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COORDINATE FINITE TYPE INVARIANT SURFACES IN SOL SPACES

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ABSTRACT. In the present paper, we study surfaces invariant under the 1-parameter subgroup in Sol space Sol_3. Also, we characterize the surfaces in Sol_3 whose coordinate functions of an immersion of the surface are eigenfunctions of the Laplacian Δ of the surface.
Keywords: Sol space, finite type surface, invariant surface.
MSC(2010): Primary: 53C30; Secondary: 53B25.

1. Introduction

Let $\vec{x} : M \longrightarrow \mathbb{E}^m$ be an immersion from an n-dimensional connected Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$. Denote the Laplacian operator of M with the induced metric by Δ. Then, the immersion $\vec{x}$ is of finite type if each component of the position vector field $\vec{x}$ of M in $\mathbb{E}^m$ can be written as a finite sum of eigenfunctions of the Laplacian operator, that is, the position vector $\vec{x}$ of M can be expressed in the form $\vec{x} = c + \sum_{i=1}^{k} x_i$, where c is a constant map and $x_1, x_2, \dots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i,\ i = 1, 2, \dots, k$. Moreover, if all eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ are mutually different, then M is said to be of k-type [4].

For the mean curvature vector field $\overrightarrow{H}$ of M, it is well known that $\Delta\vec{x} = -n\overrightarrow{H}$. In [9] Takahashi proved that an n-dimensional submanifold M of $\mathbb{E}^m$ is of 1-type (i.e., $\Delta\vec{x} = \lambda\vec{x}$) if and only if it is either a minimal submanifold in $\mathbb{E}^m$ or a minimal submanifold of hypersphere of $\mathbb{E}^m$ and Garay [6] extended it to submanifolds, that is, he studied submanifolds in $\mathbb{E}^m$ whose position vector field $\vec{x}$ satisfies the differential equation

$$\Delta\vec{x} = A\vec{x} \tag{1.1}$$

for some $m \times m$-diagonal matrix A. Garay called such submanifolds coordinate finite type submanifolds. Actually coordinate finite type submanifolds are finite

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type submanifolds whose type numbers are at most m. In other words, each coordinate function of a coordinate finite type submanifold submanifold M is of 1-type in the sense of Chen [4]. Recently, Bayram, Arslan, Önen and Bulca [3] studied rotational surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$.

Related to Garay condition (1.1), Dillen, Pas and Verstraelen [5] investigated surfaces in $\mathbb{E}^3$ whose immersion satisfies the equation

$$\Delta\vec{x} = A\vec{x} + B, \tag{1.2}$$

where $A \in \mathrm{Mat}(3,\mathbb{R})$ is a 3×3-real matrix and $B \in \mathbb{R}^3$. This equation (1.2) generalizes Garay condition (1.1) in a coordinate independent way. Several results were obtained when the ambient space is the Lorentz-Minkowski space ([1, 2, 7]), the pseudo-Galilean space ([12]) and the Heisenberg group ([11]).

In this paper we study invariant surfaces in Sol space Sol_3, in particular we classify invariant surfaces satisfying equation (1.2) with $B = 0$.

2. **Preliminaries**

The space Sol_3 is a simply connected homogenous 3-dimensional manifold whose isometry group has dimension 3 and it is one of the eight models of geometry of Thurston [10]. This is a solvable but not nilpotent Lie group which can be seen as the subgroup of 3×3-matrices given by

$$\mathrm{Sol}_3 = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\} \subset GL(3,\mathbb{R}).$$

So, we can view Sol_3 as $\mathbb{R}^3$ and the group operation $*$ is defined by

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}e^{-z}, y + \bar{y}e^{z}, z + \bar{z}\right).$$

On the other hand, the left-multiplication by $p = (x, y, z)$ in Sol_3, $L_p : q \mapsto p*q$, has tangent map

$$T_qL_p = \begin{pmatrix} e^{-z} & 0 & 0 \\ 0 & e^{z} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

in the canonical coordinates (x, y, z) of $\mathbb{R}^3$. Let $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ denote the canonical frame fields in $\mathbb{R}^3$. Then from (2.1) we have that an orthonormal basis of left-invariant vector fields in Sol_3 is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \quad E_2 = e^{z}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

and the left-invariant Riemannian metric $\langle\ ,\ \rangle$ in Sol_3 is given by

$$\langle\ ,\ \rangle = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

On the other hand, the Lie brackets are given as follows

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_3, E_2] = E_2$$

and the Livi-Civita connection $\tilde{\nabla}$ of Sol_3 is expressed as

$$\begin{aligned} &\tilde{\nabla}_{E_1}E_1 = -E_3, \quad \tilde{\nabla}_{E_1}E_2 = 0, \quad \tilde{\nabla}_{E_1}E_3 = E_1, \\ &\tilde{\nabla}_{E_2}E_1 = 0, \quad \tilde{\nabla}_{E_2}E_2 = E_3, \quad \tilde{\nabla}_{E_2}E_3 = -E_2, \\ &\tilde{\nabla}_{E_3}E_1 = 0, \quad \tilde{\nabla}_{E_3}E_2 = 0, \quad \tilde{\nabla}_{E_3}E_3 = 0. \end{aligned} \tag{2.2}$$

The following properties are well-known and can be found in [8], for example. Equipped with the left-invariant Riemannian metric, the Sol space Sol_3 is a homogenous Riemannian manifold whose group of isometrics $\mathfrak{I}(\mathrm{Sol}_3)$ has dimension 3. Also, the connected component $\mathfrak{I}_0(\mathrm{Sol}_3)$ of the identity is generated by the following three families of isometrics:

$$\begin{aligned} T_1^t(x,y,z) &= (x+t,y,z)\,, \\ T_2^t(x,y,z) &= (x,y+t,z)\,, \\ T_3^t(x,y,z) &= (e^{-t}x, e^t y, z+t), \end{aligned}$$

where $t \in \mathbb{R}$ is a real parameter. These isometrics are left multiplications by elements in Sol_3 and they are left-translations with respect to the structure of Lie group.

The corresponding Killing vector fields associated to these families of isometrics are given by

$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y}, \quad F_3 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

A surface in Sol_3 is said to be a T_i- invariant surface if it is invariant under the action of 1-parameter subgroups T_i^t of isometries generated by the Killing vector fields F_i with $i = 1,2,3$.

By the isometry $\phi(x,y,z) = (y,x,-z)$ in Sol_3, an invariant surface under the 1-parameter subgroup T_1^t is congruent to an invariant surface under the 1-parameter subgroup T_2^t. Thus, for the study of invariant surfaces in Sol_3, we may restrict our attention to surfaces invariant under the 1-parameter subgroup T_1^t or T_3^t [8].

It is well known that in terms of local coordinates $\{u_1, u_2\}$ and the coefficients g_{ij} of the first fundamental form of a surface M the Laplacian operator Δ is defined by

$$\Delta = \frac{1}{\sqrt{|\mathfrak{g}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i}(\sqrt{|\mathfrak{g}|} g^{ij} \frac{\partial}{\partial u_j}), \tag{2.3}$$

where $\mathfrak{g} = \det(g_{ij})$ and $(g^{ij}) = (g^{ij})^{-1}$.

3. Coordinate finite type $\mathbf{T_1}$-invariant surfaces

In this section, we investigate coordinate finite type T_1-invariant surfaces in Sol_3. We must also deal with T_3- invariant surfaces. However we have not been able to obtain reliable results due to the fact that the computations are very complicated and difficult to manage for this type of surfaces.

Let M be a surface invariant under the 1-parameter subgroup T_1^t. Then the parametrization of M is given by

$$\begin{aligned} \vec{x}(s,t) &= (t,0,0) * (0,y(s),z(s)) \\ &= (t,y(s),z(s)) \,. \end{aligned} \tag{3.1}$$

We assume that a curve $\alpha(s) = (0, y(s), z(s)), s \in I$ is a unit speed curve. Then we have

$$y'(s) = e^{z(s)} \cos\theta(s), \quad z'(s) = \sin\theta(s), \tag{3.2}$$

where θ is a smooth function.
From (3.1) and (3.2) we obtain an orthogonal basis

$$e_1 := \frac{\partial x}{\partial s} = \cos\theta E_2 + \sin\theta E_3, \quad e_2 := \frac{\partial x}{\partial t} = e^z E_1,$$

which implies the coefficients of the induced metric of the surface to be

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = e^{2z}.$$

Let U be a unit normal vector field of M. Then U is given by

$$U = \sin\theta E_2 - \cos\theta E_3.$$

The values of $\tilde{\nabla}_{e_i} e_j$ (for $i = 1, 2$)

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -\sin\theta(\theta' + \cos\theta)E_2 + \cos\theta(\theta' + \cos\theta)E_3, \\ \tilde{\nabla}_{e_1} e_2 &= \sin\theta e^z E_1, \\ \tilde{\nabla}_{e_2} e_2 &= -e^{2z} E_3, \end{aligned}$$

which imply the coefficients of the second fundamental form of M to be

$$\begin{aligned} h_{11} &= \langle \tilde{\nabla}_{e_1} e_1, U\rangle = \theta' + \cos\theta, \\ h_{12} &= \langle \tilde{\nabla}_{e_1} e_2, U\rangle = 0, \\ h_{22} &= \langle \tilde{\nabla}_{e_2} e_2, U\rangle = -e^{2z}\cos\theta. \end{aligned}$$

Thus, the Gaussian curvature K and the mean curvature H of M are given by, respectively

$$K = -\cos\theta(\theta' + \cos\theta), \quad H = \frac{1}{2}\theta'.$$

By (2.3), the Laplacian operator Δ of M can be expressed as

$$\Delta = -z' \frac{\partial}{\partial s} - \frac{\partial^2}{\partial s^2} - e^{-2z} \frac{\partial^2}{\partial t^2}.$$

By a straightforward computation, the Laplacian operator $\Delta\vec{x}$ of $\vec{x}$ with the help of (2.2) and (3.2) turns out to be

$$\Delta\vec{x} = \theta' \sin\theta E_2 - \theta' \cos\theta E_3. \tag{3.3}$$

First of all, we consider a harmonic surface in Sol_3, that is, a surface M which satisfies the equation $\Delta\vec{x} = 0$. In this case, we have from (3.3) that $\theta' = 0$, which implies that M is a minimal surface.

Consequently, by [8, Theorem 3.1] we have

Theorem 3.1. *Let M be a T_1-invariant surface in* Sol *space* Sol_3. *If M is a harmonic surface if and only if it is a plane or the surface generated by $z = \ln((\tan\theta_0)y)$.*

Now, we consider coordinate finite type T_1-invariant surface in Sol_3, that is, a surface M which satisfies the equation $\Delta\vec{x} = A\vec{x}$ for some diagonal matrix A with diagonal entries a_1, a_2 and a_3. Then from (3.3) we have the following

$$a_1 e^z t = 0, \tag{3.4}$$

$$a_2 y = \theta' \sin\theta e^z, \tag{3.5}$$

$$a_3 z = -\theta' \cos\theta. \tag{3.6}$$

From (3.4) we get $a_1 = 0$. If $\cos\theta = 0$ on an open interval I, then y is a constant function, it follows that M is a vertical plane. We consider $\cos\theta \neq 0$ on an open interval. Then from (3.2) we have

$$\theta' = \pm\frac{z''}{\sqrt{1-(z')^2}}, \tag{3.7}$$

because $\cos\theta = \pm\sqrt{1-(z')^2}$. From here, we have two values for θ'. Without loss of generality, we take the sign + in the above expression. The reasoning is analogous with the choice −.)

We discuss four cases according to the constants a_2 and a_3.

Case 1. $a_2 = a_3 = 0$.
In such case, $\theta' = 0$, that is, M is a minimal surface.

Case 2. $a_2 = 0$ and $a_3 \neq 0$.
From (3.2) and (3.7), equation (3.5) becomes $\frac{z'z''}{\sqrt{1-(z')^2}}e^z = 0$, it follows that $z' = 0$. This means that M is a horizontal plane.

Case 3. $a_2 \neq 0$ and $a_3 = 0$.
From (3.6), we have $z'' = 0$, that is $z = as + b$, $a, b \in \mathbb{R}$. In this case, $\theta' = 0$ and $y = 0$. So, M is a vertical plane.

Case 4. $a_2 \neq 0$ and $a_3 \neq 0$.

From (3.7), equation (3.6) becomes

$$z'' + a_3 z = 0,$$

whose general solution is

$$z(s) = c_1 \cos(\sqrt{a_3}s + d_1), \text{ if } a_3 > 0$$

or

$$z(s) = c_2 \cosh(\sqrt{-a_3}s + d_2), \text{ if } a_3 < 0,$$

where $c_1, c_2, d_1, d_2 \in \mathbb{R}$. Therefore, from (3.5) the function $y(s)$ is given by

$$y(s) = \frac{c_1^2 a_3^{\frac{3}{2}} \sin(\sqrt{a_3}s + d_1)\cos(\sqrt{a_3}s + d_1)}{a_2\sqrt{1 - c_1^2 a_3 \sin^2(\sqrt{a_3}s + d_1)}} e^{c_1 \cos(\sqrt{a_3}s + d_1)}$$

or

$$y(s) = \frac{c_2^2 (-a_3)^{\frac{3}{2}} \sinh(\sqrt{-a_3}s + d_2)\cosh(\sqrt{-a_3}s + d_2)}{a_2\sqrt{1 + c_2^2 a_3 \sinh^2(\sqrt{-a_3}s + d_2)}} e^{c_2 \cosh(\sqrt{-a_3}s + d_2)}.$$

We conclude with the following:

Theorem 3.2. *Let M be a T_1-invariant surface in* Sol *space* Sol_3. *Then M is a coordinate finite type surface, that is, it satisfies the equation*

$$\Delta\vec{x} = A\vec{x}, \quad A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in Mat(3, \mathbb{R})$$

if and only if M is a minimal surface or parametrized as

$$\vec{x}(s,t) = (t, y(s), z(s)),$$

where

(1) *either*

$$y(s) = \frac{c_1^2 a_3^{\frac{3}{2}} \sin(\sqrt{a_3}s + d_1)\cos(\sqrt{a_3}s + d_1)}{a_2\sqrt{1 - c_1^2 a_3 \sin^2(\sqrt{a_3}s + d_1)}} e^{c_1 \cos(\sqrt{a_3}s + d_1)}, \text{ and}$$

$$z(s) = c_1 \cos(\sqrt{a_3}s + d_1) \text{ with } c_1,\ d_1 \in \mathbb{R},\ a_3 > 0$$

(2) *or*

$$y(s) = \frac{c_2^2 (-a_3)^{\frac{3}{2}} \sinh(\sqrt{-a_3}s + d_2)\cosh(\sqrt{-a_3}s + d_2)}{a_2\sqrt{1 + c_2^2 a_3 \sinh^2(\sqrt{-a_3}s + d_2)}} e^{c_2 \cosh(\sqrt{-a_3}s + d_2)}, \text{ and}$$

$$z(s) = c_2 \cosh(\sqrt{-a_3}s + d_2) \text{ with } c_2,\ d_2 \in \mathbb{R},\ a_3 < 0.$$

4. $\mathbf{T_1}$-invariant surfaces satisfying $\Delta\vec{x} = A\vec{x}$ for some matrix A

In this section we study T_1-invariant surfaces in Sol_3 satisfying

$$\Delta\vec{x} = A\vec{x}, \tag{4.1}$$

where A is a general matrix, that is, $A = (a_{ij})$ with $i, j = 1, 2, 3$.
From (3.3) and (4.1), we have

$$a_{11}t + a_{12}y + a_{13}z = 0,$$

$$a_{21}t + a_{22}y + a_{23}z = e^z \theta' \sin\theta,$$

$$a_{31}t + a_{32}y + a_{33}z = -\theta' \cos\theta,$$

which implies $a_{11} = a_{21} = a_{31} = 0$.

$$a_{12}y + a_{13}z = 0, \tag{4.2}$$

$$a_{22}y + a_{23}z = e^z\theta'\sin\theta, \tag{4.3}$$

$$a_{32}y + a_{33}z = -\theta'\cos\theta. \tag{4.4}$$

Case 1. Suppose $a_{12}a_{13} \neq 0$. Differentiating (4.2) with respect to s and using (3.2) we get

$$e^z = -\frac{a_{13}}{a_{12}}\frac{z'}{\sqrt{1-(z')^2}}. \tag{4.5}$$

By using (3.7) and (4.2), equation (4.4) becomes

$$z'' - \left(\frac{a_{13}a_{32} - a_{12}a_{33}}{a_{12}}\right)z = 0. \tag{4.6}$$

On the other hand, from (3.2), (3.7) and (4.5) equation (4.3) can be rewritten as the form:

$$(z')^2 z'' = \left(\frac{a_{13}a_{22} - a_{12}a_{23}}{a_{13}}\right)(z - z(z')^2). \tag{4.7}$$

Combining (4.6) and (4.7) we have

$$\left(\frac{a_{13}a_{32} - a_{12}a_{33}}{a_{12}} + \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{13}}\right)(z')^2 = \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{13}},$$

whose general solution is given by

$$z(s) = a_1 s + b_1$$

for some constants a_1 and b_1. In such case θ is a constant function. From (4.2) we also get

$$y(s) = a_2 s + b_2$$

for some constants a_2 and b_2. Thus, a surface M is generated by $z = ay + b$ with $a, b \in \mathbb{R}$.

Case 2. If $a_{12} = 0, a_{13} \neq 0$ or $a_{12} \neq 0, a_{13} = 0$, a surface is a vertical or horizontal plane.

Case 3. Assume $a_{12} = 0,\ a_{13} = 0$.

Subcase 1. If $a_{32} = 0$, then equation (4.4) maybe written as

$$z'' + a_{33}z = 0$$

which has the general solution

$$z(s) = c_1 s + d_1, \quad \text{if } a_{33} = 0,$$

$$z(s) = c_2\cos(\sqrt{a_{33}}s + d_2),\ \text{if } a_{33} > 0$$

or

$$z(s) = c_3\cosh(\sqrt{-a_{33}}s + d_3),\ \text{if } a_{33} < 0,$$

where $c_i, d_i \in \mathbb{R}$ $(i = 1, 2, 3)$. In this case, the function $y(s)$ corresponding to $z(s)$ is given by

$$\begin{aligned} y(s) &= -\frac{a_{23}}{a_{22}}(c_1 s + d_1), \\ y(s) &= -\frac{c_2 a_{23}}{a_{22}}\cos(\sqrt{a_{33}}s + d_2) \\ &\quad + \frac{c_2^2 a_{33}^{\frac{3}{2}} \sin(\sqrt{a_{33}}s + d_2)\cos(\sqrt{a_{33}}s + d_2)}{a_{22}\sqrt{1 - c_2^2 a_{33}\sin^2(\sqrt{a_{33}}s + d_2)}} e^{c_2\cos(\sqrt{a_{33}}s + d_2)} \end{aligned}$$

or

$$\begin{aligned} y(s) = &-\frac{c_3 a_{23}}{a_{22}}\cosh(\sqrt{-a_{33}}s + d_3) \\ &+ \frac{c_3^2(-a_{33})^{\frac{3}{2}}\sinh(\sqrt{-a_{33}}s + d_3)\cosh(\sqrt{-a_{33}}s + d_3)}{a_{22}\sqrt{1 + c_3^2 a_{33}\sinh^2(\sqrt{-a_{33}}s + d_3)}} e^{c_3\cosh(\sqrt{-a_{33}}s + d_3)}. \end{aligned}$$

Subcase 2. Suppose $a_{32} \neq 0$. From (4.4) we obtain

$$y = -\frac{1}{a_{32}}(z'' + a_{33}z). \tag{4.8}$$

Differentiating (4.8) with respect to s and using (3.2) and (3.7) we get

$$e^z = -\frac{1}{a_{32}\sqrt{1 - (z')^2}}(z''' + a_{33}z').$$

From this, equation (4.3) can be rewritten as the following ODE

$$\begin{aligned} &z'z''z''' + (a_{22} + a_{33})(z')^2 z'' \\ &\qquad - a_{22}z'' + (a_{22}a_{33} - a_{23}a_{32})(z(z')^2 - z') = 0. \end{aligned} \tag{4.9}$$

If $z' = 0$, then from (4.8) y is also a constant. This contradicts the regularity of M.

We assume that $z' \neq 0$ on an open interval. Then equation (4.9) becomes

$$z''z''' + (a_{22} + a_{33})z'z'' - a_{22}\frac{z''}{z'} + (a_{22}a_{33} - a_{23}a_{32})(zz' - 1) = 0,$$

or equivalently

$$\frac{d}{ds}(z'')^2 + (a_{22} + a_{33})\frac{d}{ds}(z')^2 - 2a_{22}\frac{z''}{z'} + (a_{22}a_{33} - a_{23}a_{32})(\frac{d}{ds}(z^2) - 2) = 0.$$

From this we have

$$\begin{aligned} &(z'')^2 + (a_{22} + a_{33})(z')^2 - 2a_{22}\ln|z'| \\ &\qquad + (a_{22}a_{33} - a_{23}a_{32})(z^2 - 2s) + c_1 = 0, \end{aligned} \tag{4.10}$$

where $c_1 \in \mathbb{R}$.

Theorem 4.1. *A T_1-invariant surface in* Sol *space* Sol_3 *satisfies the equation*

$$\Delta\vec{x} = A\vec{x}, \quad A = (a_{ij}) \in Mat(3, \mathbb{R})$$

if and only if it is a vertical or horizontal plane or can be parameterized as

$$\vec{x}(s,t) = (t, y(s), z(s)),$$

where

(1) *either* $y(s) = as + b$ *and* $z(s) = cs + d$,

(2) *or* $y(s) = -\frac{c_2 a_{23}}{a_{22}} \cos(\sqrt{a_{33}}s + d_2)$

$$+\frac{c_2^2 a_{33}^{\frac{3}{2}} \sin(\sqrt{a_{33}}s+d_2)\cos(\sqrt{a_{33}}s+d_2)}{a_{22}\sqrt{1-c_2^2 a_{33}\sin^2(\sqrt{a_{33}}s+d_2)}} e^{c_2\cos(\sqrt{a_{33}}s+d_2)}$$

and $z(s) = c_2\cos(\sqrt{a_{33}}s + d_2)$,

(3) *or* $y(s) = -\frac{c_3 a_{23}}{a_{22}} \cosh(\sqrt{-a_{33}}s + d_3)$

$$+\frac{c_3^2(-a_{33})^{\frac{3}{2}} \sinh(\sqrt{-a_{33}}s+d_3)\cosh(\sqrt{-a_{33}}s+d_3)}{a_{22}\sqrt{1+c_3^2 a_{33}\sinh^2(\sqrt{-a_{33}}s+d_3)}} e^{c_3\cosh(\sqrt{-a_{33}}s+d_3)}$$

and $z(s) = c_3\cosh(\sqrt{-a_{33}}s + d_3)$,

(4) *or the functions* $y(s)$ *and* $z(s)$ *satisfies equations* (4.8) *and* (4.10), *respectively.*

Remark 4.2. To find special solution of (4.9) we take $a_{22} = 0$, $a_{23} = 0$ and $c_1 = 0$. Then we have the ODE $(z'')^2 + a_{33}(z')^2 = 0$ and its solution is $z(s) = \pm\frac{1}{\sqrt{-a_{33}}} e^{\pm\sqrt{-a_{33}}(s+c_1)} + c_2$ with $a_{33} < 0$ and $c_1, c_2 \in \mathbb{R}$. Thus, from (4.8) the function $y(s)$ is given by $y(s) = \pm\frac{2\sqrt{-a_{33}}}{a_{32}} e^{\pm\sqrt{-a_{33}}(s+c_1)} + c_2$. In such case, a matrix A is of the form $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$ with $a_{32}, a_{33} \neq 0$.

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