

ISSN: 1017-060X (Print)



ISSN: 1735-8515 (Online)

**Bulletin of the**  
**Iranian Mathematical Society**

Vol. 43 (2017), No. 3, pp. 707–725

**Title:**

**Theory of hybrid differential equations on time scales**

**Author(s):**

**S. Sun, Y. Zhao, P. Zhao and Z. Han**

Published by the Iranian Mathematical Society  
<http://bims.ims.ir>

## THEORY OF HYBRID DIFFERENTIAL EQUATIONS ON TIME SCALES

S. SUN, Y. ZHAO, P. ZHAO AND Z. HAN\*

(Communicated by Asadollah Aghajani)

**ABSTRACT.** In this paper, we develop the theory of hybrid differential equations on time scales. An existence theorem for hybrid differential equations on time scales is given under Lipschitz conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions. Our results in this paper extend and improve some known results.

**Keywords:** Differential inequalities, existence theorem, comparison principle, time scales.

**MSC(2010):** Primary: 34N05; Secondary: 34A12.

### 1. Introduction

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger in his Ph.D. Thesis in 1988, see [12]. Since then a rapidly expanding body of literature has sought to unify, extend and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$  and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations.

Many other interesting time scales exist, and they give rise to many applications, see [1]. Not only does the new theory of the so-called dynamic equations unify the theories of differential equations and difference equations, but also it extends these classical cases to cases “in between”, for example, to the so-called  $q$ -difference equations when  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$  (which has important applications in quantum theory) and can be applied to different types of

---

Article electronically published on 30 June, 2017.

Received: 8 May 2015, Accepted: 20 January 2016.

\*Corresponding author.

time scales like  $\mathbb{T} = h\mathbb{N}$ ,  $\mathbb{T} = \mathbb{N}^2$  and  $\mathbb{T} = \mathbb{T}_n$  the space of harmonic numbers. A book on the subject of time scales by Bohner and Peterson [1] summarizes and organizes much of the time scale calculus. For advances of dynamic equations on the time scales we refer the reader to [2, 8–10, 14].

In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [3–5, 7, 15]. Dhage and Lakshmikantham [7] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt} \left[ \frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), & a.e. t \in [t_0, t_0 + a], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where  $[t_0, t_0 + a]$  is a bounded interval in  $\mathbb{R}$  for some  $t_0, a \in \mathbb{R}$  with  $a > 0$ ,  $f \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$  and  $g \in \mathcal{C}([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$ , where  $\mathcal{C}([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$  denote the Carathéodory class of functions on  $[t_0, t_0 + a] \times \mathbb{R}$ . They established the existence and uniqueness results and some fundamental differential inequalities for hybrid differential equations and gave the theory of inequalities, its existence of extremal solutions and a comparison result.

Zhao et al. [15] discussed fractional hybrid differential equations involving Riemann–Liouville differential operators of order  $0 < q < 1$ ,

$$\begin{cases} D^q \left[ \frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), & a.e. t \in [0, T], \\ x(0) = 0, \end{cases}$$

where  $f \in C([0, T] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$  and  $g \in \mathcal{C}([0, T] \times \mathbb{R}, \mathbb{R})$ , where  $\mathcal{C}([0, T] \times \mathbb{R}, \mathbb{R})$  denote the Carathéodory class of functions on  $[0, T] \times \mathbb{R}$ . They gave the existence and uniqueness results and some fundamental differential inequalities for fractional hybrid differential equations and established the theory of inequalities, its existence of extremal solutions and a comparison result.

As far as we know, there are no results for the theory of hybrid differential equations on time scales. From the above works, we develop the theory of hybrid differential equations on time scales. An existence theorem for hybrid differential equations on time scales is given under Lipschitz conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions. Our results in this paper extend and improve some known results.

The paper is organized as follows: In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales which we rely on in the later section. In Section 3, by the fixed point theorem

in Banach algebra due to Dhage, an existence theorem for hybrid differential equations on time scales is given under Lipschitz conditions. In Section 4, we give a fundamental result relative to strict inequalities for hybrid differential equations on time scales. In Section 5, existence results of maximal and minimal solutions for hybrid differential equations on time scales are given. In Section 6, the comparison principle for hybrid differential equations on time scales is proved. In Section 7, we obtain the existence results of extremal positive solutions for hybrid differential equations on time scales.

## 2. Hybrid differential equation on time scales

On any time scale  $\mathbb{T}$  we define the forward and the backward jump operators by:

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\},$$

where  $\inf \emptyset = \sup \mathbb{T}$  and  $\sup \emptyset = \inf \mathbb{T}$ . A point  $t \in \mathbb{T}$  is said to be left-dense if  $\rho(t) = t$ , right-dense if  $\sigma(t) = t$ , left-scattered if  $\rho(t) < t$  and right-scattered if  $\sigma(t) > t$ . The graininess function  $\mu$  for a time scale  $\mathbb{T}$  is defined by  $\mu(t) = \sigma(t) - t$ .

For a function  $f : \mathbb{T} \rightarrow \mathbb{R}$ , the  $\Delta$  derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if  $f$  is continuous at  $t$  and  $t$  is right-scattered. If  $t$  is right-dense, then the derivative is defined by

$$f^\Delta(t) = \lim_{s \rightarrow t^+} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \rightarrow t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be rd-continuous provided  $f$  is continuous at right-dense points and there exists a finite left limit at all left-dense points in  $\mathbb{T}$ .

Let  $\mathbb{T}$  be a time scale and  $J = [t_0, t_0 + a]_{\mathbb{T}} = [t_0, t_0 + a] \cap \mathbb{T}$  be a bounded interval in  $\mathbb{T}$  for some  $t_0$ , and  $a \in \mathbb{R}$  with  $a > 0$ . Let  $C_{rd}(J \times \mathbb{R}, \mathbb{R})$  denote the class of rd-continuous functions  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ .

We consider hybrid differential equations (in short HDE) on time scales

$$(2.1) \quad \begin{cases} \left[ \begin{array}{c} x(t) \\ f(t, x(t)) \end{array} \right]^\Delta = g(t, x(t)), & a.e. \ t \in J, \\ x(t_0) = x_0, \end{cases}$$

where  $f \in C_{rd}(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$  and  $g \in C_{rd}(J \times \mathbb{R}, \mathbb{R})$ .

By a solution of the HDE (2.1) on time scales we mean a  $\Delta$ -differentiable function  $x$  such that

- (i) the function  $t \mapsto \frac{x}{f(t, x)}$  is  $\Delta$ -differentiable for each  $x \in \mathbb{R}$ , and
- (ii)  $x$  satisfies the equations in (2.1).

The theory of strict and nonstrict differential inequalities related to the ODEs and hybrid differential equations is available in the literature, see [7,13]. It is known that differential inequalities are useful for proving the existence of extremal solutions of the ODEs and hybrid differential equations defined on  $J$ .

### 3. Existence result

In this section, we prove the existence results for the HDE (2.1) on time scales. We place the HDE (2.1) in the space  $C_{rd}(J, \mathbb{R})$  of rd-continuous functions defined on  $J$ . Define a supremum norm  $\|\cdot\|$  in  $C_{rd}(J, \mathbb{R})$  by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and a multiplication in  $C_{rd}(J, \mathbb{R})$  by

$$(xy)(t) = x(t)y(t)$$

for  $x, y \in C_{rd}(J, \mathbb{R})$ . Clearly  $C_{rd}(J, \mathbb{R})$  is a Banach algebra with respect to the above norm and multiplication in it. By  $L^1(J, \mathbb{R})$  denote the space of Lebesgue  $\Delta$ -integrable functions on  $J$  equipped with the norm  $\|\cdot\|_{L^1}$  defined by

$$\|x\|_{L^1} = \int_{t_0}^{t_0+a} |x(s)| \Delta s.$$

The following fixed point theorem in Banach algebra due to Dhage [6] is fundamental in the proofs of our main results.

**Lemma 3.1** ([6]). *Let  $S$  be a non-empty, closed convex and bounded subset of the Banach algebra  $X$  and let  $A : X \rightarrow X$  and  $B : S \rightarrow X$  be two operators such that*

- (a)  $A$  is Lipschitzian with a Lipschitz constant  $\alpha$ ,
- (b)  $B$  is completely continuous,
- (c)  $x = AxBy \Rightarrow x \in S$  for all  $y \in S$ , and
- (d)  $\alpha M < 1$ , where  $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$ .

*Then the operator equation  $AxBx = x$  has a solution in  $S$ .*

This Lemma can be obtained by taking  $\phi(r) = \alpha r$  in [6, Theorem 2.1].

We consider the following hypotheses.

- (A<sub>0</sub>) The function  $x \mapsto \frac{x}{f(t, x)}$  is increasing in  $\mathbb{R}$  almost everywhere for  $t \in J$ .
- (A<sub>1</sub>) There exists a constant  $L > 0$  such that

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

for all  $t \in J$  and  $x, y \in \mathbb{R}$ .

(A<sub>2</sub>) There exists a function  $h \in L^1(J, \mathbb{R})$  such that

$$|g(t, x)| \leq h(t) \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$ .

The following lemma is useful in what follows.

**Lemma 3.2.** *Assume that hypothesis (A<sub>0</sub>) holds. Then for any  $y \in L^1(J, \mathbb{R})$ , the  $\Delta$ -differentiable function  $x$  is a solution of the HDE*

$$(3.1) \quad \left[ \frac{x(t)}{f(t, x(t))} \right]^\Delta = y(t), \quad \text{a.e. } t \in J,$$

and

$$(3.2) \quad x(t_0) = x_0,$$

if and only if  $x$  satisfies the hybrid integral equation (HIE)

$$(3.3) \quad x(t) = [f(t, x(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t y(s) \Delta s \right), \quad t \in J.$$

*Proof.* Let  $x$  be a solution of the problem (3.1) and (3.2). Applying  $\Delta$ -integral to (3.1) from  $t_0$  to  $t$ , then we have

$$\frac{x(t)}{f(t, x(t))} - \frac{x_0}{f(t_0, x_0)} = \int_{t_0}^t y(s) \Delta s,$$

i.e.,

$$x(t) = [f(t, x(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t y(s) \Delta s \right), \quad t \in J.$$

Thus, (3.3) holds.  $\square$

Conversely, assume that  $x$  satisfies HIE (3.3). Dividing by  $f(t, x(t))$  and applying  $\Delta$ -derivative on both sides of (3.3) gives (3.1). Then substituting  $t = t_0$  in (3.3) yields

$$\frac{x(t_0)}{f(t_0, x(t_0))} = \frac{x_0}{f(t_0, x_0)}.$$

Since the map  $x \mapsto \frac{x}{f(t, x)}$  is increasing in  $\mathbb{R}$  almost everywhere for  $t \in J$ , the map  $x \mapsto \frac{x}{f(t_0, x)}$  is injective in  $\mathbb{R}$  and  $x(t_0) = x_0$ . Hence (3.2) also holds. The proof is completed.  $\square$

Now we will give the following existence theorem for HDE (2.1).

**Theorem 3.3.** *Assume that hypotheses (A<sub>0</sub>)–(A<sub>2</sub>) hold. Further, if*

$$(3.4) \quad L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) < 1,$$

then the HDE (2.1) has a solution defined on  $J$ .

*Proof.* Set  $X = C_{rd}(J, \mathbb{R})$  and define a subset  $S$  of  $X$  defined by

$$(3.5) \quad S = \{x \in X \mid \|x\| \leq N\},$$

where  $N = \frac{F_0(|\frac{x_0}{f(t_0, x_0)}| + \|h\|_{L^1})}{1 - L(|\frac{x_0}{f(t_0, x_0)}| + \|h\|_{L^1})}$  and  $F_0 = \sup_{t \in J} |f(t, 0)|$ .

Clearly  $S$  is a closed, convex and bounded subset of the Banach space  $X$ . By Lemma 3.2, HDE (2.1) is equivalent to the nonlinear HIE

$$(3.6) \quad x(t) = [f(t, x(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \right), \quad t \in J.$$

Define two operators  $A : X \rightarrow X$  and  $B : S \rightarrow X$  by

$$(3.7) \quad Ax(t) = f(t, x(t)), \quad t \in J,$$

and

$$(3.8) \quad Bx(t) = \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s, \quad t \in J.$$

Then the HIE (3.6) is transformed into the operator equation as

$$(3.9) \quad Ax(t)Bx(t) = x(t), \quad t \in J.$$

We shall show that the operators  $A$  and  $B$  satisfy all the conditions of Lemma 3.1.

First, we show that  $A$  is a Lipschitz operator on  $X$  with the Lipschitz constant  $L$ . Let  $x, y \in X$ . Then by hypothesis (A<sub>1</sub>),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq L|x(t) - y(t)| \leq L\|x - y\|,$$

for all  $t \in J$ . Taking supremum over  $t$ , we obtain

$$\|Ax - Ay\| \leq L\|x - y\|,$$

for all  $x, y \in X$ .

Next, we show that  $B$  is a compact and continuous operator on  $S$  into  $X$ . First we show that  $B$  is continuous on  $S$ . Let  $\{x_n\}$  be a sequence in  $S$  converging to a point  $x \in S$ . Then by Lebesgue dominated convergence

theorem adapted to time scale,

$$\begin{aligned}
\lim_{n \rightarrow \infty} Bx_n(t) &= \lim_{n \rightarrow \infty} \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x_n(s)) \Delta s \right) \\
&= \frac{x_0}{f(t_0, x_0)} + \lim_{n \rightarrow \infty} \int_{t_0}^t g(s, x_n(s)) \Delta s \\
&= \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t \left[ \lim_{n \rightarrow \infty} g(s, x_n(s)) \right] \Delta s \\
&= \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \\
&= Bx(t),
\end{aligned}$$

for all  $t \in J$ . This shows that  $B$  is a continuous operator on  $S$ .

Next we show that  $B$  is a compact operator on  $S$ . It is enough to show that  $B(S)$  is a uniformly bounded and equicontinuous set in  $X$ . On the one hand, let  $x \in S$  be arbitrary. Then by hypothesis  $(A_2)$ ,

$$\begin{aligned}
|Bx(t)| &= \left| \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \right| \\
&\leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \int_{t_0}^t |g(s, x(s))| \Delta s \\
&\leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \int_{t_0}^t h(s) \Delta s \\
&\leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1},
\end{aligned}$$

for all  $t \in J$ . Taking supremum over  $t$ , we have

$$\|Bx\| \leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1}$$

for all  $x \in S$ . This shows that  $B$  is uniformly bounded on  $S$ .

On the other hand, let  $t_1, t_2 \in J$ , with  $t_1 < t_2$ . Then for any  $x \in S$ , one has

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left| \int_{t_0}^{t_1} g(s, x(s)) \Delta s - \int_{t_0}^{t_2} g(s, x(s)) \Delta s \right| \\
&\leq \left| \int_{t_2}^{t_1} |g(s, x(s))| \Delta s \right| \\
&\leq \left| \int_{t_2}^{t_1} h(s) \Delta s \right| \\
&= |p(t_1) - p(t_2)|
\end{aligned}$$



where  $p(t) = \int_{t_0}^t h(s)\Delta s$ . Since the function  $p$  is continuous on compact  $J$ , it is uniformly continuous. Hence, for  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \varepsilon,$$

for all  $t_1, t_2 \in J$  and for all  $x \in S$ . This shows that  $B(S)$  is an equicontinuous set in  $X$ . Now the set  $B(S)$  is uniformly bounded and equicontinuous set in  $X$ , so it is compact by Arzela–Ascoli Theorem. As a result,  $B$  is a complete continuous operator on  $S$ .

Next, we show that hypothesis (c) of Lemma 3.1 is satisfied. Let  $x \in X$  and  $y \in S$  be arbitrary such that  $x = AxBy$ . Then, by assumption  $(A_1)$ , we have

$$\begin{aligned} |x(t)| &= |Ax(t)||By(t)| \\ &= |f(t, x(t))| \left| \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, y(s))\Delta s \right| \\ &\leq [ |f(t, x(t)) - f(t, 0)| + |f(t, 0)| ] \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \int_{t_0}^t |g(s, y(s))|\Delta s \right) \\ &\leq [L|x(t)| + F_0] \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \int_{t_0}^t h(s)\Delta s \right) \\ &\leq [L|x(t)| + F_0] \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right). \end{aligned}$$

Thus,

$$|x(t)| \leq \frac{F_0 \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right)}{1 - L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right)}.$$

Taking supremum over  $t$ , we obtain

$$\|x\| \leq \frac{F_0 \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right)}{1 - L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right)} = N.$$

This shows that hypothesis (c) of Lemma 3.1 is satisfied. Finally, we have

$$M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1},$$

so,

$$\alpha M \leq L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) < 1.$$

Thus, all the conditions of Lemma 3.1 are satisfied and hence the operator equation  $AxBx = x$  has a solution in  $S$ . As a result, the HDE (2.1) has a solution defined on  $J$ . This completes the proof.  $\square$

#### 4. Hybrid differential inequalities on time scales

In this section, we will give a fundamental result relative to strict inequalities for the HDE (2.1) on time scales.

**Theorem 4.1.** *Assume that hypothesis (A<sub>0</sub>) holds. Suppose that there exist  $\Delta$ -differentiable functions  $y$  and  $z$  such that*

$$(4.1) \quad \left[ \frac{y(t)}{f(t, y(t))} \right]^\Delta \leq g(t, y(t)), \quad \text{a.e. } t \in J$$

and

$$(4.2) \quad \left[ \frac{z(t)}{f(t, z(t))} \right]^\Delta \geq g(t, z(t)), \quad \text{a.e. } t \in J,$$

one of the inequalities being strict. Then  $y(t_0) < z(t_0)$  implies

$$(4.3) \quad y(t) < z(t)$$

for all  $t \in J$ .

*Proof.* Suppose that the inequality (4.2) is strict. Assume that the claim is false. Then there exists a  $t_1 \in J$ , with  $t_1 > t_0$  such that  $y(t_1) = z(t_1)$  and  $y(t) < z(t)$  for  $t_0 \leq t < t_1$ .

Define

$$Y(t) = \frac{y(t)}{f(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t)}{f(t, z(t))}$$

for all  $t \in J$ . Then we have  $Y(t_1) = Z(t_1)$  and by (A<sub>0</sub>), we get  $Y(t) < Z(t)$  for all  $t < t_1$ .

From  $Y(t_1) = Z(t_1)$ , we have

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

for sufficiently small  $h < 0$ . The above inequality implies that

$$Y^\Delta(t_1) \geq Z^\Delta(t_1)$$

because of hypotheses (A<sub>0</sub>). Then we get

$$g(t_1, y(t_1)) \geq Y^\Delta(t_1) \geq Z^\Delta(t_1) > g(t_1, z(t_1)).$$

This contradicts  $y(t_1) = z(t_1)$ . Hence the conclusion (4.3) is valid and the proof is complete.  $\square$

The next result is concerned with nonstrict differential inequalities on time scales which requires a kind of one sided Lipschitz condition.

**Theorem 4.2.** *Assume that the conditions of Theorem 4.1 hold with inequalities (4.1) and (4.2). Suppose that there exists a real number  $L > 0$  such that*

$$(4.4) \quad g(t, x_1) - g(t, x_2) \leq L \left( \frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right), \quad \text{a.e. } t \in J$$

for all  $x_1, x_2 \in \mathbb{R}$  with  $x_1 \geq x_2$ . Then  $y(0) \leq z(0)$  implies

$$(4.5) \quad y(t) \leq z(t)$$

for all  $t \in J$ .

*Proof.* Let  $\varepsilon > 0$  and  $L > 0$  a real number. We set

$$\frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} = \frac{z(t)}{f(t, z(t))} + \varepsilon e^{2Lt},$$

so that we have

$$\frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} > \frac{z(t)}{f(t, z(t))} \Rightarrow z_\varepsilon(t) > z(t).$$

Let  $Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))}$  so that  $Z(t) = \frac{z(t)}{f(t, z(t))}$  for  $t \in J$ . By (4.2), then we have

$$Z_\varepsilon^\Delta(t) = Z^\Delta(t) + 2L\varepsilon e^{2Lt} \geq g(t, z(t)) + 2L\varepsilon e^{2Lt}.$$

From (4.4), then we obtain

$$g(t, z_\varepsilon(t)) - g(t, z(t)) \leq L \left( \frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} - \frac{z(t)}{f(t, z(t))} \right)$$

for all  $t \in J$ , then we get

$$Z_\varepsilon^\Delta(t) \geq g(t, z_\varepsilon(t)) - L\varepsilon e^{2Lt} + 2L\varepsilon e^{2Lt} > g(t, z_\varepsilon(t)),$$

i.e.,

$$\left[ \frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} \right]^\Delta > g(t, z_\varepsilon(t)),$$

for all  $t \in J$ . Also, we have  $z_\varepsilon(t_0) > z(t_0) > y(t_0)$ . Hence, by an application of Theorem 4.1 with  $z = z_\varepsilon$ , it yields that  $y(t) < z_\varepsilon(t)$  for all  $t \in J$ . By the arbitrariness of  $\varepsilon > 0$ , taking the limits as  $\varepsilon \rightarrow 0$ , we have  $y(t) \leq z(t)$  for all  $t \in J$ . This completes the proof.  $\square$

*Remark 4.3.* The conclusions of Theorems 4.1 and 4.2 also remain true if we replace the derivatives in the inequalities (4.1) and (4.2) by Dini-derivative  $D_{\pm}^\pm$  of the function  $\frac{x(t)}{f(t, x(t))}$  on the bounded interval  $J$ .

### 5. Existence of maximal and minimal solutions

In this section, we will prove the existence of maximal and minimal solutions for the HDE (2.1) on  $J = [t_0, t_0 + a]_{\mathbb{T}}$ . We need the following definition in what follows.

**Definition 5.1.** A solution  $r$  of the HDE (2.1) is said to be maximal if for any other solution  $x$  to the HDE (2.1) one has  $x(t) \leq r(t)$ , for all  $t \in J$ . Similarly, a solution  $\rho$  of the HDE (2.1) is said to be minimal if  $\rho(t) \leq x(t)$ , for all  $t \in J$ , where  $x$  is any solution of the HDE (2.1) on  $J$ .

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrary small real number  $\varepsilon > 0$ , consider the following initial value problem of HDE on time scales,

$$(5.1) \quad \begin{cases} \left[ \frac{x(t)}{f(t, x(t))} \right]^\Delta = g(t, x(t)) + \varepsilon, & a.e. \ t \in J, \\ x(t_0) = t_0 + \varepsilon, \end{cases}$$

where  $f \in C_{rd}(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$  and  $g \in C_{rd}(J \times \mathbb{R}, \mathbb{R})$ .

An existence theorem for the HDE (5.1) on time scales can be stated as follows:

**Theorem 5.2.** *Assume that hypotheses (A<sub>0</sub>)–(A<sub>2</sub>) hold. Suppose that the inequality (3.4) holds. Then for every small number  $\varepsilon > 0$ , the HDE (5.1) has a solution defined on  $J$ .*

*Proof.* By hypothesis, since

$$L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) < 1,$$

there exists an  $\varepsilon_0 > 0$  such that

$$L \left( \left| \frac{x_0 + \varepsilon}{f(t_0, x_0 + \varepsilon)} \right| + \|h\|_{L^1} + \varepsilon a \right) < 1,$$

for all  $0 < \varepsilon \leq \varepsilon_0$ . Now the rest of the proof is similar to Theorem 3.3.  $\square$

Our main result is the following existence theorem for maximal solution for the HDE (2.1) on time scales.

**Theorem 5.3.** *Assume that hypotheses (A<sub>0</sub>)–(A<sub>2</sub>) hold. Furthermore, if condition (3.4) holds, then the HDE (2.1) has a maximal solution defined on  $J$ .*

*Proof.* Let  $\{\varepsilon_n\}_0^\infty$  be a decreasing sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ , where  $\varepsilon_0$  is a positive real number satisfying the inequality

$$(5.2) \quad L \left( \left| \frac{x_0 + \varepsilon_0}{f(t_0, x_0 + \varepsilon_0)} \right| + \|h\|_{L^1} + \varepsilon_0 a \right) < 1,$$

The number  $\varepsilon_0$  exists in view of the inequality (3.4). Then for any solution  $u$  of the HDE (2.1) on time scales, by Theorem 5.2, we have

$$u(t) < r(t, \varepsilon_n)$$

for all  $t \in J$  and  $n \in \mathbb{N} \cup \{0\}$ , where  $r(t, \varepsilon_n)$  which is defined on  $J$  is a solution of the HDE

$$(5.3) \quad \begin{cases} \left[ \frac{x(t)}{f(t, x(t))} \right]^\Delta = g(t, x(t)) + \varepsilon_n, & a.e. \ t \in J, \\ x(t_0) = t_0 + \varepsilon_n. \end{cases}$$

By Theorem 4.2,  $\{r(t, \varepsilon_n)\}$  is a decreasing sequence of positive real numbers, and the limit

$$(5.4) \quad r(t) = \lim_{n \rightarrow \infty} r(t, \varepsilon_n)$$

exists. We show that the convergence in (5.5) is uniform on  $J$ . To finish, it is enough to prove that the sequence  $r(t, \varepsilon_n)$  is equicontinuous in  $C_{rd}(J, \mathbb{R})$ . Let  $t_1, t_2 \in J$  with  $t_1 < t_2$  be arbitrary. Then,

$$\begin{aligned} & |r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)| \\ &= \left| [f(t_1, r(t_1, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s \right) \right. \\ &\quad \left. - [f(t_2, r(t_2, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_2} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_2} \varepsilon_n \Delta s \right) \right| \\ &\leq \left| [f(t_1, r(t_1, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s \right) \right. \\ &\quad \left. - [f(t_2, r(t_2, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s \right) \right| \\ &\quad + \left| [f(t_2, r(t_2, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s \right) \right. \\ &\quad \left. - [f(t_2, r(t_2, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^{t_2} g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^{t_2} \varepsilon_n \Delta s \right) \right| \\ &\leq |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| \left( \left| \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} \right| + \|h\|_{L^1} + \varepsilon_n a \right) \\ &\quad + F[|p(t_1) - p(t_2)| + |t_1 - t_2| \varepsilon_n], \end{aligned}$$

where  $F = \sup_{(t,x) \in J \times [-N,N]} |f(t, x)|$  and  $p(t) = \int_{t_0}^t h(s) \Delta s$ .

Since  $f$  is continuous on the compact set  $J \times [-N, N]$ , it is uniformly continuous there. Hence,

$$|f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2$$

uniformly for all  $n \in \mathbb{N}$ . Similarly, since the function  $p$  is continuous on the compact set  $J$ , it is uniformly continuous and hence

$$|p(t_1) - p(t_2)| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2.$$

Therefore, from the above inequality, it follows that

$$|r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2$$

uniformly for all  $n \in \mathbb{N}$ . Therefore,

$$r(t, \varepsilon_n) \rightarrow r(t) \quad \text{as } n \rightarrow \infty$$

for all  $t \in J$ .

Next, we show that the function  $r(t)$  is a solution of the HDE (2.1) defined on  $J$ . Now, since  $r(t, \varepsilon_n)$  is a solution of the HDE (5.3), we have

$$(5.5) \quad r(t, \varepsilon_n) = [f(t, r(t, \varepsilon_n))] \left( \frac{x_0 + \varepsilon_n}{f(t_0, x_0 + \varepsilon_n)} + \int_{t_0}^t g(s, r(s, \varepsilon_n)) \Delta s + \int_{t_0}^t \varepsilon_n \Delta s \right)$$

for all  $t \in J$ . Taking the limit as  $n \rightarrow \infty$  in the above equation (5.5) yields

$$r(t) = [f(t, r(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, r(s)) \Delta s \right)$$

for all  $t \in J$ . Thus, the function  $r$  is a solution of the HDE (2.1) on  $J$ . Finally, from the inequality (5.3), it follows that  $u(t) \leq r(t)$  for all  $t \in J$ . Hence, the HDE (2.1) on time scales has a maximal solution on  $J$ . This completes the proof.  $\square$

## 6. Comparison theorems on time scales

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the HDE (2.1) on time scales. In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to HDE (2.1) on  $J = [t_0, t_0 + a]_{\mathbb{T}}$ .

**Theorem 6.1.** *Assume that hypotheses  $(A_0)$ – $(A_2)$  and condition (3.4) hold. Suppose that there exists a  $\Delta$ -differentiable function  $u$  such that*

$$(6.1) \quad \begin{cases} \left[ \frac{u(t)}{f(t, u(t))} \right]^\Delta \leq g(t, u(t)), & \text{a.e. } t \in J, \\ u(t_0) \leq x_0. \end{cases}$$

Then

$$(6.2) \quad u(t) \leq r(t)$$

for all  $t \in J$ , where  $r$  is a maximal solution of the HDE (2.1) on  $J$ .

*Proof.* Let  $\varepsilon > 0$  be arbitrarily small. By Theorem 5.3,  $r(t, \varepsilon)$  is a maximal solution of the HDE (5.1) and the limit

$$(6.3) \quad r(t) = \lim_{\varepsilon \rightarrow 0} r(t, \varepsilon)$$

is uniform on  $J$  and the function  $r$  is a maximal solution of the HDE (2.1) on  $J$ . Hence, we obtain

$$\begin{cases} \left[ \frac{r(t, \varepsilon)}{f(t, r(t, \varepsilon))} \right]^\Delta = g(t, r(t, \varepsilon)) + \varepsilon, & \text{a.e. } t \in J, \\ r(t_0, \varepsilon) = x_0 + \varepsilon. \end{cases}$$

From the above inequality it follows that

$$(6.4) \quad \begin{cases} \left[ \frac{r(t, \varepsilon)}{f(t, r(t, \varepsilon))} \right]^\Delta > g(t, r(t, \varepsilon)), & \text{a.e. } t \in J, \\ r(t_0, \varepsilon) > x_0. \end{cases}$$

Now we apply Theorem 4.2 to the inequalities (6.1) and (6.4) and conclude that  $u(t) < r(t, \varepsilon)$  for all  $t \in J$ . This further in view of limit (6.3) implies that inequality (6.2) holds on  $J$ . This completes the proof.  $\square$

**Theorem 6.2.** *Assume that hypotheses (A<sub>0</sub>)–(A<sub>2</sub>) and condition (3.4) hold. Suppose that there exists a  $\Delta$ -differentiable function  $v$  such that*

$$\begin{cases} \left[ \frac{v(t)}{f(t, v(t))} \right]^\Delta \geq g(t, v(t)), & \text{a.e. } t \in J, \\ v(t_0) \geq x_0. \end{cases}$$

Then

$$\rho(t) \leq v(t)$$

for all  $t \in J$ , where  $\rho$  is a minimal solution of the HDE (2.1) on  $J$ .

*Proof.* The proof can be given in a similar way as that of Theorem 6.1. So we omit it here.  $\square$

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for the HDE (2.1) on  $J$ . The following theorem is a result in this direction.

**Theorem 6.3.** *Assume that hypotheses (A<sub>0</sub>)–(A<sub>2</sub>) and condition (3.4) hold. Suppose that there exists a function  $G : J \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that*

$$g(t, x_1) - g(t, x_2) \leq G \left( t, \frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right), \quad \text{a.e. } t \in J$$

for all  $x_1, x_2 \in \mathbb{R}$  with  $x_1 \geq x_2$ . If identically zero function is the only solution of the differential equation

$$(6.5) \quad m^\Delta(t) = G(t, m(t)) \quad \text{a.e. } t \in J, \quad m(t_0) = 0,$$

then the HDE (2.1) has a unique solution on  $J$ .

*Proof.* By Theorem 3.3, the HDE (2.1) has a solution defined on  $J$ . Suppose that there are two solutions  $u_1$  and  $u_2$  of the HDE (2.1) existing on  $J$  with  $u_1 > u_2$ . Define a function  $m : J \rightarrow \mathbb{R}^+$  by

$$m(t) = \frac{u_1(t)}{f(t, u_1(t))} - \frac{u_2(t)}{f(t, u_2(t))}.$$

In view of hypothesis (A<sub>0</sub>), we conclude that  $m(t) > 0$ . Then we have

$$\begin{aligned} m^\Delta(t) &= \left[ \frac{u_1(t)}{f(t, u_1(t))} \right]^\Delta - \left[ \frac{u_2(t)}{f(t, u_2(t))} \right]^\Delta \\ &= g(t, u_1) - g(t, u_2) \\ &\leq G \left( t, \frac{u_1}{f(t, u_1)} - \frac{u_2}{f(t, u_2)} \right) \\ &= G(t, m(t)) \end{aligned}$$

for almost everywhere  $t \in J$ , and that  $m(t_0) = 0$ .

Now, we apply Theorem 6.1 with  $f(t, x) \equiv 1$  to get that  $m(t) \leq 0$  for all  $t \in J$ , where identically zero function is the only solution of the differential equation (6.5). However,  $m(t) \leq 0$  contradicts  $m(t) > 0$ . Then we can get  $u_1 = u_2$ . This completes the proof.  $\square$

## 7. Existence of extremal solutions in vector segment

Sometimes it is desirable to have knowledge of the existence of extremal positive solutions for the HDE (2.1) on  $J$ . In this section, we shall prove the existence of maximal and minimal positive solutions for the HDE (2.1) on time scales between the given upper and lower solutions on  $J = [t_0, t_0 + a]_{\mathbb{T}}$ . We use a hybrid fixed point theorem of Dhage [5] in ordered Banach spaces for establishing our results. We need the following preliminaries in what follows.

A nonempty closed set  $K$  in a Banach algebra  $X$  is called a cone with vertex 0, if

- (i)  $K + K \subseteq K$ ,
- (ii)  $\lambda K \subseteq K$  for  $\lambda \in \mathbb{R}$ ,  $\lambda \geq 0$ ,
- (iii)  $(-K) \cap K = 0$ , where 0 is the zero element of  $X$ ,
- (iv) A cone  $K$  is called positive if  $K \circ K \subseteq K$ , where  $\circ$  is the multiplication composition in  $X$ .

We introduce an order relation  $\leq$  in  $X$  as follows. Let  $x, y \in X$ . Then  $x \leq y$  if and only if  $y - x \in K$ . A cone  $K$  is called to be normal if the norm  $\|\cdot\|$  is semi-monotone increasing on  $K$ , that is, there is a constant  $N > 0$  such that  $\|x\| \leq N\|y\|$  for all  $x, y \in K$  with  $x \leq y$ . It is known that if the cone  $K$  is normal in  $X$ , then every order-bounded set in  $X$  is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [11].

**Lemma 7.1** ([5]). *Let  $K$  be a positive cone in a real Banach algebra  $X$  and let  $u_1, u_2, v_1, v_2 \in K$  be such that  $u_1 \leq v_1$  and  $u_2 \leq v_2$ . Then  $u_1 u_2 \leq v_1 v_2$ .*

For any  $a, b \in X$ , the order interval  $[a, b]$  is a set in  $X$  given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

**Definition 7.2.** A mapping  $Q : [a, b] \rightarrow X$  is said to be nondecreasing or monotone increasing if  $x \leq y$  implies  $Qx \leq Qy$  for all  $x, y \in [a, b]$ .



We use the following fixed point theorems of Dhage [4] for proving the existence of extremal solutions for the IVP (2.1) under certain monotonicity conditions.

**Lemma 7.3** ([4]). *Let  $K$  be a cone in a Banach algebra  $X$  and let  $a, b \in X$  be such that  $a \leq b$ . Suppose that  $A, B : [a, b] \rightarrow K$  are two nondecreasing operators such that*

- (a)  *$A$  is Lipschitzian with a Lipschitz constant  $\alpha$ ,*
- (b)  *$B$  is completely continuous,*
- (c)  *$AxBx \in [a, b]$  for each  $x \in [a, b]$ .*

*Further, if the cone  $K$  is positive and normal, then the operator equation  $AxBx = x$  has a least and a greatest positive solution in  $[a, b]$ , whenever  $\alpha M < 1$ , where  $M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\}$ .*

We equip the space  $C_{rd}(J, \mathbb{R})$  with the order relation  $\leq$  with the help of cone  $K$  defined by

$$(7.1) \quad K = \{x \in C_{rd}(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}.$$

It is well known that the cone  $K$  is positive and normal in  $C_{rd}(J, \mathbb{R})$ . We need the following definitions in what follows.

**Definition 7.4.** A  $\Delta$ -differentiable function  $a$  is called a lower solution of the HDE (2.1) defined on  $J$  if it satisfies (4.3). Similarly, a  $\Delta$ -differentiable function  $b$  is called an upper solution of the HDE (2.1) defined on  $J$  if it satisfies (4.4). A solution to the HDE (2.1) is a lower as well as an upper solution for the HDE (2.1) defined on  $J$  and vice versa.

We consider the following set of assumptions:

- (B<sub>0</sub>)  $f : J \times \mathbb{R} \rightarrow \mathbb{R}^+ - \{0\}$ ,  $g : J \times \mathbb{R} \rightarrow \mathbb{R}^+$ .
- (B<sub>1</sub>) The HDE (2.1) has a lower solution  $a$  and an upper solution  $b$  defined on  $J$  with  $a \leq b$ .
- (B<sub>2</sub>) The function  $x \mapsto \frac{x}{f(t, x)}$  is increasing in the interval  $[\min_{t \in J} a(t), \max_{t \in J} b(t)]$  almost everywhere for  $t \in J$ .
- (B<sub>3</sub>) The functions  $f(t, x)$  and  $g(t, x)$  are nondecreasing in  $x$  almost everywhere for  $t \in J$ .
- (B<sub>4</sub>) There exists a function  $k \in L^1(J, \mathbb{R})$  such that  $g(t, b(t)) \leq k(t)$ .

We remark that hypothesis (B<sub>4</sub>) holds in particular if  $f$  is continuous and  $g$  is  $L^1$  on  $J \times \mathbb{R}$ .

**Theorem 7.5.** *Suppose that assumptions (A<sub>1</sub>) and (B<sub>0</sub>)–(B<sub>4</sub>) hold. Furthermore, if*

$$(7.2) \quad L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|k\|_{L^1} \right) < 1,$$

then the HDE (2.1) has a minimal and a maximal positive solution defined on  $J$ .

*Proof.* Now, the HDE (2.1) is equivalent to integral equation (3.6) defined on  $J$ . Let  $X = C_{rd}(J, \mathbb{R})$ . Define two operators  $A$  and  $B$  on  $X$  by (3.7) and (3.8), respectively. Then the integral equation (3.6) is transformed into an operator equation  $Ax(t)Bx(t) = x(t)$  in the Banach algebra  $X$ . Notice that hypothesis (B<sub>0</sub>) implies  $A, B : [a, b] \rightarrow K$ . Since the cone  $K$  in  $X$  is normal,  $[a, b]$  is a norm bounded set in  $X$ . Now it is shown, as in the proof of Theorem 3.3, that  $A$  is Lipschitzian with the Lipschitz constant  $L$  and  $B$  is a completely continuous operator on  $[a, b]$ . Again, hypothesis (B<sub>3</sub>) implies that  $A$  and  $B$  are nondecreasing on  $[a, b]$ . To see this, let  $x, y \in [a, b]$  be such that  $x \leq y$ . Then, by hypothesis (B<sub>3</sub>),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all  $t \in J$ . Similarly, we have

$$\begin{aligned} Bx(t) &= \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \\ &\leq \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, y(s)) \Delta s \\ &= By(t) \end{aligned}$$

for all  $t \in J$ . So  $A$  and  $B$  are nondecreasing operators on  $[a, b]$ . Lemma 7.1 and hypothesis (B<sub>3</sub>) together imply that

$$\begin{aligned} a(t) &\leq [f(t, a(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \right) \\ &\leq [f(t, x(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \right) \\ &\leq [f(t, b(t))] \left( \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^t g(s, x(s)) \Delta s \right) \\ &\leq b(t) \end{aligned}$$

for all  $t \in J$  and  $x \in [a, b]$ . As a result  $a(t) \leq Ax(t)Bx(t) \leq b(t)$ , for all  $t \in J$  and  $x \in [a, b]$ . Hence,  $AxBx \in [a, b]$  for all  $x \in [a, b]$ . Again,

$$M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\} \leq \left| \frac{x_0}{f(t_0, x_0)} \right| + \|k\|_{L^1}$$

and so,

$$\alpha M \leq L \left( \left| \frac{x_0}{f(t_0, x_0)} \right| + \|k\|_{L^1} \right) < 1.$$

Now, we apply Lemma 7.3 to the operator equation  $AxBx = x$  to obtain that the HDE (2.1) has a minimal and a maximal positive solution in  $[a, b]$  defined on  $J$ . This completes the proof.  $\square$

*Remark 7.6.* The main results in this paper extend and improve some known results in [7].

## 8. Conclusion

In this paper, we develop the theory of hybrid differential equations on time scales. By the fixed point theorem in Banach algebra due to Dhage, an existence theorem for hybrid differential equations (2.1) on time scales is given under Lipschitz conditions. We give a fundamental result relative to strict inequalities for hybrid differential equations (2.1) by the properties of  $\Delta$  derivative on time scales. The existence results of maximal and minimal solutions for hybrid differential equations on time scales are given by fixed point theorem in Banach algebra due to Dhage. We prove the comparison principle for hybrid differential equations (2.1) by fundamental differential inequalities on time scales. The existence results of extremal positive solutions for hybrid differential equations on time scales are also established. Our results in this paper extend and improve some known results.

## Acknowledgments

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

This research is supported by the Natural Science Foundation of China (61374074, 11571202, 61374002), and supported by Shandong Provincial Natural Science Foundation (ZR2013AL003).

## REFERENCES

- [1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser Basel, Boston, 2001.
- [2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Basel, Boston, 2003.
- [3] B.C. Dhage, On  $\alpha$ -condensing mappings in Banach algebras, *Math. Student* **63** (1994), no. 1-4, 146–152.
- [4] B.C. Dhage, Fixed point theorems in ordered Banach algebras and applications, *PanAmer. Math. J.* **9** (1999), no. 4, 83–102.
- [5] B.C. Dhage, A nonlinear alternative in Banach algebras with applications to functional differential equations, *Nonlinear Funct. Anal. Appl.* **9** (2004), no. 4, 563–575.
- [6] B.C. Dhage, On a fixed point theorem in Banach algebras with applications, *Appl. Math. Lett.* **18** (2005), no. 3, 273–280.
- [7] B.C. Dhage and V. Lakshmikantham, Basic results on hybrid differential equations, *Nonlinear Anal. Hybrid Syst.* **4** (2010), no. 3, 414–424.

- [8] A. Dogan, Existence of multiple positive solutions for  $p$ -Laplacian multipoint boundary value problems on time scales, *Adv. Difference Equ.* **2013** (2013), no. 238, 23 pages.
- [9] A. Dogan, Existence of positive solutions for  $p$ -Laplacian an  $m$ -point boundary value problem involving the derivative on time scales, *Electron. J. Differential Equations* **2014** (2014), no. 37, 10 pages.
- [10] A. Dogan, Triple positive solutions for  $m$ -point boundary-value problems of dynamic equations on time scales with  $p$ -Laplacian, *Electron. J. Differential Equations* **2015** (2015), no. 131, 12 pages.
- [11] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinues Differential Equations*, Marcel Dekker Inc. New York, 1994.
- [12] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56.
- [13] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Academic Press, New York, 1969.
- [14] H. Li, Z. Han and Y. Wang, Nonoscillatory solutions for super-linear Emden–Fowler type dynamic equations on time scales, *Electron. J. Qual. Theory Differ. Equ.* **2015** (2015), no. 53, 13 pages.
- [15] Y. Zhao, S. Sun, Z. Han and Q. Li, Theory of fractional hybrid differential equations, *Comput. Math. Appl.* **62** (2011), no. 3, 1312–1324.

(Shurong Sun) SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF JINAN, JINAN, SHANDONG 250022, P.R. CHINA.

*E-mail address:* [sshrong@163.com](mailto:sshrong@163.com)

(Yige Zhao) SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF JINAN, JINAN, SHANDONG 250022, P.R. CHINA.

*E-mail address:* [zhaoeager@126.com](mailto:zhaoeager@126.com)

(Ping Zhao) SCHOOL OF ELECTRICAL ENGINEERING, UNIVERSITY OF JINAN, JINAN, SHANDONG 250022, P.R. CHINA.

*E-mail address:* [cse.zhaop@ujn.edu.cn](mailto:cse.zhaop@ujn.edu.cn)

(Zhenlai Han) SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF JINAN, JINAN, SHANDONG 250022, P.R. CHINA.

*E-mail address:* [hanzhenlai@163.com](mailto:hanzhenlai@163.com)