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ON NON-NORMAL NON-ABELIAN SUBGROUPS OF FINITE GROUPS

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Abstract. In this paper we prove that a finite group G having at most three conjugacy classes of non-normal non-abelian proper subgroups is always solvable except for $G \cong \mathrm{A}_5$, which extends Theorem 3.3 in [Some sufficient conditions on the number of non-abelian subgroups of a finite group to be solvable, Acta Math. Sinica (English Series) 27 (2011) 891–896.]. Moreover, we show that a finite group G with at most three same order classes of non-normal non-abelian proper subgroups is always solvable except for $G \cong \mathrm{A}_5$.
Keywords: Non-abelian subgroup, non-normal, conjugacy class, same order class.
MSC(2010): Primary: 20D05; Secondary: 20D10.

1. Introduction

Abelian subgroups are a class of special subgroups in finite groups, and they play an important role in characterizing the structure of finite groups. Zassenhaus [8] proved that if in a finite group G the normalizer of every abelian subgroup is also its centralizer, then G is abelian. As an extension, S. Li, J. Shi and X. He [3] obtained that if in a finite group G the normalizer of every abelian subgroup of prime-power order generated by at most two elements, and every elementary abelian subgroup, is also its centralizer, then G is abelian. In [5], J. Shi and the author investigated the influence of some quantitative properties of abelian subgroups on the solvability of finite groups.

We are also interested in investigating the influence of non-abelian subgroups on the structure of finite groups. J. Shi and the author [6] investigated the number of conjugacy classes of non-abelian proper subgroups of finite groups, and we obtained the following result:

Theorem 1.1 ([6, Theorem 3.3]). *Let G be a finite group.*

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(1) *Suppose that G has at most two conjugacy classes of non-abelian proper subgroups, then G is solvable.*

(2) *G is a non-solvable group having exactly three conjugacy classes of non-abelian proper subgroups if and only if $G \cong \mathrm{A}_5$.*

In this paper, as an extension of [6, Theorem 3.3], we only study the number of conjugacy classes of non-normal non-abelian proper subgroups of finite groups, we have the following result, the proof of which is given in Section 2.

Theorem 1.2. *Let G be a finite group.*

(1) *Suppose that G has at most two conjugacy classes of non-normal non-abelian proper subgroups, then G is solvable.*

(2) *G is a non-solvable group having exactly three conjugacy classes of non-normal non-abelian proper subgroups if and only if $G \cong \mathrm{A}_5$.*

We say that two non-abelian proper subgroups H_1 and H_2 of G belong to the same order class if H_1 and H_2 have the same order, that is $|H_1| = |H_2|$. It is obvious that any two conjugate subgroups must have the same order, but two subgroups of the same order might not be conjugate.

In [7, Theorems 3.1 and 3.2], J. Shi and the author showed that if a finite group G has at most two same order classes of non-nilpotent proper subgroups then G is solvable, and G is a non-solvable group with exactly three same order classes of non-nilpotent proper subgroups if and only if $G \cong \mathrm{A}_5$ or $\mathrm{SL}_2(5)$. Note that any non-nilpotent subgroup must be a non-abelian subgroup. Thus as a direct corollary of [7, Theorems 3.1 and 3.2], we have the following result.

Theorem 1.3. *Let G be a finite group.*

(1) *Suppose that G has at most two same order classes of non-abelian proper subgroups, then G is solvable.*

(2) *Suppose that G is a non-solvable group having exactly three same order classes of non-abelian proper subgroups, then $G \cong \mathrm{A}_5$.*

As an extension of Theorem 1.3, we investigate the number of same order classes of non-normal non-abelian proper subgroups. We prove the following result whose proof is given in Section 3.

Theorem 1.4. *Let G be a finite group.*

(1) *Suppose that G has at most two same order classes of non-normal non-abelian proper subgroups, then G is solvable.*

(2) *Suppose that G is a non-solvable group with exactly three same order classes of non-normal non-abelian proper subgroups, then $G \cong \mathrm{A}_5$.*

Without applying Theorem 1.3, consider the set of indices of all non-abelian proper subgroups in a finite group. We have the following result.

Theorem 1.5. *Let G be a finite group and let $\pi_a(G)$ be the set of indices of all non-abelian proper subgroups in G. Then $\pi_a(G) = \pi_a(\mathrm{A}_5)$ if and only if $G \cong \mathrm{A}_5$.*

The proof of Theorem 1.5 is given in Section 4. Motivated by Theorem 1.5, we propose the following question.

Question 1.6. (1) Let G be a finite group and N a non-abelian simple group. Is it always true that $\pi_a(G) = \pi_a(N)$ if and only if $G \cong N$?

(2) Let G be a finite group and S a solvable group. Suppose that $\pi_a(G) = \pi_a(S)$. Is it true that G is also solvable?

2. **Proof of Theorem 1.2**

Proof. (1) Suppose that G is a group having at most two conjugacy classes of non-normal non-abelian proper subgroups. Assume that G is non-solvable. Since any finite group with an abelian maximal subgroup is solvable, every maximal subgroup of G is non-abelian. Then by the hypothesis, G has at most two conjugacy classes of non-normal maximal subgroups. By [1, Theorem 2], G is solvable, a contradiction. Hence our assumption is not true and then G is solvable.

(2) We need to prove only the necessity part. Suppose that G is a non-solvable group having exactly three conjugacy classes of non-normal non-abelian proper subgroups. Since G is non-solvable, every maximal subgroup of G is non-abelian. Then by [1, Theorem 2] and the hypothesis, G has exactly three conjugacy classes of non-normal maximal subgroups.

Let M be any maximal subgroup of G. By the hypothesis, every non-abelian proper subgroup of M must be normal in G. In particular, every non-abelian proper subgroup of M is normal in M. By conclusion (1), M is solvable. Then G is a minimal non-solvable group, which follows that $G/\Phi(G)$ is a minimal non-abelian simple group, where $\Phi(G)$ is the Frattini subgroup of G.

Claim: Every non-abelian proper subgroup of G is non-normal.

Otherwise, assume that N is a normal non-abelian proper subgroup of G. Since $G/\mathit{\Phi}(G)$ is a non-abelian simple group, we have $N\Phi(G) = \Phi(G)$. That is $N \leq \Phi(G)$, which implies that $\Phi(G)$ is non-abelian. Then for every proper subgroup H of G satisfying $H > \Phi(G)$, H is a non-normal non-abelian proper subgroup of G. It is obvious that $G/\Phi(G)$ has more than three conjugacy classes of non-trivial subgroups, since $G/\Phi(G)$ is a non-abelian simple group, which implies that G has more than three conjugacy classes of non-normal non-abelian proper subgroups, a contradiction. So every non-abelian proper subgroup of G is non-normal.

Thus by the hypothesis G has exactly three conjugacy classes of non-abelian proper subgroups. It follows that $G \cong \mathrm{A}_5$ by Theorem 1.1(2). □

3. **Proof of Theorem 1.4**

Proof. (1) Let G be a group having at most two same order classes of non-normal non-abelian proper subgroups. Assume that G is non-solvable. Let G be a counterexample of the smallest order. Then G is a minimal non-solvable group. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group. It is obvious that $G/\Phi(G)$ also has at most two same order classes of non-normal non-abelian proper subgroups. Since $G/\Phi(G)$ has no normal non-abelian proper subgroups, $G/\Phi(G)$ has at most two same order classes of non-abelian proper subgroups. By Theorem 1.3(1), $G/\Phi(G)$ is solvable, a contradiction. Hence G is solvable.

(2) Let G be a non-solvable group with exactly three same order classes of non-normal non-abelian proper subgroups. Let $M \leq G$ such that M is a minimal non-solvable group. Then $M/\Phi(M)$ is a minimal non-abelian simple group.

Claim: $\Phi(M)$ is abelian.

Otherwise, if $\Phi(M)$ is non-abelian. Then for every subgroup K of M such that $\Phi(M) < K < M$ it follows that K is a non-normal non-abelian proper subgroup of M. It is obvious that $M/\Phi(M)$ has more than three same order classes of non-trivial subgroups since $M/\Phi(M)$ is a non-abelian simple group. It follows that M has more than three same order classes of non-normal non-abelian proper subgroups, a contradiction. So $\Phi(M)$ is abelian.

It follows that M has no normal non-abelian proper subgroups. By the hypothesis and Theorem 1.3(1), M must have exactly three same order classes of non-abelian proper subgroups. Then by Theorem 1.3(2), one has $M \cong \mathrm{A}_5$.

Claim: $M = G$.

Otherwise, assume that $M < G$. Let L be a subgroup of G such that M is maximal in L. If M is not normal in L. Then M is not normal in G. It follows that G has at least four same order classes of non-normal non-abelian proper subgroups, a contradiction. If M is normal in L, then M is a normal maximal subgroup of L. Since $M \cong \mathrm{A}_5$, one has $L \cong S_5$ or $\mathrm{A}_5 \times \mathbb{Z}_p$ for some prime p. However, it is easy to see that both S_5 and $\mathrm{A}_5 \times \mathbb{Z}_p$ have more than three same order classes of non-normal non-abelian proper subgroups, also a contradiction. Hence $M = G$. That is $G \cong \mathrm{A}_5$. □

4. **Proof of Theorem 1.5**

Proof. We only need to prove the necessity part. Suppose that G is a group with $\pi_a(G) = \pi_a(\mathrm{A}_5) = \{5, 6, 10\}$. Let H be a non-abelian proper subgroup of G such that $|G : H| = 6$. Note that H must be a maximal subgroup of G by the hypothesis. We have that G/H_G is isomorphic to a subgroup of the symmetric group S_6, where H_G is the largest normal subgroup of G that is contained in H. In particular, G/H_G is isomorphic to a non-solvable subgroup of S_6 since every maximal subgroup of a solvable group has prime power index and $|G/H_G : H/H_G| = 6$ is not a prime power. It is easy to see that $\pi_a(G/H_G) \subseteq \pi_a(G) = \{5, 6, 10\}$ and $\mathrm{A}_5, S_5, \mathrm{A}_6$ and S_6 are all non-solvable subgroups of S_6 by [2]. Then we must have $G/H_G \cong \mathrm{A}_5$.

Claim: $H_G = 1$.

Otherwise, assume $H_G \neq 1$. Let M_1 and M_2 be two maximal subgroups of G such that $M_1/H_G \cong \mathbb{Z}_3 : \mathbb{Z}_2$ and $M_2/H_G \cong \mathbb{Z}_5 : \mathbb{Z}_2$. Since $\pi_a(G) = \{5,\ 6,\ 10\}$, one has that M_1 and M_2 must be two minimal non-abelian groups. By [4], we can easily get that H_G is a cyclic 2-group. Let N be a subgroup of H_G such that $H_G/N \cong \mathbb{Z}_2$. Then $(G/N)/(H_G/N) \cong G/H_G \cong \mathrm{A}_5$. We have $G/N \cong \mathrm{A}_5 \times \mathbb{Z}_2$ or $\mathrm{SL}(2,5)$. However, $\pi_a(\mathrm{A}_5 \times \mathbb{Z}_2) = \{2,5,6,10\} \nsubseteq \{5,\ 6,\ 10\}$ and $\pi_a(\mathrm{SL}(2,5)) = \{5,6,10,15\} \nsubseteq \{5,\ 6,\ 10\}$, a contradiction.

So $H_G = 1$ and then $G \cong \mathrm{A}_5$. $\square$

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