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Author(s):

N.V. Thin

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UNIQUENESS OF MEROMORPHIC FUNCTIONS AND $Q-$ DIFFERENCE POLYNOMIALS SHARING SMALL FUNCTIONS

N.V. THIN

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Abstract. The paper concerns interesting problems related to the field of Complex Analysis, in particular, Nevanlinna theory of meromorphic functions. We have studied certain uniqueness problem on differential polynomials of meromorphic functions sharing a small function. Outside, in this paper, we also consider the uniqueness of $q-$ shift difference - differential polynomials of meromorphic functions sharing small function or a set in the complex plane. Our results generalize some previous results in this trend.
Keywords: Uniqueness theorem, q-shift differential polynomials, value distribution, meromorphic function.
MSC(2010): 30D35.

1. Introduction and main results

A meromorphic function means meromorphic in the whole complex plane. We assume that the reader is used to the standard notations and fundamental results of Nevanlinna theory. Let f, g be two meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share $a - CM$ if $f - a$ and $g - a$ have the same zeros with multiplicities. Furthermore, if $f - a$ and $g - a$ have the same zeros without counting multiplicities, then we say that f and g share $a - IM$.

Let m and p be positive integers. We denote by $\overline{N}_{(m}(r, a; f)$ $(\overline{N}_{(m}(r, \dfrac{1}{f-a}))$ the reduced counting function of a-point of f whose multiplicities are not less than m, and $\overline{N}_{m)}(r, a; f)$ $(\overline{N}_{m)}(r, \dfrac{1}{f-a}))$ the reduced counting function of

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a-point of f whose multiplicities are at most m.

$$N_p(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(p}(r, \frac{1}{f-a}), a \in \mathbb{C};$$
$$N_2(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f).$$

We define

$$\delta_p(0, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, 1/f)}{T(r, f)};$$
$$\delta_{p)}(0, f) = 1 - \limsup_{r\to\infty} \frac{N_{p)}(r, 1/f)}{T(r, f)}.$$

We denote by the set of small functions of f in $\mathbb{C}$ by $\mathcal{M}_f(\mathbb{C})$. When f is an entire function, we replace $\mathcal{M}_f(\mathbb{C})$ by $\mathcal{A}_f(\mathbb{C})$. We say that f, $g \in \mathcal{M}(\mathbb{C})$ *share a function* α *CM* if $f - \alpha$ and $g - \alpha$ have the same zeros with multiplicities. If $f - \alpha$ and $g - \alpha$ have the same zeros without counting multiplicities, then we say that f, $g \in \mathcal{M}(\mathbb{C})$ *share a function* α *IM*.

Let S be a subset of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\bigcup_{a\in S}\{z : f(z) = a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ then we say that f and g share the set $S-CM$. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say that f and g share the set $S - IM$. We see that if $S = \{a\}$, then f and g share the set $S - CM$ implies f and g share $a - CM$, and f and g share the set $S - IM$ implies f and g share $a - IM$.

In 2014, L. R. Jie et al. ([5]) proved the following result.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let n, k and m be three positive integers with $n > 4m + 9k + 14$. Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ or $P(z) \equiv c_0$, where $a_0 \neq 0$, $a_1, \ldots, a_{m-1}, a_m \neq 0$, $c_0 \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $a(z) - IM$, then*

(i) *When $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, one of the following two cases hold:*

(i1) *$f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.*

(i2) *f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \cdots + a_0) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \cdots + a_0)$.*

(ii) *When $P(z) \equiv c_0$, $f(z) = tg(z)$ for constant t such that $t^n = 1$.*

(iii) *$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2(z)$.*

Furthermore, if $\max\{\chi_1, \chi_2\} < 0$, *where*

$$\chi_1 = \frac{2m}{n+m-2k} + \frac{m+1}{n+m+2k} + \frac{2k+m}{n+m+k} + 1 - \delta_{k)}(0, P(f)) - \delta_{k-1)}(0, P(f)),$$

$$\chi_2 = \frac{2m}{n+m-2k} + \frac{m+1}{n+m+2k} + \frac{2k+m}{n+m+k} + 1 - \delta_{k)}(0, P(g)) - \delta_{k-1)}(0, P(g)),$$

then the statement (iii) is not happening.

In this paper, we improve Theorem 1.1. Namely, we prove

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions whose zeros and poles are of multiplicities at least s, p respectively, where s, p are positive integers and $\alpha(z) \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero; let n, m, v and $k \geqslant 2$ be four positive integers satisfying*

$n \geqslant k+1$;

$n + m > \dfrac{4k+7}{p} + 2\{m + \dfrac{k+2}{s}\} + 3\{m + \dfrac{k+1}{s}\}$;

and let

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 = (z - b_1)^{m_1} \ldots (z - b_v)^{m_v} Q(z),$$

where $m_i \geqslant k+1$ for $i = 1, \ldots, v$, $v \geqslant 1 + \dfrac{1}{p}$, $m = \deg Q + \sum_{i=1}^{v} m_i$, where $a_0 \neq 0$, $a_1, \ldots, a_{m-1}$, $a_m \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $\alpha(z) - IM$, then either $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_0)$.

Remark 1.3. Theorem 1.2 is an improvement of Theorem 1.1 when the polynomial $P(z)$ has the form

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 = (z - b_1)^{m_1} \ldots (z - b_v)^{m_v} Q(z),$$

where $m_i \geqslant k+1$ for $i = 1, \ldots, v$, $v \geqslant 1 + \dfrac{1}{p}$, and $m = \deg Q + \sum_{i=1}^{v} m_i$. Indeed, if we take $s = p = 1$, we see $n > 4m + 9k + 14$, and we get Theorem 1.1. In the case $n > 4m + 9k + 14$, then the statements of Theorem 1.2 is true without the condition $\max\{\chi_1, \chi_2\} < 0$.

Next, we consider the uniqueness of q-shift difference polynomials of meromorphic functions. In 2015, Q. Zhao and J. Zhang [15] proved the following results.

Theorem 1.4. *Let $f(z)$ be a transcendental meromorphic function with zero order, and let n, k be positive integers. If $n > k+5$, then $(f^n(z) f(qz+c))^{(k)} - 1$ has infinitely many zeros.*

Theorem 1.5. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 2k+5$, and $(f^n(z) f(qz+c))^{(k)}$ and $(g^n(z) g(qz+c))^{(k)}$ share z or $1 - CM$, then $f \equiv tg$ for a constant t with $t^{n+1} = 1$.*

Theorem 1.6. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 5k + 11$, and $(f^n(z)f(qz + c))^{(k)}$ and $(g^n(z)g(qz + c))^{(k)}$ share z or $1 - IM$, then $f \equiv tg$ for a constant t with $t^{n+1} = 1$.*

In 2013, Z. Huang ([3]) obtained the following result.

Theorem 1.7. *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and q be a nonzero complex constant, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. Then for $n > 2m + 3$ (resp. $n > m$), $P(f(z))f(qz) - a(z)$ has infinitely many zeros, where $a(z) \not\equiv 0$ is a small function of f.*

Theorem 1.8. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic (resp. entire) functions of zero order and q be a nonzero complex constant, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. If $n > 3m + 4$ (resp. $n > 2m + 1$) and $P(f(z))f(qz)$ and $P(g(z))g(qz)$ share $1, \infty - CM$, then one of the following two results holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of λ_j $(j = 0, 1, \ldots, n)$, and*

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

(2) *$f(z)$ and $g(z)$ satisfy algebraic equation $R(f(z), g(z)) = 0$, where*

$$R(w_1, w_2) = P(w_1)w_1(qz) - P(w_2)w_2(qz).$$

Remark 1.9. In the proof of Theorem 1.7 and Theorem 1.8, Z. Huang used the inequality $\overline{N}(r, P(f)) \leq mT(r, f) + S(r, f)$, where $P(z)$ is polynomial with $m-$ distinct zero points. We see that the inequality is very weak. Indeed, we have the equality $\overline{N}(r, P(f)) = \overline{N}(r, f)$.

Thus, Theorem 1.7 and Theorem 1.8 may be improved in the case of meromorphic functions as follows.

Theorem 1.10. *Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero complex constant, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. Then for $n \geq m+5$, $P(f(z))f(qz) - a(z)$ has infinitely many zeros, where $a(z) \not\equiv 0$ is a small function of f.*

Theorem 1.11. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions of zero order and q be a nonzero complex constant, and let $P(z) = a_n z^n +$*

$a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ *be a nonconstant polynomial with constant coefficients* $a_0, a_1, \ldots, a_{n-1}$, $a_n \neq 0$, *and* m *be the number of distinct zeros of* $P(z)$. *If* $n \geq 2m+6$ *and* $P(f(z))f(qz)$ *and* $P(g(z))g(qz)$ *share* $1, \infty - CM$, *then one of the following two results holds:*

(1) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^d = 1$, *where* $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ *denotes the lowest common multiple of* λ_j $(j = 0, 1, \ldots, n)$, *and*

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

(2) $f(z)$ *and* $g(z)$ *satisfy algebraic equation* $R(f(z), g(z)) = 0$, *where*

$$R(w_1, w_2) = P(w_1)w_1(qz) - P(w_2)w_2(qz).$$

In 2014, X. Qi and L. Yang ([8]) gave the following result.

Theorem 1.12. *Let* $S_1 = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$, *where* $n \geq 2m+4$, *and* $m \geq 2$ *are integers such that* n *and* $n-m$ *have no common factors,* $S_2 = \{\infty\}$, *and let* a, b *be two non-zero constants such that the algebraic equation* $\omega^n + a\omega^{n-m} + b = 0$ *has no multiple roots. Suppose* f *is a non-constant zero order meromorphic function such that* $E_{f(z)}(S_j) = E_{f(qz)}(S_j)$ *for* $j = 1, 2$, *and* $q \in \mathbb{C} \setminus \{0\}$, *then* $f(z) = f(qz)$, $|q| = 1$.

Now, connecting Theorem 1.4 to Theorem 1.8, we prove some results for uniqueness of q-shift difference-differential of meromorphic functions sharing the small function $a(z)$ or sets for higher derivative. Our results are given in the following.

Theorem 1.13. *Let* $f(z)$ *be a transcendental meromorphic (resp. entire) function of zero order,* q *and* c *be complex constants,* $q \neq 0$ *and* k *be a positive integer, and let* $P(z) = a_nz^n + a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ *be a nonconstant polynomial with constant coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, *and* m *be the number of distinct zeros of* $P(z)$. *Then for* $n \geq m(k+1)+5$ *(resp.* $n \geq m(k+1)+3$*),* $(P(f(z))f(qz+c))^{(k)} - a(z)$ *has infinitely many zeros, where* $a(z) \not\equiv 0$ *is a small function of* f.

Remark 1.14. In Theorem 1.13, when $m = 1$, we get Theorem 1.4. Thus, Theorem 1.13 is an extension of Theorem 1.4.

Theorem 1.15. *Let* $f(z)$ *be a transcendental meromorphic function of zero order,* q *and* c *be complex constants,* $q \neq 0$ *and* k *be a positive integer, and let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ *be a nonconstant polynomial with constant coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, *and* m *be the number of distinct zeros of* $P(z)$. *Then for* $n \geq \dfrac{3}{2}m+3$, $(P(f(z))f(qz+c))^{(k)} - 1$ *has infinitely many zeros.*

Remark 1.16. In Theorem 1.15, when $m = 1$, we get the improvement of Theorem 1.4. Futhermore, the number n is independent of k.

Theorem 1.17. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order, q and c be complex constants, $q \neq 0$, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic (resp. entire) small function and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of the distinct zeros of $P(z)$. If $n \geq 2m(k+1) + 2k + 6$ (resp. $n \geq 2m(k+1) + 4$) and $(P(f(z))f(qz+c))^{(k)}$ and $(P(g(z))g(qz+c))^{(k)}$ share $a(z)$, $\infty - CM$, then one of the following two results holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of λ_j $(j = 0, 1, \ldots, n)$, and*
$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

(2) *$f(z)$ and $g(z)$ satisfy algebraic equation $R(f(z), g(z)) = 0$, where*
$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

We see that Theorem 1.17 is a supplement of Theorem 1.8 for differential polynomial and an extension of Theorem 1.5 for meromorphic functions.

Theorem 1.18. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, q and c be complex constants, $q \neq 0$, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic (resp. entire) small function and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. If $n \geq 2m(k+2) + 3m(k+1) + 8k + 21$ and $(P(f(z))f(qz+c))^{(k)}$ and $(P(g(z))g(qz+c))^{(k)}$ share $a(z) - IM$, then one of the following three results holds:*

(1) $(P(f(z))f(qz+c))^{(k)}.(P(g(z))g(qz+c))^{(k)} \equiv a^2(z)$.

(2) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of λ_j $(j = 0, 1, \ldots, n)$, and*
$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

(3) *$f(z)$ and $g(z)$ satisfy algebraic equation $R(f(z), g(z)) = 0$, where*
$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

We see that Theorem 1.18 is an extension of Theorem 1.6 for meromorphic functions. By an argument as in Theorem 1.12, we will prove Theorem 1.19 below.

Theorem 1.19. *Let $S_1 = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$, where $n \geq 2m+4$, $m \geq 2$ are integers such that n and $n-m$ have no common factors, $S_2 = \{\infty\}$, and let a, b be two non-zero constants such that the algebraic equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Suppose f is a non-constant zero order meromorphic function such that $E_{f(z)}(S_j) = E_{f(qz+c)}(S_j)$ for $j = 1, 2$, and $q \in \mathbb{C} \setminus \{0\}$, then $f(z) = f(qz+c)$.*

Theorem 1.20. *Let $S_1 = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$, where $n \geq 2m+4$, $m \geq 2$ are integers such that n and $n-m$ have no common factors, let a, b be two non-zero constants such that the algebraic equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots, $S_2 = \{\infty\}$, and let q and c be two constants complex, $q \neq 0$. Suppose f is a transcendental zero order meromorphic function such that $E_{(f^l)^{(k)}(z)}(S_j) = E_{(f^l)^{(k)}(qz+c)}(S_j)$ for $j = 1, 2$, where $l \geq 4, k$ are positive integers, then $f(z) = tf(qz+c)$, $t^l = 1$.*

We see that Theorem 1.20 is a supplement of Theorem 1.12 for derivative with order k.

2. Some lemmas

Lemma 2.1 ([4])**.** *Let f be a non-constant meromorphic function, and let p and k be two positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f);$$
$$N_p(r, \frac{1}{f^{(k)}}) \leqslant k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.2 ([12])**.** *Let f and g be two non-constant meromorphic functions, and let $a(z)$ ($a \not\equiv 0, \infty$) be a small function of both f and g. If f and g share $a(z) - IM$, one of the following three cases holds:*

(i) $$\begin{aligned} T(r, f) \leqslant & N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) \\ & + 2(\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f)) + (\overline{N}(r, \frac{1}{g}) + \overline{N}(r, g)) \\ & + S(r, f) + S(r, g), \end{aligned}$$

and similar inequality for $T(r, g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv a^2$.

Lemma 2.3. *Let f and g be two transcendental meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share $\alpha(z) - IM$ and*

$$\begin{aligned} (2k+4)\Theta(\infty, f) + \delta_{k+2}(0, f) + 2\delta_{k+1}(0, f) + (2k+3)\Theta(\infty, g) + \delta_{k+2}(0, g) \\ + \delta_{k+1}(0, g) > 4k + 11. \end{aligned}$$

then $f \equiv g$ or $f^{(k)}g^{(k)} = \alpha(z)^2$, where $\alpha(z)$ is a small function of f and g. Furthermore, in the case $k = 1$, the statement of this lemma holds when f' and g' are non-constant meromorphic functions.

Proof. Since $f^{(k)}$ and $g^{(k)}$ share $a(z) - IM$, by Lemma 2.2, we suppose that the inequality is true. Thus

$$\begin{aligned}T(r,f^{(k)}) \leqslant & N_2(r,f^{(k)}) + N_2\left(r,\frac{1}{f^{(k)}}\right) + N_2(r,g^{(k)}) + N_2\left(r,\frac{1}{g^{(k)}}\right)\\ & + 2(\overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}\left(r,f^{(k)}\right)) + \left(\overline{N}(r,\frac{1}{g^{(k)}}\right) + \overline{N}\left(r,g^{(k)}\right)\Big)\\ & + S(r,f) + S(r,g),\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}T(r,g^{(k)}) \leqslant & N_2(r,f^{(k)}) + N_2(r,\frac{1}{f^{(k)}}) + N_2(r,g^{(k)}) + N_2\left(r,\frac{1}{g^{(k)}}\right)\\ & + 2(\overline{N}\left(r,\frac{1}{g^{(k)}}\right) + \overline{N}(r,g^{(k)})) + \left(\overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}(r,f^{(k)})\right)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{2.2}$$

From (2.1), by using Lemma 2.1, we have

$$\begin{aligned}T(r,f^{(k)}) \leqslant & N_2(r,f^{(k)}) + T(r,f^{(k)}) - T(r,f) + N_{k+2}(r,\frac{1}{f}) + N_2(r,g^{(k)})\\ & + k\overline{N}(r,g) + N_{k+2}(r,\frac{1}{g}) + 2\left(k\overline{N}(r,f) + N_{k+1}(r,\frac{1}{f}) + \overline{N}(r,f)\right)\\ & + \left(k\overline{N}(r,g) + N_{k+1}(r,\frac{1}{g}) + \overline{N}(r,g)\right) + S(r,f) + S(r,g).\end{aligned}$$

Hence,

$$\begin{aligned}T(r,f) \leqslant & (2k+4)\overline{N}(r,f) + N_{k+2}(r,\frac{1}{f}) + 2N_{k+1}(r,\frac{1}{f}) + (2k+3)\overline{N}(r,g)\\ & + N_{k+2}(r,\frac{1}{g}) + N_{k+1}(r,\frac{1}{g}) + S(r,f) + S(r,g).\end{aligned} \tag{2.3}$$

Similarly, from (2.2), we obtain

$$\begin{aligned}T(r,g) \leqslant & (2k+4)\overline{N}(r,g) + N_{k+2}(r,\frac{1}{g}) + 2N_{k+1}(r,\frac{1}{g}) + (2k+3)\overline{N}(r,f)\\ & + N_{k+2}(r,\frac{1}{f}) + N_{k+1}(r,\frac{1}{f}) + S(r,f) + S(r,g).\end{aligned} \tag{2.4}$$

Without loss of generality, we suppose that there exists a set I with an infinite measure such that $T(r,g) \leqslant T(r,f)$ for $r \in I$. Therefore, from (2.3), we see that

$$\begin{aligned} T(r,f) \leqslant & (2k+4)(1-\Theta(\infty,f))T(r,f) + (1-\delta_{k+2}(0,f))T(r,f) \\ & + 2(1-\delta_{k+1}(0,f))T(r,f) + (2k+3)(1-\Theta(\infty,g))T(r,f) \\ & + (1-\delta_{k+2}(0,g))T(r,f) + (1-\delta_{k+1}(0,g))T(r,f) + S(r,f). \end{aligned} \tag{2.5}$$

Thus, we obtain

$$\begin{aligned} ((2k+4)\Theta(\infty,f) + \delta_{k+2}(0,f) + 2\delta_{k+1}(0,f) + (2k+3)\Theta(\infty,g) + \delta_{k+2}(0,g) \\ + \delta_{k+1}(0,g) - (4k+11))T(r,f) \leqslant S(r,f) \end{aligned}$$

for $r \in I$, which contradicts

$$\begin{aligned} (2k+4)\Theta(\infty,f) + \delta_{k+2}(0,f) + 2\delta_{k+1}(0,f) + (2k+3)\Theta(\infty,g) + \delta_{k+2}(0,g) \\ + \delta_{k+1}(0,g) > 4k+11. \end{aligned}$$

By Lemma 2.2, we have $f^{(k)} \equiv g^{(k)}$ or $f^{(k)}g^{(k)} = \alpha(z)^2$. If $f^{(k)} \equiv g^{(k)}$, then $f(z) = g(z) + P(z)$, where $P(z)$ is a polynomial of degree at most $k-1$. In the case $k = 1$, we have $f = g + c$, where c is a constant. If $P(z) \not\equiv 0$, then by the Second Main Theorem for small function, we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f-P(z)}) + S(r,f) \\ &\leqslant \overline{N}(r,f) + N_{k+2}(r,\frac{1}{f}) + N_{k+2}(r,\frac{1}{g}) + S(r,f) \\ &\leqslant (3-(\Theta(\infty,f) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g)))T(r,f) + S(r,f) \end{aligned} \tag{2.6}$$

From (2.6), we get

$$(\Theta(\infty,f) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) - 2)T(r,f) \leqslant S(r,f) \tag{2.7}$$

for $r \in I$.

On the other hand, from the condition

$$\begin{aligned} (2k+4)\Theta(\infty,f) + \delta_{k+2}(0,f) + 2\delta_{k+1}(0,f) + (2k+3)\Theta(\infty,g) + \delta_{k+2}(0,g) \\ + \delta_{k+1}(0,g) > 4k+11 \end{aligned}$$

for $r \in I$, we conclude that

$$\Theta(\infty,f) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) > (4k+11) - (4k+9) = 2.$$

From (2.7), we get the contradiction. Thus, we obtain $P(z) \equiv 0$, that is, $f \equiv g$. this completes the proof.

□

Lemma 2.4 ([11]). *Let $f(z)$ be a nonconstant meromorphic function of zero order. Then on a set of lower logarithmic density* 1, *we have*

$$T(r, f(qz+c)) = (1+o(1))T(r,f) + O(\log r),$$

where $q \in \mathbb{C} \setminus \{0\}$ and c are complex constants

Lemma 2.5 ([11]). *Let $f(z)$ be a nonconstant meromorphic function of zero order. Then on a set of lower logarithmic density 1, we have*

$$N(r, f(qz+c)) = (1+o(1))N(r,f) + O(\log r),$$

where $q \in \mathbb{C} \setminus \{0\}$ and c are complex constants.

Lemma 2.6 ([2]). *Let f be a transcendental meromorphic function in the complex plane, $k \geq 1$ be an integer, and $\varepsilon > 0$. Then we have*

$$(1-\varepsilon)T(r,f) \leq \overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}-1}) + S(r,f).$$

3. Proof of Theorem 1.2

Proof. Set $P(z) = (z-b_1)^{m_1} \dots (z-b_v)^{m_v} Q(z)$, where $m_i \geqslant k+1$ for $i = 1, \dots, v$, $v \geqslant 1 + \dfrac{1}{p}$, and $m = \deg Q + \sum_{i=1}^{v} m_i$. Let $F = f^n P(f)$, and $G = g^n P(g)$. By hypothesis, we get that $F^{(k)}$ and $G^{(k)}$ share $\alpha(z) - IM$. We have

$$\begin{aligned}
\delta_{k+2}(0,G) &= 1 - \limsup_{r\to\infty} \frac{N_{k+2}(r, \frac{1}{G})}{T(r,G)} \\
&= 1 - \limsup_{r\to\infty} \frac{N_{k+2}(r, \frac{1}{g^n P(g)})}{(n+m)T(r,g)} \\
&\geqslant 1 - \limsup_{r\to\infty} \frac{(k+2)\overline{N}(r, \frac{1}{g}) + mT(r,g)}{(n+m)T(r,g)} \\
&\geqslant 1 - \limsup_{r\to\infty} \frac{\frac{k+2}{s}T(r,g) + mT(r,g)}{(n+m)T(r,g)} \\
&= 1 - \frac{m + \frac{k+2}{s}}{n+m}. \qquad (3.1)
\end{aligned}$$

Similarly, we have

$$\delta_{k+2}(0,F) \geqslant 1 - \frac{m + \frac{k+2}{s}}{n+m}. \qquad (3.2)$$

$$\delta_{k+1}(0,G)=1-\limsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{G})}{T(r,G)}$$
$$=1-\limsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{g^nP(g)})}{(n+m)T(r,g)}$$
$$\geqslant 1-\limsup_{r\to\infty}\frac{(k+1)\overline{N}(r,\frac{1}{g})+mT(r,g)}{(n+m)T(r,g)}$$

$$\geqslant 1-\limsup_{r\to\infty}\frac{\frac{k+1}{s}T(r,g)+mT(r,g)}{(n+m)T(r,g)}$$
$$=1-\frac{m+\frac{k+1}{s}}{n+m}. \tag{3.3}$$

Similarly, we obtain

$$\delta_{k+1}(0,F)\geqslant 1-\frac{m+\frac{k+1}{s}}{n+m}, \tag{3.4}$$

and

$$\Theta(\infty,G)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,G)}{T(r,G)}$$
$$=1-\limsup_{r\to\infty}\frac{\overline{N}(r,g)}{(n+m)T(r,g)}$$
$$\geqslant 1-\limsup_{r\to\infty}\frac{\frac{1}{p}N(r,g)}{(n+m)T(r,g)}$$
$$\geqslant 1-\limsup_{r\to\infty}\frac{\frac{1}{p}T(r,g)}{(n+m)T(r,g)}$$
$$=1-\frac{1}{p(n+m)}. \tag{3.5}$$

Also similarly,

$$\Theta(\infty,F)\geqslant 1-\frac{1}{p(n+m)}. \tag{3.6}$$

By Lemma 2.3, we obtain

$$
\begin{aligned}
&(2k+4)\Theta(\infty,F)+\delta_{k+2}(0,F)+2\delta_{k+1}(0,F)+(2k+3)\Theta(\infty,G)\\
&\quad+\delta_{k+2}(0,G)+\delta_{k+1}(0,G)\\
&=(4k+7)\Big(1-\frac{1}{p(n+m)}\Big)+2\left(1-\frac{m+\dfrac{k+2}{s}}{n+m}\right)\\
&\quad+3\left(1-\frac{m+\dfrac{k+1}{s}}{n+m}\right). \qquad (3.7)
\end{aligned}
$$

From $m+n>\dfrac{4k+7}{p}+2\{m+\dfrac{k+2}{s}\}+3\{m+\dfrac{k+1}{s}\}$ and (3.7), we get

$$
\begin{aligned}
(2k+4)\Theta(\infty,F)+\delta_{k+2}(0,F)+2\delta_{k+1}(0,F)+(2k+3)\Theta(\infty,G)+\delta_{k+2}(0,G)\\
+\delta_{k+1}(0,G)>4k+11.
\end{aligned}
$$

Hence, we get $F\equiv G$ or $F^{(k)}G^{(k)}\equiv(\alpha(z))^2$. Let $F\equiv G$, that is

$$
f^n(a_mf^m+\cdots+a_0)=g^n(a_mg^m+\cdots+a_0). \qquad (3.8)
$$

Let $h=\dfrac{f}{g}$. If h is a constant, then substituting $f=hg$ into (3.8), we obtain

$$
a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+\cdots+a_0g^n(h^n-1)=0,
$$

which implies that $h^d=1$, where $d=(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=0,1,\ldots m$. Then $f\equiv tg$ for a constant t such that $t^d=1$. If h is not constant, from (3.8), we see that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\cdots+a_0)-w_2^n(a_mw_2^m+a_{m-1}w_2^{m-1}+\cdots+a_0)$.

Let

$$
F^{(k)}G^{(k)}\equiv(\alpha(z))^2. \qquad (3.9)
$$

We denote by $\sum$ the zeros and poles of $(\alpha(z))^2$. We will show that $F^{(k)}$ and $G^{(k)}$ admit zeros and poles outside of $\sum$. Indeed, suppose that all the zeros and poles of $F^{(k)}$ belong to $\sum$. Therefore

$$
\overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})\leqslant 2T(r,\alpha(z))^2,
$$

which implies that

$$
\overline{N}(r,f)+\sum_{i=1}^{v}\overline{N}(r,\frac{1}{f-b_i})\leqslant S(r,f).
$$

By the Second Main Theorem, we have

$$(v-1)T(r,f) \leqslant \overline{N}(r,f) + \sum_{i=1}^{v} \overline{N}(r, \frac{1}{f-b_i}) + S(r,f),$$

then $(v-1)T(r,f) \leqslant S(r,f)$, which is a contradiction. Hence, $F^{(k)}$ and $G^{(k)}$ admit zeros and poles outside of $\sum$. We suppose that z_0 is zero of f of order s_1 which does not belong to $\sum$, then z_0 is a zeros of $[f^n P(f)]^{(k)}$. Hence z_0 is a pole of $[g^n P(g)]^{(k)}$. This will lead to that z_0 is pole of g of order $p_1 \geqslant p$. From (3.9), we have $ns_1 - k = (n+m)p_1 + k$, then $s_1 \geqslant \frac{(n+m)p+2k}{n}$. Similarly, we suppose that z_i is a b_i point of f of order s_i, then z_i is a pole of g of order p_i, $i = 1, \ldots, v$. Then $s_i \geqslant \frac{(n+m)p+2k}{m_i}$, $i = 1, \ldots, v$. By the Second Main Theorem, we obtain

$$\begin{aligned}(1+\frac{1}{p})T(r,f) \leqslant vT(r,f) &\leqslant \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \sum_{i=1}^{v} \overline{N}(r, \frac{1}{f-b_i}) + S(r,f) \\ &\leqslant (\frac{1}{p} + \frac{n}{(n+m)p+2k} + \sum_{i=1}^{v} \frac{m_i}{(n+m)p+2k})T(r,f) + S(r,f).\end{aligned}$$

Therefore,

$$\left(1 - \frac{n + \sum_{i=1}^{v} m_i}{(n+m)p+2k}\right) T(r,f) \leqslant S(r,f),$$

which is a contradiction. This implies that equality (3.9) is impossible. □

4. **Proof of Theorem 1.13**

Proof. First, from Lemma 2.4 and the First Main Theorem, we have

$$\begin{aligned} nT(r,f) &= T(r, P(f)) \\ &= T\left(r, P(f)(z)f(qz+c).\frac{1}{f(qz+c)}\right) \\ &\leq T(r, P(f)(z)f(qz+c)) + T\left(r, \frac{1}{f(qz+c)}\right) \\ &= T(r, P(f)(z)f(qz+c)) + T(r,f) + S(r,f). \end{aligned}$$

This implies

$$(n-1)T(r,f) \leq T(r, P(f)(z)f(qz+c)) + S(r,f). \tag{4.1}$$

Take $F = P(f)(z)f(qz + c)$, by the Second Main Theorem for small function, we have

$$T(r, F^{(k)}) \leq \overline{N}(r, F^{(k)}) + \overline{N}(r, \frac{1}{F^{(k)}}) + \overline{N}\left(r, \frac{1}{F^{(k)} - a(z)}\right) + S(r, f). \tag{4.2}$$

By Lemma 2.1 we see

$$\overline{N}(r, \frac{1}{F^{(k)}}) \leq T(r, F^{(k)}) - T(r, F) + N_{k+1}(r, \frac{1}{F}) + S(r, f).$$

Thus, (4.2) implies

$$T(r, F) \leq \overline{N}(r, F^{(k)}) + N_{k+1}(r, \frac{1}{F}) + \overline{N}\left(r, \frac{1}{F^{(k)} - a(z)}\right) + S(r, f). \tag{4.3}$$

By simple computing, and from Lemma 2.5, we obtain

$$\begin{aligned}\overline{N}(r, F^{(k)}) &= \overline{N}(r, P(f)f(qz + c)) \leq \overline{N}(r, f) + \overline{N}(r, f(qz + c)) \\ &\leq 2T(r, f) + S(r, f),\end{aligned}$$

and

$$\begin{aligned}N_{k+1}(r, \frac{1}{F}) &\leq (k + 1)\overline{N}(r, \frac{1}{P(f)}) + N\left(r, \frac{1}{f(qz + c)}\right) \\ &\leq (k + 1)mT(r, f) + T(r, f) + S(r, f).\end{aligned}$$

Thus, combining with (4.1) and (4.3), we get

$$\begin{aligned}(n - 1)T(r, f) &\leq 3T(r, f) + (k + 1)mT(r, f) + \overline{N}\left(r, \frac{1}{F^{(k)} - a(z)}\right) + S(r, f) \\ &\leq (m(k + 1) + 3)T(r, f) + \overline{N}\left(r, \frac{1}{F^{(k)} - a(z)}\right) + S(r, f).\end{aligned}$$

From $n \geq m(k+1)+5$, we obtain that $(P(f)f(qz+c))^{(k)} - a(z)$ has infinitely many zeros. □

5. **Proof of Theorem 1.15**

Proof. Take $F(z) = P(f)f(qz + c)$. Apply $\varepsilon = \frac{1}{3}$ in Lemma 2.6 for and transcendental meromorphic function F, to get

$$\frac{2}{3}T(r, F) \leq \overline{N}(r, \frac{1}{F}) + N\big(r, \frac{1}{F^{(k)} - 1}\big) + S(r, F). \tag{5.1}$$

From (5.1), we have

$$\frac{2}{3}(n - 1)T(r, f) \leq (m + 1)T(r, f) + N\big(r, \frac{1}{F^{(k)} - 1}\big) + S(r, f).$$

By $n \geq \frac{3}{2}m+3$, we see that $(P(f)f(qz+c))^{(k)}-1$ has infinitely many zeros. □

6. **Proof of Theorem 1.17**

Proof. Take $F = P(f(z))f(qz+c)$ and $G = P(g(z))g(qz+c)$. Since $(P(f(z))f(qz+c))^{(k)}$ and $(P(g(z))g(qz+c))^{(k)}$ share $a(z), \infty - CM$, then there exists the holomorphic function $\alpha(z)$ satisfying

$$\frac{\dfrac{(P(f(z))f(qz+c))^{(k)}}{a(z)} - 1}{\dfrac{(P(g(z))g(qz+c))^{(k)}}{a(z)} - 1} = e^{\alpha(z)}, \tag{6.1}$$

since the function of left side of (6.1) has order zero. Thus, $e^{\alpha(z)} \equiv A$, where $A \neq 0$ is a constant complex. The equality (6.1) implies

$$(P(f)f(qz+c))^{(k)} = A(P(g)g(qz+c))^{(k)} + a(z)(1-A). \tag{6.2}$$

Now, we will prove that $A = 1$. Conversely, if $A \neq 1$, by an argument as in Theorem 1.13, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + N_{k+1}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F^{(k)} - (1-A)a(z)}\right) + S(r,f) \\ &= \overline{N}(r,F) + N_{k+1}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G^{(k)}}) + S(r,f) \\ &\leq \overline{N}(r,F) + N_{k+1}(r,\frac{1}{F}) + k\overline{N}(r,G) + N_{k+1}(r,\frac{1}{G}) \\ &\quad + S(r,f) + S(r,g), \end{aligned} \tag{6.3}$$

since $S(r,G) = S(r,g), S(r,F) = S(r,f)$. We see

$$\begin{aligned} &\overline{N}(r,F) \leq 2T(r,f) + S(r,f), \\ &\overline{N}(r,G) \leq 2T(r,g) + S(r,g), \\ &N_{k+1}(r,\frac{1}{F}) \leq (m(k+1)+1)T(r,f) + S(r,f), \\ &N_{k+1}(r,\frac{1}{G}) \leq (m(k+1)+1)T(r,g) + S(r,g). \end{aligned} \tag{6.4}$$

Combining (4.1), (6.3) and (6.4), we get

$$\begin{aligned} (n-1)T(r,f) &\leq (m(k+1)+3)T(r,f) + (m(k+1)+2k+1)T(r,g) \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{6.5}$$

Similarly, we obtain

$$\begin{aligned} (n-1)T(r,g) &\leq (m(k+1)+3)T(r,g) + (m(k+1)+2k+1)T(r,f) \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{6.6}$$

From (6.5) and (6.6), we get

$$(n - (2m(k+1)+2k+5))(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g).$$

This contradicts $n \geq 2m(k+1) + 2k + 6$. Thus, we get $A = 1$. Hence, we have

$$(P(f)f(qz+c))^{(k)} = (P(g)g(qz+c))^{(k)}.$$

This implies

$$P(f)f(qz+c) = P(g)g(qz+c) + Q(z),$$

where $Q(z)$ is a polynomial of degree at most $k-1$. Next, we prove $Q(z) \equiv 0$. Indeed, if $Q(z) \not\equiv 0$, by the Second Main Theorem for small function, we have

$$\begin{aligned}(n-1)T(r,f) &\leq T(r,F) + S(r,f)\\ &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-Q(z)}) + S(r,f)\\ &= \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,f).\end{aligned}$$

Since $\overline{N}(r,\frac{1}{F}) \leq (m+1)T(r,f) + S(r,f)$ (because $P(z)$ has m-distinct zero points). Thus, we deduce that

$$(n-1)T(r,f) \leq (m+3)T(r,f) + (m+1)T(r,g) + S(r,f) + S(r,g).$$

This implies

$$(n-m-4)T(r,f) \leq (m+1)T(r,g) + S(r,f) + S(r,g). \tag{6.7}$$

Similarly, we get

$$(n-m-4)T(r,g) \leq (m+1)T(r,f) + S(r,f) + S(r,g). \tag{6.8}$$

From (6.7) and (6.8), we have

$$(n-2m-5)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g).$$

This is a contradiction since $n \geq 2m(k+1)+2k+6 \geq 2m+6$. Hence $Q(z) \equiv 0$. By an argument as in [3], it is easy to see that f and g satisfy of the following statements:

(1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, \ldots, n)$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0,\\ n+1 & \text{if } a_j = 0.\end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

Note, when f and g are transcendental entire functions, we have $N(r,F) = N(r,G) = 0$. By computing similar to the case of meromorphic functions, it is easy to get the statements of Theorem 1.17 with $n \geq 2m(k+1)+4$. □

7. Proof of Theorem 1.18

Proof. Take $F = P(f)f(qz+c), G = P(g)g(qz+c)$. We see that $F^{(k)}$ and $G^{(k)}$ share $a(z)-IM$. By Lemma 2.2, and by an argument as in the proof of Lemma 2.3, we have

$$\begin{aligned} T(r,F) \leqslant &(2k+4)\overline{N}(r,F) + N_{k+2}(r,\frac{1}{F}) + 2N_{k+1}(r,\frac{1}{F}) + (2k+3)\overline{N}(r,G) \\ &+ N_{k+2}(r,\frac{1}{G}) + N_{k+1}(r,\frac{1}{G}) + S(r,f) + S(r,g), \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} T(r,G) \leqslant &(2k+4)\overline{N}(r,G) + N_{k+2}(r,\frac{1}{G}) + 2N_{k+1}(r,\frac{1}{G}) + (2k+3)\overline{N}(r,F) \\ &+ N_{k+2}(r,\frac{1}{F}) + N_{k+1}(r,\frac{1}{F}) + S(r,f) + S(r,g). \end{aligned} \tag{7.2}$$

□

We have

$$\begin{aligned} \overline{N}(r,F) &\le \overline{N}(r,f) + N(r,f(qz+c)) \\ &\le 2T(r,f) + S(r,f), \\ N_{k+2}(r,\frac{1}{F}) &\le N_{k+2}\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{f(qz+c)}\right) \\ &\le (m(k+2)+1)T(r,f) + S(r,f), \\ N_{k+1}(r,\frac{1}{F}) &\le N_{k+1}\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{f(qz+c)}\right) \\ &\le (m(k+1)+1)T(r,f) + S(r,f). \end{aligned} \tag{7.3}$$

Similarly, we obtain

$$\begin{aligned} \overline{N}(r,G) &\le \overline{N}(r,g) + N(r,g(qz+c)) \\ &\le 2T(r,g) + S(r,g), \\ N_{k+2}(r,\frac{1}{G}) &\le N_{k+2}\left(r,\frac{1}{P(g)}\right) + N\left(r,\frac{1}{g(qz+c)}\right) \\ &\le (m(k+2)+1)T(r,g) + S(r,g), \\ N_{k+1}(r,\frac{1}{G}) &\le N_{k+1}\left(r,\frac{1}{P(g)}\right) + N\left(r,\frac{1}{g(qz+c)}\right) \\ &\le (m(k+1)+1)T(r,g) + S(r,g). \end{aligned} \tag{7.4}$$

Combining (7.1) and (7.4), we get

$$(n - (2m(k+2) + 3m(k+1) + 8k + 20))(T(r,f) + T(r,g)) \le S(r,f) + S(r,g).$$

This contradicts $n \geq 2m(k+2)+3m(k+1)+8k+21$. Thus, we have $F^{(k)}G^{(k)} \equiv a^2(z)$ or $F^{(k)} = G^{(k)}$. By an argument as in Theorem 1.17 and [3], it is easy to see that f and g satisfy one of the following two statements:

(1) (2) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of λ_j ($j = 0, 1, \ldots, n$), and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

8. Proof of Theorem 1.20

Proof. Apply Theorem 1.19 to the meromorphic functions $(f^l)^{(k)}(z)$ and $(f^l)^{(k)}(qz+c)$, we have

$$(f^l)^{(k)}(z) = (f^l)^{(k)}(qz+c).$$

Thus

$$f^l(z) = f^l(qz+c) + Q(z),$$

where $Q(z)$ is a polynomial with degree at most $l-1$. If $Q(z) \not\equiv 0$, by the Second Main Theorem for small function and Lemma 2.4, we have

$$\begin{aligned} lT(r,f) &= T(r,f^l) \\ &\leq \overline{N}(r,f^l) + \overline{N}(r,\frac{1}{f^l}) + \overline{N}\left(r, \frac{1}{f^l - Q(z)}\right) + S(r,f) \\ &= \overline{N}(r,f^l) + \overline{N}(r,\frac{1}{f^l}) + \overline{N}\left(r, \frac{1}{f^l(qz+c)}\right) + S(r,f) \\ &\leq 3T(r,f) + S(r,f). \end{aligned}$$

This contradicts $l \geq 4$. Thus, $Q(z) \equiv 0$. This implies

$$f^l(z) = f^l(qz+c),$$

and then $f(z) = tf(qz+c)$, $t^l = 1$.

□

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(Nguyen Van Thin) Department of Mathematics, Thai Nguyen University of Education, Luong Ngoc Quyen Street, Thai Nguyen City, Vietnam.

E-mail address: thinmath@gmail.com