} = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt},$$

or the equivalent norm

$$\|u\|_p = \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2}.$$

Moreover the space $H^1_{0,p}(0,+\infty)$ is reflexive. Indeed

Lemma 2.2. (a) *The operator*

$$\begin{aligned} T: H^1_{0,p}(0,+\infty) &\longrightarrow T(H^1_{0,p}(0,+\infty)) \subset L^2(0,+\infty)\times L^2(0,+\infty) := L^2_2(0,+\infty) \\ u &\longrightarrow T(u) = (u,\sqrt{p}u') \end{aligned}$$

is an isometric isomorphism.
(b) *$H^1_{0,p}(0,+\infty)$ is a reflexive space.*

Proof. (a) It is clear that T is a linear operator and that T preserves norm, i.e.,

$$\forall\, u \in H^1_{0,p}(0,+\infty), \quad \|Tu\|_{L^2_2} = \|u\|_p.$$

Indeed

$$\begin{aligned}\|Tu\|_{L_2^2} &= \|(u,\sqrt{p}u')\|_{L_2^2}\\ &= \|u\|_{L^2}+\|\sqrt{p}u'\|_{L^2}\\ &= \|u\|_p.\end{aligned}$$

(b) Since $L^2(0,+\infty)$ is a reflexive Banach space, the cartesian product $L_2^2(0,+\infty)$ is also a reflexive Banach space with respect to the norm

$$\|u\|_{L_2^2}=\|u_1\|_{L^2}+\|u_2\|_{L^2},\quad \text{where } u=(u_1,u_2)\in L_2^2(0,+\infty).$$

From part (a), $T(H^1_{0,p}(0,+\infty))$ is a closed subspace of $L_2^2(0,+\infty)$; then by [9, Theorem 4.10.5], the space $T(H^1_{0,p}(0,+\infty))$ is reflexive. Consequently $H^1_{0,p}(0,+\infty)$ is also reflexive, see [9, Lemma 4.10.4]. □

Lemma 2.3. *On $H^1_{0,p}(0,+\infty)$, the quantity $\|u\|=\sqrt{\int_0^{+\infty}p(t)u'^2(t)dt}$ is a norm which is equivalent to the $H^1_{0,p}(0,+\infty)$-norm.*

Proof. Given $u\in H^1_{0,p}(0,+\infty)$, in view of Lemma 2.1, we have

$$\int_0^{+\infty}|u(t)|^2dt\le M\|u\|^2.$$

Then

$$\int_0^{+\infty}p(t)u'^2(t)dt\le\int_0^{+\infty}\left(u^2(t)+p(t)u'^2(t)\right)dt\le(1+M)\int_0^{+\infty}p(t)u'^2(t)dt,$$

that is

$$\|u\|\le\|u\|_{0,p}\le\sqrt{1+M}\|u\|.$$

□

Lemma 2.4. $(H^1_{0,p}(0,+\infty),\|\cdot\|)$ *embeds in* $(C_0[0,+\infty),\|u\|_\infty)$*, where* $C_0[0,+\infty)=\{u\in C([0,+\infty),\mathbb{R})\mid \lim_{t\to+\infty}u(t)=0\}$ *and* $\|u\|_\infty=\sup_{t\in[0,+\infty)}|u(t)|$.

Proof. For $u\in H^1_{0,p}(0,+\infty)$, we have

$$\begin{aligned}|u(t)|=|u(t)-u(0)| &= \left|\int_0^t u'(s)ds\right|=\left|\int_0^t\sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}ds\right|\\ &\le \left(\int_0^t p(s)u'^2(s)ds\right)^{\frac12}\left(\int_0^t\frac{1}{p(s)}ds\right)^{\frac12}\\ &\le \left(\int_0^{+\infty} p(s)u'^2(s)ds\right)^{\frac12}\left(\int_0^{+\infty}\frac{1}{p(s)}ds\right)^{\frac12}.\end{aligned}$$

Hence

$$\|u\|_\infty\le\sqrt{\left\|\frac{1}{p}\right\|_{L^1}}\|u\|.$$

□

Corollary 2.5. $H^1_{0,p}(0,+\infty)$ *embeds continuously in* $C_0[0,+\infty)$ *and in* $L^2(0,+\infty)$.

To prove that $H^1_{0,p}(0,+\infty)$ embeds compactly in $C_0[0,+\infty)$, we appeal to Corduneanu's compactness criterion:

Lemma 2.6 ([8])**.** *Let $\mathcal{H} \subset C_0([0,+\infty),\mathbb{R})$ be a bounded set. Then $\mathcal{H}$ is relatively compact if the following conditions hold:*

(a) *$\mathcal{H}$ is equicontinuous on any compact sub-interval of $[0,+\infty)$, i.e.,*

$$\forall\, J \subset [0,+\infty) \;\; \textit{compact subinterval}\;, \forall\, \varepsilon > 0,\, \exists\, \delta > 0,\, \forall\, t_1, t_2 \in J : \\ |t_1 - t_2| < \delta \Longrightarrow |u(t_1) - u(t_2)| \le \varepsilon,\; \forall\, u \in \mathcal{H},$$

(b) *$\mathcal{H}$ is equiconvergent at $+\infty$, i.e.,*

$$\forall\, \varepsilon > 0, \exists\, T = T(\varepsilon) > 0 \;\textit{such that} \\ \forall\, t_1,\, t_2 \in J : t_1, t_2 \ge T(\varepsilon) \Longrightarrow |u(t_1) - u(t_2)| \le \varepsilon,\;\; \forall\, u \in \mathcal{H}.$$

Lemma 2.7. *The embedding*

$$H^1_{0,p}(0,+\infty) \hookrightarrow C_0[0,+\infty)$$

is compact.

Proof. Let $D \subset H^1_{0,p}(0,+\infty)$ be a bounded set; then it is bounded in $C_0[0,+\infty)$ by Lemma 2.4. Let $R > 0$ be such that for all $u \in D$, $\|u\| \le R$, we have

(a) *D is equicontinuous on every compact interval of $[0,+\infty)$.* For $u \in D$ and $t_1, t_2 \in [0,+\infty)$, we have

$$\begin{aligned} |u(t_1) - u(t_2)| &= \left|\int_{t_2}^{t_1} u'(\tau)d\tau\right| = \left|\int_{t_2}^{t_1} \sqrt{p(\tau)}u'(\tau)\frac{1}{\sqrt{p(\tau)}}d\tau\right| \\ &\le \left(\int_{t_2}^{t_1} \frac{1}{p(\tau)}d\tau\right)^{\frac{1}{2}} \|u\| \\ &\le R\left(\int_{t_2}^{t_1} \frac{1}{p(\tau)}d\tau\right)^{\frac{1}{2}} \end{aligned}$$

and the right-hand side tends to 0, as $|t_1 - t_2| \to 0$ for $\frac{1}{p} \in L^1(0,+\infty)$.

(b) *D is equiconvergent.* For $t \in [0,+\infty)$ and $u \in D$, using the fact that $u(+\infty) = 0$, we have

$$\begin{aligned} |u(t) - u(+\infty)| &= |u(t)| \\ &= \left|\int_t^{+\infty} u'(\tau)d\tau\right| \\ &\le \left(\int_t^{+\infty} \frac{1}{p(\tau)}d\tau\right)^{\frac{1}{2}} \left(\int_t^{+\infty} p(\tau)u'^2(\tau)d\tau\right)^{\frac{1}{2}}. \\ &\le \left(\int_t^{+\infty} \frac{1}{p(\tau)}d\tau\right)^{\frac{1}{2}} \|u\| \\ &\le R\left(\int_t^{+\infty} \frac{1}{p(\tau)}d\tau\right)^{\frac{1}{2}} \longrightarrow 0,\; \text{as } t \to +\infty. \end{aligned}$$

The result then follows from Lemma 2.6. $\square$

2.2. **Critical point theory.** Now we recall some essential facts from critical point theory. (See [1,3,13]).

Definition 2.8. Let $(X, \|\cdot\|)$ be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \longrightarrow \mathbb{R}$ a functional. We say that J is Fréchet differentiable at $u \in \Omega$ if there exists an operator $A \in X'$ such that

$$\lim_{v\in\Omega,\, \|v\|\to 0} \frac{J(u+v) - J(u) - Av}{\|v\|} = 0.$$

The operator A, which is unique, is called the Fréchet differential of J at u and is denoted by $A = J'_F(u)$ or $A = J'(u)$ when there is no confusion.

Definition 2.9. Let X be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \longrightarrow \mathbb{R}$ a functional. We say that J is Gâteaux differentiable at $u \in \Omega$ if there exists $A \in X'$ such that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = Av,$$

for all $v \in X$. The operator A, which is unique, is denoted by $A = J'_G(u)$ or merely J'.

The mapping which sends to every $u \in \Omega$ the mapping $J'_G(u)$ is called the Gâteaux differential of J and is denoted by J'_G.

Proposition 2.10 ([1])**.** *Let X be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \longrightarrow \mathbb{R}$ a Gâteaux differentiable functional at some point $u \in \Omega$. If J'_G is continuous at u, then J is Fréchet differentiable at u and $J'_F(u) = J'_G(u)$.*

We say that $J \in C^1$ if J'_G is continuous at every $u \in \Omega$.

Definition 2.11. Let X be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \longrightarrow \mathbb{R}$ a Gâteaux differentiable functional. A point $u \in \Omega$ is called a critical point of J if $J'(u) = 0$, i.e., $J'(u)v = 0$, for every $v \in X$. If further $J(u) = c$, we say that u is a critical point of J at level c.

Clearly, every point of a local minimum of a Gâteaux differentiable functional J is a critical point of J.

Definition 2.12. Let X be a Banach space. A functional $J : X \longrightarrow \mathbb{R}$ is called coercive if, for every sequence $(u_k)_{k\in\mathbb{N}} \subset X$,

$$\|u_k\| \to +\infty \Longrightarrow J(u_k) \to +\infty.$$

Definition 2.13. Let X be a Banach space. A functional $J : X \longrightarrow (-\infty, +\infty]$ is said to be sequentially weakly lower semi-continuous (*swlsc* for short) if

$$J(u) \le \liminf_{n\to+\infty} J(u_n),$$

as $u_n \rightharpoonup u$ in X, when $n \to \infty$.

Then we have:

Lemma 2.14 ([6, Minimization Principal]). *Let X be a reflexive Banach space and J a functional defined on X such that*

(1) $\lim_{\|u\| \to +\infty} J(u) = +\infty$ *(coercivity condition),*

(2) *J is sequentially weakly lower semi-continuous.*

Then J is lower bounded on X and achieves its lower bound at some point u_0.

Definition 2.15. Let X be a real Banach space, $J \in C^1(X, \mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition, (PS) condition for brevity.

Lemma 2.16 (Mountain Pass Theorem). *(*See, e.g., [15, Theorem 2.2] or [16, Theorem 3.1]). *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$ satisfy $J(0) = 0$. Assume that J satisfies (PS) and there exist positive numbers ρ and α such that*

(1) *$J(u) \geq \alpha$ if $\|u\| = \rho$,*

(2) *there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.*

Then there exists a critical point. Furthermore it is characterized by

$$J'(u) = 0, \;\; J(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \, \gamma(0) = 0, \, \gamma(1) = u_0\}.$$

3. Existence of weak solutions

Take $v \in H^1_{0,p}(0, +\infty)$, multiply the equation in problem (1.1) by v, and then integrate over $(0, +\infty)$; we get

$$-\int_0^{+\infty} (p(t)u'(t))'v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

The left-hand term is

$$\begin{aligned} -\int_0^{+\infty} (p(t)u'(t))'v(t)dt &= -\sum_{j=0}^{+\infty} \int_{t_j}^{t_{j+1}} (p(t)u'(t))'v(t)dt \\ &= \sum_{j=1}^{+\infty} h(t_j) I_j(u(t_j))v(t_j) + \int_0^{+\infty} p(t)u'(t)v'(t)dt. \end{aligned}$$

Hence

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = -\sum_{j=1}^{+\infty} h(t_j) I_j(u(t_j))v(t_j) + \int_0^{+\infty} f(t, u(t))v(t)dt.$$

This leads to the natural concept of weak solution for problem (1.1).

Definition 3.1. We say that a function $u \in H^1_{0,p}(0,+\infty)$ is a weak solution of problem (1.1) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} h(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t))v(t)dt = 0,$$

for all $v \in H^1_{0,p}(0,+\infty)$.

In order to study problem (1.1), we consider the functional $J : H^1_{0,p}(0,+\infty) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} h(t_j) \int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t))dt,$$

where

$$F(t,u) = \int_0^u f(t,s)ds.$$

3.1. **The sublinear case.**

Theorem 3.2. *Suppose that the following conditions hold:*

(H_1) *There exist a constant $\mu \in [0,1)$ and positive functions $a_1, b_1 \in L^1[0,+\infty)$ such that*

$$|f(t,x)| \le a_1(t)|x|^\mu + b_1(t), \quad \textit{for a.e. } t \in [0,+\infty) \textit{ and all } x \in \mathbb{R}.$$

(I_0) *There exist constants $k > 0$ and $\gamma \in [0,1)$ such that*

$$|I_j(s)| \le k|s|^\gamma, \ \forall\, s \in \mathbb{R}, \forall\, j \in \{1,2,\ldots\}.$$

Then problem (1.1) has at least one weak solution.

Proof. Claim 1. The functional J is well defined.
Let $d = \sqrt{\left\|\frac{1}{p}\right\|_{L^1}}$. Given $u \in H^1_{0,p}(0,+\infty)$, Assumptions ($H_1$) and ($I_0$) guarantee that

$$|F(t,u(t))| \le \frac{a_1(t)}{\mu+1}|u(t)|^{\mu+1} + b_1(t)|u(t)|.$$

Hence

$$\begin{aligned}\left|\int_0^{+\infty} F(t,u(t))dt\right| &\le \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}\int_0^{+\infty} a_1(t)dt + d\|u\|\int_0^{+\infty} b_1(t)dt \\ &\le \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}\|a_1\|_{L^1} + d\|u\|\|b_1\|_{L^1}\end{aligned}$$

and

$$\left|\sum_{j=1}^{+\infty}\int_0^{u(t_j)} h(t_j)I_j(\tau)d\tau\right| \le \tfrac{kd^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}\sum_{j=1}^{+\infty} h(t_j).$$

Then

$$\begin{aligned}|J(u)| &\le \tfrac{1}{2}\|u\|^2 + \tfrac{kd^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}\sum_{j=1}^{+\infty} h(t_j) \\ &\quad + \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}\|a_1\|_{L^1} + d\|u\|\|b_1\|_{L^1} \\ &< \infty.\end{aligned}$$

Claim 2. J is sequentially weakly lower semi-continuous.
Let $(u_n) \subset H^1_{0,p}(0,+\infty)$ be a sequence such that $u_n \rightharpoonup u$ in $H^1_{0,p}(0,+\infty)$, as $n \to \infty$. Then (u_n) converges uniformly to u on $[0,+\infty)$ and

$$\liminf_{n\to+\infty} \|u_n\| \geq \|u\|.$$

The continuity of functions f and I_j, $j \in \{1,2,\ldots\}$ together with Lebesgue Dominated Convergence Theorem yield

$$\begin{array}{rl} & \liminf\limits_{n\to+\infty} J(u_n) \\ = & \liminf\limits_{n\to+\infty} \left(\frac{1}{2}\|u_n\|^2 + \sum_{j=1}^{+\infty}\int_0^{u_n(t_j)} h(t_j)I_j(\tau)d\tau - \int_0^{+\infty} F(t,u_n(t))dt\right) \\ \geq & \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty}\int_0^{u(t_j)} h(t_j)I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t))dt = J(u). \end{array}$$

Therefore, J is sequentially weakly lower semi-continuous.
Claim 3. J is coercive.
In view of (H_1) and (I_0), Lemma 2.4 implies that

$$\begin{array}{rcl} J(u) & \geq & \frac{1}{2}\|u\|^2 - \frac{kd^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}\sum_{j=1}^{+\infty} h(t_j) \\ & & -\frac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}\|a_1\|_{L^1} - d\|u\|\|b_1\|_{L^1}. \end{array} \tag{3.1}$$

Since $\mu < 1$ and $\gamma < 1$, (3.1) implies

$$\lim_{\|u\|\longrightarrow+\infty} J(u) = +\infty.$$

Lemma 2.14 guarantees that J has a local minimum which is a critical point of J. Finally, it is easy to check that under (H_1), the functional J is Gâteaux differentiable and the Gâteaux derivative at a point $u \in X$ is given by

$$\begin{array}{rcl} (J'(u),v) & = & \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} h(t_j)I_j(u(t_j))v(t_j) \\ & & -\int_0^{+\infty} f(t,u(t))v(t)dt, \end{array} \tag{3.2}$$

for all $v \in H^1_{0,p}(0,+\infty)$. Therefore u is a weak solution of problem (1.1). $\square$

Remark 3.3. If, in addition, $u \in H^2_p(t_j,t_{j+1})$, for all $j \in \{1,2,\ldots\}$, where

$$H^2_p(t_j,t_{j+1}) = \{u \in AC[0,+\infty),\mathbb{R}) : \sqrt{p}u' \in L^2(t_j,t_{j+1}),\ (pu')' \in L^2(t_j,t_{j+1})\},$$

then u will be called a strong solution of problem (1.1).

We have:

Proposition 3.4. *In (H_1), assume further that $a_1, b_1 \in L^2(0,+\infty)$. Then every weak solution is a strong solution of problem (1.1).*

Proof. Since $u \in H^1_{0,p}(0,+\infty)$ is a critical point of J, we have, for any $v \in H^1_{0,p}(0,+\infty)$, the relation

(3.3)
$$0 = \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} h(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t))v(t)dt.$$

For $j \in \{1,2,\ldots\}$, if $v \in H^1_{0,p}(t_j,t_{j+1})$ $(v = v_j)$, then

$$\int_{t_j}^{t_{j+1}} p(t)u'(t)v'(t)dt = \int_{t_j}^{t_{j+1}} f(t,u(t))v(t)dt.$$

Thus $u_j \in H^1_{0,p}(t_j,t_{j+1})$ is solution of the equation:

$$-\left(p(t)u'\right)' = f(t,u(t)),\ t \in (t_j,t_{j+1}), \tag{3.4}$$

Since, $u \in C_0[0,+\infty)$, then by (H_1), $|f(t,u(t))|^2 \leq 2\left(a_1(t)^2\|u\|_\infty^{2\mu} + b_1(t)^2\right)$ and so $u_j \in H^2_p\ (t_j,t_{j+1})$. By (3.4), we can also get the limits $u'(t_j^+)$, $u'(t_j^-)$, $j \in \{1,2,\ldots\}$. An integration by parts in (3.3) yields

$$\begin{aligned} 0 = \quad & -\sum_{j=0}^{j=+\infty}\int_{t_j}^{t_{j+1}}(p(t)u'(t))'v(t)dt - \sum_{j=1}^{+\infty}\triangle(p(t_j)u'(t_j))v(t_j) \\ & +\sum_{j=1}^{+\infty} h(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t))v(t)dt. \end{aligned}$$

Since u satisfies the equation in problem (1.1) a.e. on $[0,+\infty)$, we obtain

$$\sum_{j=1}^{+\infty} h(t_j)I_j(u(t_j))v(t_j) = \sum_{j=1}^{+\infty}\triangle(p(t_j)u'(t_j))v(t_j), \quad \text{for all}\, v \in H^1_{0,p}.$$

Finally,

$$\triangle(p(t_j)u'(t_j)) = h(t_j)I_j(u(t_j)), \ \text{ for every } \ j \in \{1,2,\ldots\}.$$

□

In fact, u is even a classical solution, i.e., $u \in C^2(t_j,t_{j+1})$, for all $j \in \{1,2,\ldots\}$, whenever $f : [0,+\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ is further continuous.

Example 3.5. Consider the boundary value problem

$$\left\{\begin{array}{rcll} -(e^t u'(t))' & = & \frac{\sqrt{|u|}}{(1+t)^2} + \frac{1}{(1+t)^3}, & \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0) = u(+\infty) & = & 0, & \\ \triangle(e^j u'(j)) & = & \frac{\sqrt[3]{u(j)}}{1+j^2}, & j \in \{1,2,\ldots\}. \end{array}\right. \tag{3.5}$$

It can be easily checked that all conditions of Theorem 3.2 are satisfied with $f(t,x) = \frac{\sqrt{|x|}}{(1+t)^2} + \frac{1}{(1+t)^3}$, $\mu = 1/2$, $a_1(t) = \frac{1}{(1+t)^2}$, $b_1(t) = \frac{1}{(1+t)^3}$, $I_j(s) = s^{1/3}$ $\gamma = 1/3$, $k = 1$, $h(t) = \frac{1}{1+t^2}$, and $\sum_{j=1}^{\infty} h(j) = \frac{\pi}{4}$. Therefore problem (3.5) has at least one solution.

3.2. **The limit case $\mu = 1$.**

Theorem 3.6. *Assume that* (I_0) *holds both with*

(H_2) *There exist positive functions* $a_2, b_2 \in L^1(0,+\infty)$ *with* $|a_2|_{L^1} < \frac{1}{d^2}$ *and*

$$|f(t,x)| \leq a_2(t)|x| + b_2(t), \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } x \in \mathbb{R}.$$

Then problem (1.1) *has at least one weak solution.*

Proof. Arguing as in the proof of Theorem 3.2, we can prove that J is sequentially weakly lower semi-continuous. In addition, under (H_2) and (I_0), we have the estimates:

$$|F(t,u(t))| \leq \frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|,$$

$$\begin{aligned} \left|\int_0^{+\infty} F(t,u(t))dt\right| &\leq \int_0^{+\infty} \left(\frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|\right) dt \\ &\leq \frac{d^2}{2}\|u\|^2\|a_2\|_{L^1} + d\|u\|\|b_2\|_{L^1}. \end{aligned}$$

Then

$$J(u) \geq \frac{1}{2}\left(1 - d^2\|a_2\|_{L^1}\right)\|u\|^2 - d\|u\|\|b_2\|_{L^1} - \frac{kd^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}\sum_{j=1}^{+\infty} h(t_j). \tag{3.6}$$

Since $\|a_2\|_{L^1} < \frac{1}{d^2}$ and $\gamma < 1$, (3.6) implies that

$$\lim_{\|u\| \longrightarrow +\infty} J(u) = +\infty.$$

Then Lemma 2.14 guarantees that problem (1.1) has at least one weak solution. $\square$

Example 3.7. Since hypotheses (I_0) and (H_2) are satisfied, by Theorem 3.6, the boundary value problem

$$\left\{\begin{array}{rcll} -(e^t u'(t))' &=& \frac{u}{(1+t)^4} + \frac{1}{(1+t)^5}, & \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0) = u(+\infty) &=& 0, & \\ \triangle(e^j u'(j)) &=& \frac{\sqrt[4]{u(j)}}{(1+j)^3}, & j \in \{1,2,\ldots\}, \end{array}\right. \tag{3.7}$$

has at least one solution.

3.3. **Nontrivial weak solution.** Our third and last result provides existence of weak solution which is nontrivial for it is obtained by means of Mountain Pass Theorem.

Theorem 3.8. *Suppose that the following conditions hold:*

(H_3) *There exist positive functions* φ, g *such that* $\varphi \in L^1((0,+\infty),\mathbb{R})$ *and* $g \in C(\mathbb{R},\mathbb{R})$ *with*

$$|f(t,x)| \leq \varphi(t)g(x), \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } x \in \mathbb{R}.$$

(H$_4$) $\lim_{x\to 0}\frac{f(t,x)}{x}=0$, *uniformly in* $t\geq 0$.

(H$_5$) *There exist positive functions* $c_1, c_2 \in L^1((0,+\infty),[0,+\infty))$ *and* $\sigma > 2$ *such that*

(a) $F(t,x)\geq c_1(t)|x|^\sigma - c_2(t)$, *for a.e.* $t\geq 0$ *and all* $x\in\mathbb{R}$,

(b) $\sigma F(t,x)\leq xf(t,x)$, *for a.e.* $t\geq 0$ *and all* $x\in\mathbb{R}\setminus\{0\}$.

(I$_1$) *There exists* $0<\gamma\leq 2$ *such that*

$$\gamma\int_0^x I_j(s)ds \geq xI_j(x)>0,\ \forall\, x\in\mathbb{R}\setminus\{0\},\ \forall\, j\in\{1,2,\dots\}.$$

Then problem (1.1) *has at least one nontrivial weak solution.*

Proof. Claim 1. Let $0<\varepsilon<\frac{1}{M}$. *From* (H$_4$), *there exists* $\delta>0$ *such that*

$$|x|\leq\delta \implies |f(t,x)|\leq\varepsilon|x|.$$

Using Lemma 2.4, we deduce that

$$\int_0^{+\infty}|F(t,u(t))dt| \leq \frac{\varepsilon}{2}\|u\|^2_{L^2}\leq\frac{\varepsilon}{2}M\|u\|^2,\ \ \text{for a.e. } t\geq 0,$$

whenever $\|u\|_\infty\leq\delta$. Let $0<\rho\leq\frac{\delta}{d}$ and $\alpha=\frac{1}{2}(1-\varepsilon M)\rho^2$. Then for $\|u\|=\rho$, we have

$$\begin{array}{rcl} J(u) & = & \frac{1}{2}\|u\|^2+\sum_{j=1}^{+\infty}\int_0^{u(t_j)}h(t_j)I_j(\tau)d\tau-\int_0^{+\infty}F(t,u(t))dt \\ & \geq & \frac{1}{2}\|u\|^2-\int_0^{+\infty}F(t,u(t))dt \\ & \geq & \frac{1}{2}(1-\varepsilon M)\|u\|^2=\alpha. \end{array}$$

Assumption (1) in Lemma 2.16 is then satisfied.

Claim 2. From (I$_1$), *there exists* $c_3>0$ *such that*

$$\int_0^x I_j(s)ds\leq c_3|x|^\gamma,\ \ \textit{for every}\ x\in\mathbb{R}.$$

Now (H$_5$)(a) and Lemma 2.4 guarantee that for some $v_0\in H^1_{0,p}(0,+\infty)$, $v_0\neq 0$, we have

$$\begin{array}{rcl} J(\xi v_0) & = & \frac{1}{2}\xi^2\|v_0\|^2+\sum_{j=1}^{+\infty}\int_0^{\xi v_0(t_j)}h(t_j)I_j(\tau)d\tau-\int_0^{+\infty}F(t,\xi v_0(t))dt \\ & \leq & \frac{1}{2}\xi^2\|v_0\|^2+c_3\xi^{\,\gamma}d^{\,\gamma}\|u_0\|^\gamma\sum_{j=1}^{+\infty}h(t_j) \\ & & -|\xi|^\sigma\int_0^{+\infty}c_1(t)|v_0(t)|^\sigma dt+\int_0^{+\infty}c_2(t)dt. \end{array}$$

Since $\sigma>2\geq\gamma$, then for $u_0=\xi v_0$, $J(u_0)\leq 0$, as $\xi\to+\infty$. Hence assumption (2) in Lemma 2.16 is satisfied.

Claim 3. J satisfies the (PS) *condition.*

Notice first that by (H$_3$), $J\in C^1(H^1_{0,p}(0,+\infty),\mathbb{R})$. Now, let (u_n) be a sequence in $H^1_{0,p}(0,+\infty)$ such that $(J(u_n))$ is bounded and $\lim_{n\to+\infty}J'(u_n)=0$. We shall prove that the sequence (u_n) is bounded. Using (H_5)(b) and (I_1), there exists some $K>0$ such that

$$\begin{aligned} & K \geq \sigma J(u_n) - \langle J'(u_n), u_n\rangle \\ = \ & (\tfrac{\sigma}{2}-1)\|u_n\|^2 \\ & -\int_0^{+\infty} (\sigma F(t,u_n(t)) - f(t,u_n(t))u_n(t))\,dt \\ & +\sum_{j=1}^{+\infty} h(t_j)\left(\sigma \int_0^{u_n(t_j)} I_j(t)dt - I_j(u_n(t_j))u_n(t_j)\right) \\ \geq \ & (\tfrac{\sigma}{2}-1)\|u_n\|^2. \end{aligned}$$

Since $\sigma>2$, then the sequence (u_n) is bounded in $H^1_{0,p}(0,+\infty)$. Next, we prove that (u_n) converges strongly to some u in $H^1_{0,p}(0,+\infty)$.
Since (u_n) is bounded in the reflexive Banach space $H^1_{0,p}(0,+\infty)$, there exists a subsequence of (u_n), still denoted (u_n), such that (u_n) converges weakly to some u in $H^1_{0,p}(0,+\infty)$. Then (u_n) converges uniformly to u on $[0,+\infty)$ by Lemma 2.7. Thus

$$\lim_{n\to+\infty}\sum_{j=1}^{+\infty} h(t_j)(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j)-u(t_j)) = 0 \tag{3.8}$$

and

$$\lim_{n\to+\infty}\int_0^{+\infty} (f(t,u_n(t)) - f(t,u(t)))\,(u_n(t)-u(t))\,dt = 0. \tag{3.9}$$

Since $\lim_{n\to+\infty} J'(u_n)=0$ and (u_n) converges weakly to some u, we get

$$\lim_{n\to+\infty}\langle J'(u_n)-J'(u), u_n-u\rangle = 0. \tag{3.10}$$

It follows from (3.2) that

$$\begin{aligned}\langle J'(u_n)-J'(u), u_n-u\rangle \ = \ & \|u_n-u\|^2 + \sum_{j=1}^{+\infty} h(t_j)(I_j(u_n(t_j)) \\ & -I_j(u(t_j)))(u_n(t_j)-u(t_j)) \\ & -\int_0^{+\infty} (f(t,u_n(t)) - f(t,u(t)))\,(u_n(t)-u(t))\,dt. \end{aligned}$$

Hence $\lim_{n\to+\infty}\|u_n-u\|=0$. Thus (u_n) converges strongly to u in $H^1_{0,p}(0,+\infty)$. Therefore, J satisfies the (PS) condition. All conditions of Lemma 2.16 are then fulfilled; as a consequence J has a critical point which is a nontrivial weak solution of problem (1.1). □

Example 3.9. By Theorem 3.8, the boundary value problem

$$\left\{\begin{array}{rcll} -(e^t u'(t))' &=& \frac{u^3(t)}{(1+t)^3}, & \text{a.e. } t\geq 0,\ t\neq t_j, \\ u(0)=u(+\infty) &=& 0, & \\ \triangle(e^j u'(j)) &=& \frac{\sqrt[3]{u(j)}}{(1+j)^2}, & j\in\{1,2,\ldots\}, \end{array}\right. \tag{3.11}$$

has at least one nontrivial solution.

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