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SOLITONS FOR NEARLY INTEGRABLE BRIGHT SPINOR BOSE-EINSTEIN CONDENSATE

F. AHMADI ZEIDABADI* AND S.M. HOSEINI

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Abstract. Using the explicit forms of eigenstates for linearized operator related to a matrix version of Nonlinear Schrödinger equation, soliton perturbation theory is developed for the $F = 1$ bright spinor Bose-Einstein condensates. A small disturbance of the integrability condition can be considered as a small correction to the integrable equation. By choosing appropriate perturbation, the soliton solution for small deviation from the integrability condition is found. Numerical simulations exhibit good agreement with analytical results.
Keywords: Bose-Einstein condensate, Integrable 2×2 matrix, nonlinear Schrödinger equation, soliton perturbation theory, discrete and continuous eigenfunctions.
MSC(2010): 37K40.

1. Introduction

Bose-Einstein condensate (BEC) has extensively investigated by several authors in mathematical and physical disciplines (cf. [4, 9, 10, 12]; and references therein). BECs were observed for the first time in lab in 1995, when Eric Cornell and Carl Wieman were observing the results of efforts of Bose and Einstein to describe bosons with integer spin (F) for dilute vapors of sodium and rubium in ultra cooled temperature. It is well-known that time-evolution of the system containing bosonic atoms with the hyperfine spin of integer F obeys the generalized Gross-Pitaevskii (GP) equations in $(2F + 1)$-dimensions. The 3-dimensional GP equation when $F = 1$ has been the focus of several researches, due to relative simplicity.

Soliton theory is an important branch of nonlinear science. It not only describes various kinds of stable phenomena in nature, such as water waves [2], solitary signals in optical fibres [14], and etc, but also it gives many effective

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methods to obtain exact solutions of nonlinear partial differential equations. These methods include Inverse Scattering Transform (IST) [14], direct Hirota method [5], Darboux transformation method [8, 11] and Bäcklund transformation method [11]. It is noted that the most general soliton solutions *or* N-soliton solutions can be constructed via IST method.

Wadati and his co-authors [9, 12] constructed the most general soliton solutions for the matrix of Nonlinear Schrödinger (mNLS) equation

$$iQ_t + Q_{xx} + 2QQ^\dagger Q = 0, \tag{1.1}$$

where Q is a square matrix, based on Gel'fand-Levitan-Merchenko integral equations. As we shall see latter, (1.1) is a matrix member of NLS hierarchy and the GP equations can be converted to (1.1) under some conditions, called integrability conditions. The exact or nearly exact integrability conditions are commonly met in labs, while the former leads to the soliton equation (1.1) and later leads to the perturbed soliton equation

$$iQ_t + Q_{xx} + 2QQ^\dagger Q = \epsilon F, \qquad |\epsilon| \ll 1 \tag{1.2}$$

with less important terms as perturbation to (1.1). Here F is a function of $Q, Q^\dagger$ and their derivatives. The scalar counterpart of (1.1) is the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1.3}$$

which describes the modulation of weakly nonlinear wavetrains in deep water and the evolution of the slowly varying envelope of an optical pulse.

In a recent research on perturbation theory for BECs, Doktorov *et.al* [4] have constructed the N-soliton solutions by the formalism based on the Riemann-Hilbert (RH) problem via IST procedure and as a result both rank-one and rank-two soliton solutions of 3-dimensional GP equation were obtained. (See Section 2). They also showed that the rank-two solitons can be categorized in two ferromagnetic and polar solitons where former is equivalent to rank-one soliton with familiar hyperbolic secant profile. The categorization is based on the amount of determinant of the polarization matrix subjected to the normalization condition. More precisely, the ferromagnetic soliton corresponds to when the determinant is zero and the polar one corresponds to nonzero determinant. Hence it is sufficient to elaborate the soliton perturbation theory around rank-one solitons. They also consider the small deviation from the integrability condition as (1.2) (see Section 4) and applied an adiabatic approximation to determine the BEC soliton dynamics. As a result, they found temporal dynamical systems for soliton parameters.

Soliton perturbation method is a strong tools to obtain perturbed solutions to a nearly integrable system. For scalar integrable equations, the theory is well applied due to relatively simple forms of eigenfunctions of linearized operator and the Zakharov-Shabat eigenstate closure [6]. This requires that the complete set of eigenfunctions for the linearized problem, related to the nonlinear wave equation, be determined. Yang [13] constructed this set for a large class of integrable nonlinear wave equations such as the Kortewegde Vries (KdV), NLS and modified KdV equations. The same procedure can be exploited to find the eigenstates of the adjoint linearization operator. He found that the eigenfunctions for these hierarchies are the squared Jost solutions. Chen and Yang [3] developed direct soliton perturbation theory for the derivative NLS and the modified NLS equations. Using the similarity between the KdV and derivative NLS hierarchies they showed that the eigenfunctions for the linearized derivative NLS equation are the derivatives of the squared Jost solutions. This is in contrast to the counterpart for NLS, Hirota and mKdV hierarchies, where the eigenfunctions are just the squared Jost solutions. Suppressing the secular terms, they also found the slow evolution of soliton parameters and the perturbation induced radiation.

Hoseini and Marchant [7] examined bright solitary wave interaction for a focusing version of the higher order Hirota equation. A family of higher-order-embedded solitons was found by using an asymptotic transformation. When embedded solitons did not exist, soliton perturbation theory was used to determine the details of a single evolving solitary wave to first order. In particular, an integral expression was found for the first-order correction to the solitary wave profile. They also asymptotically analysed the integral expression to derive an analytical form for the tail of the solitary wave. It was shown that for the right-moving solitary wave, a steady-state tail forms, while for the left-moving wave, some transients propagate on the steady-state tail. For matrix integrable versions, the procedure is quite complicated. Recently, we developed the theory for a 2×2 matrix version of complex modified Korteweg-de Vries (CmKdV) equation

$$Q_t + Q_{xxx} + 3(Q_x Q^\dagger Q + Q Q^\dagger Q_x) = 0, \tag{1.4}$$

and established the IST procedure to find its most general N-soliton solutions [1]. (1.4) belongs to NLS hierarchy and can be constructed via AKNS procedure. Lax pair of (1.4) and the one related to (1.1) are the same, which enables us to apply a similar procedure to find the continuous and discrete eigenfunctions for linear operator related to (1.1). More importantly, the closure relation for the corresponding eigenfunctions for (1.4) was proved and hence it can, in a similar manner, be shown that the closure set corresponding to (1.1) also contains 8 continuous and 6 discrete eigenstates.

This paper is organized as following. In Section 2 we review the connections between GP equation and (1.1) and GP's one- and two-soliton solutions are outlined. The categorization of rank-two solitons based on the polarization matrix is also argued. In Section 3, we develop the soliton perturbation theory for (1.2) using the explicit forms of the eigenfunctions. In Section 4, we find the first-order solution for a perturbed equation describing a small disturbance of the integrability condition by applying the results for soliton perturbation theory found in Section 3. In Section 5 we will show that there is a good agreement between the numerical and analytical results and finally Section 6 concludes the results of the paper.

2. Gross-Pitaevskii (GP) equation and its soliton solutions

Here in the present section, we review the connection between GP equations and the soliton equation (1.1). We consider the system of GP equation (i.e., [4, 9]) as

$$\begin{aligned} i\hbar\partial_t\Phi_\pm &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_\pm + (c_0+c_2)(|\Phi_\pm|^2+|\Phi_0|^2)\Phi_\pm \\ &\quad +(c_0-c_2)|\Phi_\mp|^2\Phi_\pm + c_2\Phi_\mp^*\Phi_0^2, \\ i\hbar\partial_t\Phi_0 &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_0 + (c_0+c_2)(|\Phi_+|^2+|\Phi_-|^2)\Phi_0 \\ &\quad +c_0|\Phi_0|^2\Phi_0 + 2c_2\Phi_+\Phi_-\Phi_0^*, \end{aligned} \tag{2.1}$$

where the vector $\Phi = (\Phi_+, \Phi_0, \Phi_-)^T$ describes the hyperfine spin $F = 1$ spinor BEC. It is noted that (2.1) is applied to describe the BEC trapped in a cigar-shape region in x-direction, where a tight confinement is considered in transversal directions. We adopt the notations in [4] and [9] where $c_0 = (g_0+2g_2)/3$ and $c_1 = (g_2-g_0)/3$ control the spin-independent and spin-dependent interactions, respectively. The coupling factors g_f are determined as

$$g_f = \frac{4\hbar a_f}{ma_\perp^2}(1 - C\frac{a_f}{a_\perp})^{-1}, \tag{2.2}$$

where a_f is the length of s-wave scattering in the channel with the total hyperfine spin $f = 0, 2$. Meanwhile, $a_\perp$ and m are the size of the transverse ground state and the atom mass, respectively and C is a constant.

The system (2.1) is not integrable in the sense that is not adjusted in IST method for all ranges of the parameters. When the condition

$$c_0 = c_2 = -c > 0, \quad (2g_0 = -g_2 > 0), \tag{2.3}$$

is imposed, (1.1) appears as a compatibility condition of a system of linear equation, so-called Lax pair, i.e., (1.1) is integrable. Utilizing the independent variables x and t and the components $\Phi_{\pm,0}$, as in [9] and in presence of the integrability condition, GP equation (2.1) is arranged to (1.1) where

$$Q = \begin{pmatrix} \Phi_+ & \Phi_0 \\ \Phi_0 & \Phi_- \end{pmatrix}. \tag{2.4}$$

We develop the bright soliton perturbation theory for (1.1) when $c < 0$. It is noted that the situations $c_0 < 0$ and $c_2 < 0$ correspond to attractive mean-field and ferromagnetic spin-exchange interactions, respectively. The matrix NLS equation (1.1) admits N-soliton solutions which have been constructed by applying IST based on RH problem in [4]. Here we review the forms of the one- and two-soliton solutions; Rank one soliton solution can be simplified as

$$Q = r\begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\,\theta\,\sin\,\theta \\ \cos\,\theta\sin\,\theta & e^{i\chi}\sin^2\theta \end{pmatrix} e^{i\phi}\,\mathrm{sech}\,\, rz, \tag{2.5}$$

where

$$z = x - vt + z_0, \quad z_0 = -\frac{\rho}{r},$$
$$\phi = \frac{1}{2}vx + (r^2 - \frac{1}{4}v^2)t + \phi_0, \tag{2.6}$$

and

$$\cos\theta = \frac{|n_1|}{\sqrt{|n_1|^2 + |n_2|^2}} = \frac{|n_3|}{\sqrt{|n_3|^2 + |n_4|^2}}, \chi = \arg(n_3) - \arg(n_4),$$
$$\sigma_0 = \arg(n_1) - \arg(n_4) = \arg(n_2) - \arg(n_3). \tag{2.7}$$

The amplitude and velocity are r and v, while z_0 and ϕ_0 determine the initial position of the soliton and its initial phase, respectively. The parameters θ and χ are constants with physical meanings that determine the normalized population of atoms in different spin states and the relative phases between the components. The complex number n_i are constant also. A rank-two soliton has the explicit representation based on the polarization matrix $\Pi^{(2)}$ as

$$Q = 4\nu e^{i\phi}(\Pi^{(2)}e^z + \sigma_2 \Pi^{(2)^\dagger}\sigma_2 D e^{-z})Z^{-1}, \tag{2.8}$$
$$Z = 1 + e^{2z} + |D|^2 e^{-z}, \qquad D = \det\,\Pi^{(2)},$$
$$\Pi^{(2)} = \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and z and ϕ are determined in (2.6) and where α, β and γ are complex numbers which satisfy in normalization condition $2|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Now, the ferromagnetic state is determined when $D = 0$ which is equivalent to one-soliton solution (2.5) after some simplifications. It is noted when $D \neq 0$, the state is called polar state. For details for two-soliton derivation, we refer the reader to [4]. When the integrability condition (2.3) is not exactly valid, the soliton solutions (2.5) and (2.8) is no longer available, and hence the soliton perturbation theory can be a tool to determine the perturbed solutions.

3. Soliton perturbation theory

The application of the soliton perturbation theory for non-dimensional nearly integrable equations is well-known in soliton theory, see e.g. [6] and the references therein. The standard and popular approach is to convert the problem to a linear differential operator problem and then to expand the solution of resulting problem based on the continuous and discrete eigenfunctions of the operator. Suppression of the secular growth then allows the slow variations in the soliton parameters to be determined. In this section, we shall apply a similar procedure to the scalar counterpart to construct the linear operator problem related to (1.1).

We consider the perturbed matrix NLS equation

$$iQ_t + Q_{xx} + 2QQ^\dagger Q = \epsilon F, \qquad |\epsilon| \ll 1, \tag{3.1}$$

where F is a function of Q and its derivatives.

According to soliton perturbation theory, we let all soliton parameters to be T-dependent, where $T = \epsilon t$. Hence, we can define the solitary solution of (3.1) as

$$Q(x,t) = e^{i\phi}(\Phi(z) + \epsilon\Phi^{(1)}(z) + \epsilon^2\Phi^{(2)}(z) + \cdots), \tag{3.2}$$

where

$$\Phi(z) = r \begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix} \text{sech rz},$$

and

$$\phi = \frac{1}{2}vz + \sigma,\ z = x - \nu,\ \nu = \int_0^t v d\tau + z_0,\ \sigma = \int_0^t (r^2 + \frac{1}{4}v^2)d\tau + \phi_0. \tag{3.3}$$

When the parameters do not depend on T, *i.e.*, $\epsilon = 0$, the perturbed solution (3.2) recover the one-soliton solution (2.5).

$$(i\partial_t + L)A = \mathcal{W}, \tag{3.4}$$

where $\Psi = (\Phi_{1,1}, \Phi_{1,2}, \Phi_{2,1}, \Phi_{2,2})$ and $\Psi^{(1)} = (\Phi^{(1)}_{1,1}, \Phi^{(1)}_{1,2}, \Phi^{(1)}_{2,1}, \Phi^{(1)}_{2,2})$,

$$A = (\Psi^{(1)}, \Psi^{(1)^*})^T, \mathcal{W} = (W, -W^*)^T,$$

$$W = F_0 - i\Psi_r r_T + ix_{0T}\Psi_z - i\Psi_\theta\theta_T - i\Psi_\chi\chi_T - (\frac{1}{2}vx_{0T} - \frac{1}{2}v_T z - \phi_{0T})\Psi,$$

$$F_0 = e^{-i\phi}F(\Psi e^{i\phi}), \tag{3.5}$$

and finally, the explicit form of the differential operator L around $Q = \Phi(z)e^{ir^2t}$ is determined as

$$L = \begin{bmatrix} \mathcal{A} & \vdots & \mathcal{B} \\ \cdots & \cdots & \cdots \\ -\mathcal{B}^* & \vdots & -\mathcal{A}^* \end{bmatrix}, \tag{3.6}$$

where

$$\mathcal{A} = \partial_{zz} + 2(\Box\Phi^\dagger\Phi + \Phi\Phi^\dagger\Box) - r^2, \quad \mathcal{B} = 2\Phi\Box\Phi.$$

The "$\Box$" stands for the positions of the function $\Phi^{(1)}$ in (3.4). It is noted that, the relations (3.4)–(3.5) are exactly the same as the scalar counterparts [14].

As the main result of the paper, we find the eigenfunctions of the operator L in (3.6) and their relations. The explicit forms of the eigenfunctions $\mathcal{Z}_i$ are given in Appendix A. These eigenstates are squared Zakharov-Shabat eigenfunctions. Their closure relationship has also been proved in [1]. A short discussion of the derivation of the eigenstates has been given in [1]. The eigenstates satisfy

$$L\mathcal{Z}_j(z,k) = \lambda(k)\mathcal{Z}_j(z,k), \quad 1 \leqslant j \leqslant 4, \qquad L\mathcal{Z}_j(z,k) = -\lambda(k)\mathcal{Z}_j(z,k), \quad 5 \leqslant j \leqslant 8, \tag{3.7}$$

where

$$\lambda(k) = r^2(k^2+1). \tag{3.8}$$

It is also noted that these relations are similar to the scalar NLS equation, and the localized (discrete) eigenstates of L are

$$\begin{aligned} \mathcal{W}_1 &= (\Psi_z, \Psi_z^*)^T, & \mathcal{W}_2 &= (\Psi, -\Psi^*)^T, \\ \mathcal{W}_3 &= (\Psi_\theta, \Psi_\theta^*)^T, & \mathcal{W}_4 &= (\Psi_\chi, \Psi_\chi^*)^T, \\ \mathcal{W}_5 &= \frac{1}{2}z(\Psi, -\Psi^*)^T, & \mathcal{W}_6 &= (\Psi_r, \Psi_r^*)^T, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &L\mathcal{W}_1 = L\mathcal{W}_2 = L\mathcal{W}_3 = L\mathcal{W}_4 = 0, \\ &L\mathcal{W}_5 = \mathcal{W}_1, \quad L\mathcal{W}_6 = 2r\mathcal{W}_2. \end{aligned} \tag{3.10}$$

The number of the localized eigenstates (3.9) is different from non-dimensional NLS counterpart, as the parameters χ, θ are T-dependent. We define the inner-product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f^T(x)g(x)dx. \tag{3.11}$$

Under this inner product, $L^A = -\sigma L \sigma^{-1}$, and $\sigma = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$. It is not difficult to show that

$$\gamma_j(z,k) = \sigma^{-1}\mathcal{Z}_{j+4}(z,k), \qquad j = 1, \dots, 4, \tag{3.12}$$

$$\gamma_{j+4}(z,k) = \sigma\mathcal{Z}_j(z,k), \qquad j = 5, \dots, 8. \tag{3.13}$$

are L^A's eigenstates with

$$L^A\gamma_j(z,k) = \lambda(k)\gamma_j(z,k), \qquad j = 1, \dots, 4, \tag{3.14}$$

$$L^A\gamma_j(z,k) = -\lambda(k)\gamma_j(z,k), \quad j = 5, \dots, 8. \tag{3.15}$$

Here $\lambda(k)$ is given in (3.8). The discrete eigenfunctions

$$\begin{aligned} \mathcal{V}_1(z) &= \sigma^{-1}\mathcal{W}_1(z), & \mathcal{V}_2(z) &= \sigma^{-1}\mathcal{W}_2(z), \\ \mathcal{V}_3(z) &= \sigma^{-1}\mathcal{W}_3(z), & \mathcal{V}_4(z) &= \sigma^{-1}\mathcal{W}_4(z), \\ \mathcal{V}_5(z) &= \sigma\mathcal{W}_5(z), & \mathcal{V}_6(z) &= \sigma\mathcal{W}_6(z), \end{aligned} \tag{3.16}$$

with

$$L^A\mathcal{V}_1(z) = L^A\mathcal{V}_2(z) = L^A\mathcal{V}_3(z) = L^A\mathcal{V}_4(z) = 0, \quad L^A\mathcal{V}_5(z) = \mathcal{V}_1(z), \qquad L^A\mathcal{V}_6(z) = 2r\mathcal{V}_2(z). \tag{3.17}$$

Now, the inner product relations between L's and L^A's eigenfunctions can be shown

$$\langle Z_j(z,k), \gamma_j(z,k')\rangle = \frac{2\pi}{r}(k^2+1)^2\delta(k-k'), \quad j = 1, \ldots, 8, \tag{3.18}$$

as

$$\langle \mathcal{W}_1(z), \mathcal{V}_5(z)\rangle = \langle \mathcal{W}_5(z), \mathcal{V}_1(z)\rangle = r, \tag{3.19}$$

$$\langle \mathcal{W}_2(z), \mathcal{V}_6(z)\rangle = \langle \mathcal{W}_6(z), \mathcal{V}_2(z)\rangle = 2, \tag{3.20}$$

$$\langle \mathcal{W}_3(z), \mathcal{V}_4(z)\rangle = -\langle \mathcal{W}_4(z), \mathcal{V}_3(z)\rangle = 4ir\sin 2\theta, \tag{3.21}$$

$$\langle \mathcal{W}_4(z), \mathcal{V}_6(z)\rangle = \langle \mathcal{W}_4(z), \mathcal{V}_3(z)\rangle = -2i\cos 2\theta, \tag{3.22}$$

where other inner products are all zero.

4. **Solution for the perturbed equation**

Now, in this section, we shall apply all results in previous sections in the case when the integrability conditions (2.3) are not exactly met.
To solve (3.4), we first expand the forcing term $\mathcal{W}$ and the solution A, into the complete set of L's eigenfunctions,

$$\mathcal{W} = \sum_{i=1}^{6} c_i\mathcal{W}_i + \sum_{i=1}^{8}\int_{-\infty}^{\infty}\alpha_i(k)\mathcal{Z}_i(z,k)\mathrm{d}k, \tag{4.1}$$

$$A = \sum_{i=1}^{6} h_i(t)\mathcal{W}_i(z) + \int_{-\infty}^{\infty}\sum_{i=1}^{8} g_i(k,t)\mathcal{Z}_i(z,k)\mathrm{d}k. \tag{4.2}$$

Now, we can find that

$$c_1 = \frac{1}{r}\langle \mathcal{W}, \mathcal{V}_5\rangle, \quad c_2 = \frac{1}{2}\langle \mathcal{W}, \mathcal{V}_6\rangle + ic_4\cos 2\theta, \tag{4.3}$$

$$c_3 = \frac{1}{4ir\sin 2\theta}\langle \mathcal{W}, \mathcal{V}_4\rangle + \frac{c_6}{2r}\cot 2\theta, \quad c_4 = \frac{1}{4r\sin 2\theta}\langle \mathcal{W}, \mathcal{V}_3\rangle, \tag{4.4}$$

$$c_5 = \frac{1}{r}\langle \mathcal{W}, \mathcal{V}_1\rangle, \quad c_6 = \frac{1}{2}\langle \mathcal{W}, \mathcal{V}_2\rangle, \tag{4.5}$$

$$\alpha_j(k) = \frac{r}{2\pi(k^2+1)^2}\langle \mathcal{F}, \gamma_j\rangle, \quad j = 1, \ldots, 8, \tag{4.6}$$

where

$$\mathcal{F} = (F_0, -F_0^*)^T, \quad F_0 = (f_1, f_2, f_3, f_4)^T. \tag{4.7}$$

Suppressing the secular terms given

$$h_j(t) = 0, \qquad 1 \leqslant j \leqslant 6, \tag{4.8}$$

where the following dynamical equations for the soliton parameters are determined:

$$\begin{aligned}\frac{dr}{dT} = \int_{-\infty}^{\infty} \quad & r[\cos^2\theta \ Im\{e^{i\chi}(f_1+\bar{f}_4)\} - \frac{i}{2}\sin 2\theta \ Re\{f_2+f_3\} \\ & -Im\{e^{i\chi}\bar{f}_4\}] \ \operatorname{sech} rz \ dz,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{dv}{dT} = \int_{-\infty}^{\infty} \quad & r[2i\cos^2\theta \ Im\{e^{i\chi}(f_1-\bar{f}_4)\} + \sin 2\theta \ Re\{f_2+f_3\} \\ & +2 \ Re\{e^{i\chi}\bar{f}_4\}] \operatorname{sech} rz \ \tanh rz \ dz,\end{aligned} \tag{4.10}$$

$$\frac{d\chi}{dT} = \frac{1}{2}\int_{-\infty}^{\infty}[Re\{e^{i\chi}(f_1-\bar{f}_4)\} - \cot 2\theta \ Re\{f_2+f_3\})] \operatorname{sech} rz \ dz, \tag{4.11}$$

$$\begin{aligned}\frac{d\theta}{dT} = \frac{1}{4r\sin 2\theta}\{\int_{-\infty}^{\infty} \quad & r[\cos^2\theta Im\{e^{i\chi}(f_1-\bar{f}_4)\} + Im\{e^{i\chi}\bar{f}_4\}] \operatorname{sech} rz dz \\ & +2\cos 2\theta \ r_T\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}\frac{dz_0}{dT} = \int_{-\infty}^{\infty} \quad & [\cos^2\theta \ Im\{e^{i\chi}(f_1+\bar{f}_4)\} + \cos\theta\sin\theta \ Im\{f_2+f_3\} \\ & - \ Im\{e^{i\chi}\bar{f}_4\}]z \ \operatorname{sech} rz \ dz,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{d\phi_0}{dT} = \quad & \frac{1}{2}v\frac{dz_0}{dT} + \cos 2\theta\chi_T - \int_{-\infty}^{\infty}[i\cos^2\theta \ Im\{e^{i\chi}(f_1-\bar{f}_4)\} + \cos\theta\sin\theta \\ & Re\{f_2+f_3\} + \ Re\{e^{i\chi}\bar{f}_4\}] \operatorname{sech} rz(1 - rz\tanh rz)dz.\end{aligned} \tag{4.14}$$

Here, "Re" and "Im" are real and imaginary parts, respectively, and $\bar{f}$ represents the complex conjugate of f.

Evolution equations for the position ν and phase σ are also found as

$$\frac{d\nu}{dt} = v + \epsilon \ z_{0_T}, \tag{4.15}$$

$$\frac{d\sigma}{dt} = r^2 + \frac{1}{4}v^2 + \epsilon\phi_{0_T}, \tag{4.16}$$

$$g_j = \frac{\alpha_j(t)(1 - e^{i\lambda(k)t})}{\lambda(k)}, \quad j = 1, ..., 4, \tag{4.17}$$

$$g_j = -\frac{\alpha_j(t)(1 - e^{-i\lambda(k)t})}{\lambda(k)}, \quad j = 5, ..., 8, \tag{4.18}$$

$$\begin{aligned} A = & \int_{-\infty}^{\infty} \frac{1 - e^{ir^2(k^2+1)}}{2\pi r(k^2+1)^3} \sum_{j=1}^{4} \langle \mathcal{F}, \gamma_j \rangle \mathcal{Z}_j \ dk \\ & - \int_{-\infty}^{\infty} \frac{1 - e^{-ir^2(k^2+1)}}{2\pi r(k^2+1)^3} \sum_{j=5}^{8} \langle \mathcal{F}, \gamma_j \rangle \mathcal{Z}_j \ dk. \end{aligned} \tag{4.19}$$

Using (4.9)–(4.14) and (3.3) the asymptotic solution for a long time, when all the energy radiation has dispersed and the shape oscillation of the perturbed soliton stopped, can be determined as

$$A \to \int_{-\infty}^{\infty} \frac{1}{2\pi r(k^2+1)^3} \sum_{j=1}^{4} \langle \mathcal{F}, \gamma_j \rangle \mathcal{Z}_j \ - \frac{1}{2\pi r(k^2+1)^3} \sum_{j=5}^{8} \langle \mathcal{F}, \gamma_j \rangle \mathcal{Z}_j dk. \tag{4.20}$$

As an example, here we consider (1.1) with an initial nearly soliton. In other words, we consider the following initial condition:

$$Q(z, 0) = (1 + \epsilon) \begin{pmatrix} e^{-i\chi} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & e^{i\chi} \sin^2\theta \end{pmatrix} \operatorname{sech} z, \quad \epsilon \ll 1, \tag{4.21}$$

which is a slightly amplified matrix NLS soliton. Define

$$\tilde{Q} = \frac{1}{1+\epsilon} Q(z, t), \tag{4.22}$$

which satisfies the perturbed mNLS equation

$$i\tilde{Q}_t + \tilde{Q}_{xx} + 2\tilde{Q}\tilde{Q}^\dagger\tilde{Q} = -6\epsilon\tilde{Q}\tilde{Q}^\dagger\tilde{Q}. \tag{4.23}$$

Here the $O(\epsilon^2)$ term has been neglected. The initial condition for $\tilde{Q}$ is

$$\tilde{Q}(z, 0) = \begin{pmatrix} e^{-i\chi} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & e^{i\chi} \sin^2\theta \end{pmatrix} \operatorname{sech} z, \tag{4.24}$$

which is a soliton solution of the unperturbed mNLS equation with initial amplitude $r = 1$ and initial position $z_0 = 0$. Here

$$F_0 = -6r^3 \ (e^{-i\chi} \cos^2\theta, \cos\theta \sin\theta, \cos\theta \sin\theta, e^{i\chi} \sin^2\theta)^T \ \operatorname{sech}^3 rz. \tag{4.25}$$

Substituting this F_0 into dynamical equations for r, v, χ and θ, we find that

$$\frac{dr}{dT} = \frac{dv}{dT} = \frac{d\chi}{dT} = \frac{d\theta}{dT} = 0. \tag{4.26}$$

Similarly, we find that $\frac{d\nu}{dt} = 0$, and hence the position of this soliton does not change. Its phase equation (4.16) can be calculated through simple algebra to be

$$\frac{d\sigma}{dt} = r^2 + 6\epsilon r^2 = (1 + 6\epsilon) r^2. \tag{4.27}$$

Thus

$$\phi(t) = \sigma(t) = r^2(1 + 6\epsilon)t. \tag{4.28}$$

To calculate the form $\mathcal{W}$, notice from dynamical equations (4.13)–(4.14) that $\frac{dz_0}{dT} = 0$, and $\frac{d\phi_0}{dT} = r^2$. It is noted that (4.26)–(4.16) are exactly the same as results in [4] where an adiabatic perturbation theory has been applied with only a difference of factor 6ϵ and 4ϵ, as the perturbed term in (4.23) is replaced by 4ϵ in [4]. Also, we note that the model (1.1) is invariant under the transformation $(Q, x, t) \to (rQ, \frac{x}{r}, \frac{t}{r^2})$, so without loss of any generality we assume $r = 1$ in all the results presented below and in next section. Thus

$$\begin{aligned} W = \quad & F_0 + \phi_{0T}\Psi = 6(e^{-i\chi}\cos^2\theta, \cos\theta\sin\theta, \cos\theta\sin\theta, e^{i\chi}\sin^2\theta)^T \\ & \operatorname{sech} x(1 - \operatorname{sech}^2 x), \end{aligned} \tag{4.29}$$

and performing integration by Residual theorem gives

$$\frac{\langle \mathcal{F}, \gamma_1\rangle}{\cos^2\theta} = \frac{\langle \mathcal{F}, \gamma_2\rangle}{\cos\theta\sin\theta} = \frac{\langle \mathcal{F}, \gamma_3\rangle}{\cos\theta\sin\theta} = \frac{\langle \mathcal{F}, \gamma_4\rangle}{\sin^2\theta} = -\frac{3\pi}{2}(1 + k^2)^2 \ \operatorname{sech}\frac{\pi k}{2},$$

$$\frac{\langle \mathcal{F}, \gamma_5\rangle}{\cos^2\theta} = \frac{\langle \mathcal{F}, \gamma_6\rangle}{\cos\theta\sin\theta} = \frac{\langle \mathcal{F}, \gamma_7\rangle}{\cos\theta\sin\theta} = \frac{\langle \mathcal{F}, \gamma_8\rangle}{\sin^2\theta} = \frac{3\pi}{2}(1 + k^2)^2 \ \operatorname{sech}\frac{\pi k}{2},$$

so

$$\begin{aligned} \Phi^{(1)} \to \quad & \begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix} \times \\ & \int_{-\infty}^{\infty} \frac{3e^{ikx} \ sech\frac{\pi k}{2}}{4(k^2 + 1)}(k^2 - 1 + 2ik \ \tanh \ x + 2 \ \operatorname{sech}^2 x) dk \\ & = \begin{pmatrix} e^{-i\chi}cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix} \frac{3}{2} \ \operatorname{sech} \ x(1 - 2x\tanh \ x), \end{aligned} \tag{4.30}$$

and so

$$\tilde{Q}(x,t) \to \quad e^{i(1+6\epsilon)t}\left\{ \operatorname{sech} x + \frac{3}{2}\epsilon \ \operatorname{sech}\ x(1-2x\ \tanh\ x)\right\} \times$$

$$\begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix}, \tag{4.31}$$

and

$$Q(x,t) \to \quad e^{i(1+6\epsilon)t}(1+\epsilon)\left\{ \operatorname{sech} x + \frac{3}{2}\epsilon \ \operatorname{sech}\ x(1-2x\ \tanh\ x)\right\} \times$$

$$\begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix}. \tag{4.32}$$

More importantly, when the condition (3.3) is not exactly satisfied, the above results can be also obtained.

5. **Numerical comparisons**

To show validity of the evolution results for soliton parameters under perturbation, we compare the numerical simulations and analytical results obtained from soliton perturbation theory.

Here, the perturbed equation (4.23) is solved by pseudospectral method for spatial variable x, and fourth-order Runge-Kutta procedure for temporal variable t. The spatial and temporal discretizations are $\Delta x = 0.14648$, $\Delta t = 1 \times 10^{-4}$, respectively. The initial solution is a soliton (2.5) with $r = 1$. As the mNLS equation (1.1) is the Galilean invariant we can choose $v = 0$. The perturbed parameter is also $\epsilon = 0.1$.

Figure 1 shows the numerical evolution of the mNLS soliton (2.5) as an initial solution for (4.23). Shown is the comparison between the initial solution (2.5) and the numerical solution of (4.23) for $t = 10$ for different components of $\tilde{Q}$. The other parameters are $\theta = \pi/3$ and $\chi = 0$. The distinction is negligible, i.e., the amplitude and velocity of the soliton remain constant due to the perturbation. This fact has been predicted in (4.26).

Figure 2 explains the comparisons between the phase numerical and analytical predictions up to $t = 10$. The parameters are $\theta = \pi/4$ and $\chi = 0$. The comparison is excellent especially for smaller times. For instance, the analytical prediction for phase in (4.28) for $t = 9.5$ is 15.2 whilst its numerical value is about 15.988 with a difference about 5%.

Figure 3 shows the solitary wave tail amplitude versus x for different components of $\Phi^{(1)}$, for $t = 10$. For numerical solutions considered in these figures, the quantity $\epsilon^{-1}|\tilde{Q}_s - \tilde{Q_{0_s}}|$, for $s = \pm, 0$ is plotted, where Q_0 is the initial mNLS soliton solution (2.5). The parameters are the same as Figure 1. This quantity represents the appropriate comparison with perturbation solution $\Phi^{(1)}$ in

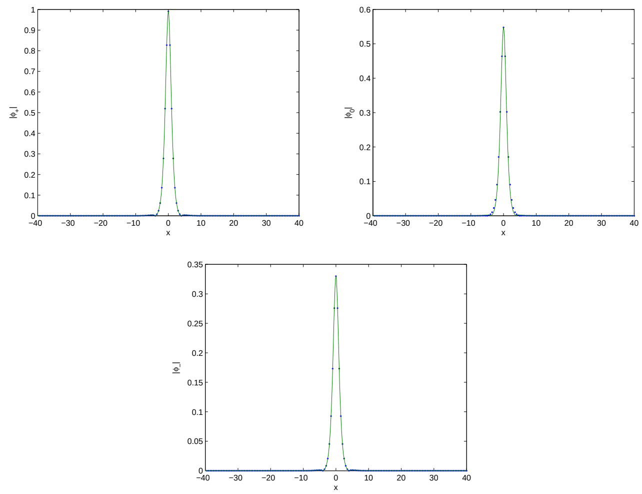

FIGURE 1. Comparison of the initial soliton (4.24) (solid line) and the numerical solution of (4.23) (dot line) versus x. The initial and perturbation parameters are in the text.

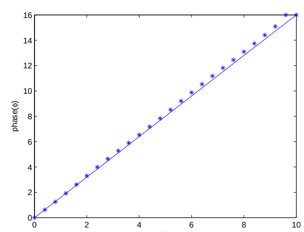

FIGURE 2. Comparison between the analytical and numerical results for phase shifts: solid line is (4.28) and "*" are numerical solution. The initial and perturbation parameters are in the text.

(4.30), in tail regions, away from the solitary wave, located at $x = 0$. It can be seen that the perturbation and numerical solutions are near the same to graphical accuracy, which confirms the validity of analytical results of soliton perturbation to the nearly integrable GP equation. It is noted that a similar figure happens in left tail region with a similar accuracy which has not been shown here.

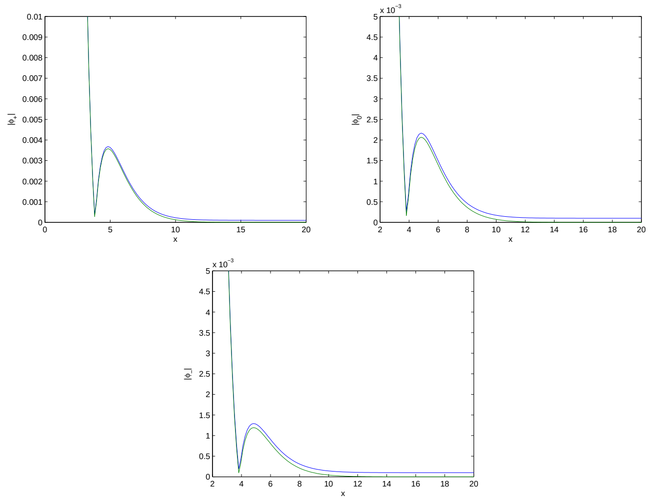

FIGURE 3. The amplitude of the right solitary wave tail, $\Phi^{(1)}$, versus x for different components. Shown are the analytical (4.30) (green curve) and numerical (blue curve).

6. **Conclusion**

In this paper, we have explained the details of the soliton perturbation theory for a matrix version of nonlinear Schrödinger (mNLS) equation, which is related to Bose-Einstein condensate (BEC). Condition (2.3) is a necessary condition for integrability *i.e.*, when (2.3) is satisfied, all N-soliton solutions can be determined. This paper provides all details to determine nearly soliton solution for BEC, when (2.3) is not exactly satisfied. For a future research, the analysis reviewed in this paper could be extended to obtain the discrete soliton to

$$(\ln(h_n))_{xy} + h_{n+1} - 2h_n + h_{n-1} = \epsilon H(h_{n-1}, h_n, h_{n+1}). \tag{6.1}$$

Where $\epsilon = 0$, (6.1) determines the integrable two-dimensional Toda lattice system. This lattice has close connection to Tzitzeica and Fordy-Gibbons systems which are both solitonic systems.

7. **Appendix A**

In this Appendix, we just give the explicit forms of L's eigenfunctions $\mathcal{Z}_i$ in (3.7).

$$\mathcal{Z}_1 = \begin{pmatrix} -\cos^4\theta\ \operatorname{sech}^2 rz \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ (ik-1+\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz)(ik-1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz\ (ik-1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \end{pmatrix} e^{irkz},$$

$$\mathcal{Z}_2 = \begin{pmatrix} -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(-ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ (-ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz)(-ik-1+\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \end{pmatrix} e^{irkz},$$

$$\mathcal{Z}_3 = \begin{pmatrix} -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(ik+1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ (ki-1+\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz)(ik+1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(-ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \end{pmatrix} e^{irkz},$$

$$\mathcal{Z}_4 = \begin{pmatrix} -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ -\sin^4\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(-ik+1-\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ (ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz)(ik-1+\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz) \end{pmatrix} e^{irkz},$$

$$\mathcal{Z}_5 = \begin{pmatrix} (ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz)(ik-1+\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(-ik+1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos^4\theta\ \operatorname{sech}^2 rz \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \end{pmatrix} e^{-irkz},$$

$$\mathcal{Z}_6 = \begin{pmatrix} \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2(rz) \\ (ik-1+\sin^2\theta\ e^{rz}\ \operatorname{sech} rz)(ik+1-\cos^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ sechrz(-ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \end{pmatrix} e^{-irkz},$$

$$\mathcal{Z}_7 = \begin{pmatrix} \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(-ik+1-\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ (ik+1-\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz)(ik-1+\cos^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(ik+1-\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ -\cos^3\theta\sin\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \end{pmatrix} e^{-irkz},$$

$$\mathcal{Z}_8 = \begin{pmatrix} -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ \cos\theta\sin\theta\ e^{rz}\ \operatorname{sech} rz(ik+1-\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz) \\ \cos\theta\sin\theta\ e^{-rz}\ \operatorname{sech} rz(-ik+1-\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ (ik+1-\sin^2\theta\ e^{-rz}\ \operatorname{sech} rz)(ik-1+\sin^2\theta\ e^{rz}\ \operatorname{sech} rz) \\ -\cos^2\theta\sin^2\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ -\cos\theta\sin^3\theta\ \operatorname{sech}^2 rz \\ -\sin^4\theta\ \operatorname{sech}^2 rz \end{pmatrix} e^{-irkz}.$$

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