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On a class of locally projectively flat Finsler metrics

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ON A CLASS OF LOCALLY PROJECTIVELY FLAT FINSLER METRICS

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ABSTRACT. In this paper we study Finsler metrics with orthogonal invariance. We find a partial differential equation equivalent to these metrics being locally projectively flat. Some applications are given. In particular, we give an explicit construction of a *new* locally projectively flat Finsler metric of vanishing flag curvature which differs from the Finsler metric given by Berwald in 1929.
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1. Introduction

A Finsler metric F on a manifold M is said to be *locally projectively flat* if at any point, there is a local coordinate system (x^i) in which the geodesics are straight lines as point sets, equivalently, F is pointwise projectively related to a locally Minkowskian metric. Why are we interested in this type of Finsler metric? Riemannian metrics of constant (sectional) curvature are locally projectively flat. The converse is also true according to Beltrami's theorem. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solution to Hilbert's fourth problem [4].

Recently, many Finslerian geometers have made efforts in the characterization and construction of locally projectively flat Finsler metrics [2,11,13,17,20]. Chern-Shen showed that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed [2]. In 2007, Z. Shen investigated the necessary and sufficient condition for (α, β)-metrics to be projectively flat apart from some special cases [8,18]. Very recently, H. Zhu has discussed a certain class of Finsler metrics with orthogonal invariance. She showed that these metrics are locally projectively flat if and only if they are

of scalar curvature [20]. It is known that every locally projectively flat Finsler metric is of scalar curvature. The converse may not be true. For example, Bao-Shen's examples of Randers metrics with constant flag curvature 1 are non-locally projectively flat [1].

Recall that a Finsler metric on $\mathbb{B}^n(\nu)$ is said to be *orthogonally invariant* (*spherically symmetric* in an alternative terminology in [5,12]) if it satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $x \in \mathbb{B}^n(\nu)$, $y \in T_x\mathbb{B}^n(\nu)$ and $A \in O(n)$. Finsler metrics with orthogonal invariance form a rich class of Finsler metrics. Many classical Finsler metrics with nice curvature properties are orthogonally invariant, such as the Bryant metric, the metric introduced by Berwald and the Chern-Shen's metric [2]. In [6], Huang-Mo showed the following: A Finsler metric F on $\mathbb{B}^n(\nu)$ is of orthogonal invariance if and only if there is a function $\phi : [0, \nu) \times \mathbb{R} \to \mathbb{R}$ such that $F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y \rangle}{|y|}\right)$ where $(x, y) \in T\mathbb{B}^n(\nu) \setminus \{0\}$.

In this paper, we are going to study the locally projective flatness of Finsler metrics of orthogonal invariance without additional condition. We find a partial differential equation equivalent to a class of Finsler metrics being locally projectively flat. More precisely, we prove the following:

Theorem 1.1. *Let* $F(x, y) = |y|\phi(r, s)$ *be a spherically symmetric Finsler metric on* $\mathbb{B}^n(\nu)\,(n > 2)$, *where* $r := |x|$, *and* $s := \frac{\langle x, y \rangle}{|y|}$. *Then* F *is locally projectively flat if and only if* $\phi = \phi(r, s)$ *satisfies*

$$(1.1) \qquad \left[(r^2 - s^2)Q - 1\right] r\phi_{ss} - s\phi_{rs} + \phi_r + rQ(\phi - s\phi_s) = 0,$$

where $Q = Q(r, s)$ *is given by*

$$(1.2) \qquad Q(r, s) = f(r) + \frac{2rf(r)^2 + f'(r)}{r + 2f(r)r^3}s^2,$$

where $f = f(r)$ *is a differential function.*

Let us take a look at the special case: when $f = 0$. Then we get the partial differential equation (2.3) equivalent to F being projectively flat [5,10].

As we mentioned above, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However the situation is much more complicated for Finsler metrics. In fact, there are lots of locally projectively flat Finsler metrics which are not of constant flag curvature [13]. Conversely, there are infinitely many non-locally projectively flat Finsler metrics with constant flag curvature [1]. Therefore it is an interesting problem to characterize and construct locally projectively flat Finsler metrics with constant flag curvature. As an application of Theorem 1.1, we find two equations that characterize locally projectively flat spherically symmetric metrics with

constant flag curvature. (See Theorem 2.3 below.) Then we construct explicitly locally projectively flat spherically symmetric metrics with constant flag curvature using Theorem 1.1.

Theorem 1.2. *The following spherically symmetric Finsler metrics are locally projectively flat*

$$(1.3) \qquad F(x,\,y) = \frac{\left[\xi(1 + 4|x|^2) \pm 2\langle x,\,y\rangle\right]^2}{\xi(1 + 4|x|^2)},$$

where ξ is given by

$$(1.4) \qquad \xi(x,\,y) := \sqrt{\frac{(1 + 4|x|^2)|y|^2 - 4\langle x,\,y\rangle^2}{1 + 4|x|^2}}.$$

Moreover, F is of constant flag curvature $K = 0$.

The flag curvature in Finsler geometry is the most important Riemannian quantity and it is an analogue of sectional curvature in the Riemannian geometry [2].

Let $F(x,\,y) = |y|\phi\left(|x|,\,\frac{\langle x,y\rangle}{|y|}\right)$ be a Finsler metric of constant flag curvature K. When $K \neq 0$, one can solve ϕ by using (2.4b). However, the situation is much more complicated for F to be of vanishing constant flag curvature. In the proof of Theorem 1.2 we give a *new* approach to construct explicitly spherically symmetric Finsler metrics with zero flag curvature. (See Section 4 below.) In [7, Theorem 4], the author claims that on a convex domain $\mathcal{U} \subset \mathbb{R}^n$, a spherically symmetric Finsler metric F is locally projectively flat with constant flag curvature $K = 0$ if and only if F is given by

$$(1.5) \qquad F = \frac{|y|^4}{\Phi(\langle x,\,y\rangle \pm \Phi)^2},$$

where $\Phi = \sqrt{(1 - c^2|x|^2)|y|^2 + c^2\langle x,\,y\rangle^2}$, and c is a nonzero constant. Actually, we can prove (1.3) is also a locally projectively flat spherically symmetric Finsler metric with vanishing flag curvature which differs from the Finsler metric (1.5).

In Theorem 4.2, we explicitly construct news locally projectively flat Finsler metric of orthogonal invariance. These metrics are of constant flag curvature $K = -1$.

2. **Locally projectively flat Finsler metrics**

In this section we are going to characterize locally projectively flat Finsler metrics with orthogonal invariance.

Theorem 2.1. *Let $F(x,\,y) = |y|\phi\,(r,\,s)$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\nu)\,(n > 2)$, where $r := |x|$, and $s := \frac{\langle x,y\rangle}{|y|}$. Then the following assertions are equivalent:*

(i) F *is locally projectively flat;*

(ii) $\phi = \phi(r,\,s)$ *satisfies* (1.1) *where* $Q = Q(r,\,s)$ *is given in* (1.2);

(iii) $\phi = \phi(r,\,s)$ *satisfies* (1.1) *where* $Q = Q(r,\,s)$ *is a polynomial in* s *and* F *is of scalar curvature.*

Proof. According to Douglas' result, the Finsler metric $F(x,\,y)$ on $\mathbb{B}^n(\nu)$ $(n > 2)$ is locally projectively flat if and only if F has vanishing Weyl curvature and Douglas curvature [3]. On the other hand, note that $F = |y|\phi\left(|x|,\,\frac{\langle x,\,y\rangle}{|y|}\right)$ is spherically symmetric, we have the following:

(1) F has vanishing Douglas curvature if and only if $\phi = \phi(r,\,s)$ satisfies (1.1), where by [12]

(2.1) $$Q(r,\,s) = f(r) + g(r)s^2.$$

(2) If $\phi = \phi(r,\,s)$ satisfies (1.1) where $Q = Q(r,\,s)$ is a polynomial in s. Then F has vanishing Weyl curvature if and only if (2.1) holds and by [9, Propositions 3.2 and 4.1]

(2.2) $$2f^2 + \frac{1}{r}f' - g + 2r^2 fg = 0.$$

Now our theorem is an immediate conclusion of (1), (2) and Douglas' result. Recall that a Finsler metric has vanishing Weyl curvature if and only if it is of scalar curvature [10]. $\square$

As a consequence of Theorem 2.1, by taking $f = 0$ in (1.2), we obtain the following result obtained by Huang and the first author. (See [6, Theorem 1.1].)

Corollary 2.2. *Let* $F = |y|\phi\left(|x|,\,\frac{\langle x,\,y\rangle}{|y|}\right)$ *be a spherically symmetric Finsler metric on* $\mathbb{B}^n(\nu)$. *Then* $F = F(x,\,y)$ *is projectively flat if and only if* $\phi = \phi(r,\,s)$ *satisfies*

(2.3) $$s\phi_{rs} + r\phi_{ss} - \phi_r = 0.$$

Theorem 2.1 also generalizes a result previously only known in the case of Q being a polynomial in s [20].

Proof of Theorem 1.1. This is an immediate consequence of Theorem 2.1. $\square$

Combine [14, Theorem 2.3] with Theorem 1.1, we have

Theorem 2.3. *Let* $F(x,\,y) = |y|\phi(r,\,s)$ *be a spherically symmetric Finsler metric on* $\mathbb{B}^n(\nu)$ *where* $n \geq 3$; $r := |x|$ *and* $s := \frac{\langle x,\,y\rangle}{|y|}$. *Then* F *is locally projectively flat with constant flag curvature* K *if and only if*

(2.4a) $$r\phi_{ss} + s\phi_{rs} - \phi_r = 2r\left[f(r) + g(r)s^2\right]\left[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}\right],$$

$$(2.4b) \qquad P^2 - \frac{1}{r}(sP_r + rP_s) + 2Q[1 + sP + (r^2 - s^2)P_s] = K\phi^2$$

where P and Q are defined in (3.2) and (3.1) respectively; f and g satisfy (2.2).

3. Some lemmas

In this section we are going to recall and establish the lemmas required in the proof of Theorem 1.2.

Lemma 3.1 ([12])**.** *Let $F = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\nu) \subset \mathbb{R}^n$. Let $x^1, \ldots, x^n$ be coordinates on $\mathbb{R}^n$ and let $y = \sum y^i \partial/\partial x^i$. Then its geodesic coefficients are given by*

$$G^i = |y|Py^i + |y|^2 Qx^i$$

where

$$(3.1) \qquad Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \qquad r := |x|, \quad \phi = \phi(r,\,s)$$

and

$$(3.2) \qquad P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}\left[s\phi + (r^2 - s^2)\phi_s\right].$$

Lemma 3.2 ([14])**.** *Let $P(r,s)$ and $Q(r,s)$ be differentiable functions. If there exists a differentiable function $\phi = \phi(r,s)$ such that (3.1) and (3.2) hold, then*

$$(3.3) \qquad\qquad\qquad A + B\Phi = 0,$$

where

$$\begin{aligned} A :=& s[2P_r - sP_{rs} + s(2Q_r - sQ_{rs})] - rsP_{ss} + 2rsQP_{ss}\Psi \\ & - 2rQ_s(s + r^2 P) + rs(Q_s - sQ_{ss})(1 - 2Q\Psi) + 2rP(P - sP_s), \\ B :=& 4rsQ(1 + sP) - s[sP_r - (2Q_r - sQ_{rs})\Psi] \\ & + r(1 - 2Q\Psi)[(Q_s - sQ_{ss})\Psi - (P + sP_s)], \\ & \Phi := \frac{2P - sP_s + s(2Q - sQ_s)}{1 - \Psi(2Q - sQ_s) + sP}, \qquad \Psi := r^2 - s^2. \end{aligned}$$

Lemma 3.3. *Let $P(r,s)$ and $Q(r,s)$ be differentiable functions. If there exists a differentiable function $\phi = \phi(r,s)$ such that (3.1) and (3.2) hold, then*

$$(3.4a) \qquad\qquad (\log \phi)_s = \frac{2P - sP_s + s(2Q - sQ_s)}{E},$$

$$(3.4b) \qquad (\log \phi)_r = \frac{r}{E}\left[P_s + 2P^2 + 2sPQ + sQ_s + 2\Psi(PQ_s - QP_s)\right]$$

where

$$(3.5) \qquad\qquad E := 1 - \Psi(2Q - sQ_s) + sP.$$

Proof. The formula (3.4a) is immediate from [14, Lemma 6.2]. Now we are going to show (3.4b). By using (3.4a), we have

$$(3.6) \qquad U := s + \Psi(\log \phi)_s = \frac{s + r^2 P + \Psi(P - sP_s)}{E}.$$

Define W by

$$(3.7) \qquad W := \frac{s\phi_r + r\phi_s}{\phi}.$$

Combining this with (3.1), (3.2) and (3.6), we have

$$W = 2r(P + QU) = 2r\left[P + Q\frac{s + r^2 P + \Psi(P - sP_s)}{E}\right].$$

It follows that

$$(\log \phi)_r = \frac{W - r(\log \phi)_s}{s}$$
$$= \frac{2r}{s}\left[P + Q\frac{s + r^2 P + \Psi(P - sP_s)}{E}\right] - \frac{r}{s}\frac{2P - sP_s + s(2Q - sQ_s)}{E}$$
$$(3.8) \qquad = \frac{r}{sE}(I),$$

where we have made use of (3.4a) and (3.7), and

$$(I) := 2PE + 2Q\left[s + r^2 P + \Psi(P - sP_s)\right] - [2P - sP_s + s(2Q - sQ_s)]$$
$$(3.9) \qquad = s\left[P_s + 2P^2 + 2sPQ + sQ_s + \Psi(PQ_s - QP_s)\right].$$

Plugging (3.9) into (3.8) yields (3.4b). □

4. **Proof of Theorem 1.2**

In order to show Theorem 1.2, we need the following characterization of spherically symmetric Finsler metrics of constant flag curvature.

Theorem 4.1 ([14]). *Let* $F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y \rangle}{|y|}\right)$ *be a spherically symmetric Finsler metric on* $\mathbb{B}^n(\nu)$*. Then* F *is of isotropic flag curvature (or constant flag curvature when* $n \geq 3$*) if and only if*

$$(4.1a) \quad 2Q(2Q - sQ_s) + \frac{1}{r}(2Q_r - sQ_{rs} - rQ_{ss}) + (r^2 - s^2)(2QQ_{ss} - Q_s^2) = 0,$$

$$(4.1b) \qquad \frac{1}{r}P_r - \frac{s}{r}P_{rs} - P_{ss} + 2Q(P - sP_s) + 2(r^2 - s^2)QP_{ss} = 0,$$

where P *and* Q *are given in* (3.1) *and* (3.2)*. In this case, the flag curvature* K *of* F *satisfies* (2.4b)*.*

Let us consider the Douglas spherically symmetric metric $F(x, y) = |y|\phi(|x|,$
$\frac{\langle x, y\rangle}{|y|})$. Then $Q = a(r) + b(r)s^2$ [12]. Assume that F is of scalar curvature.
We have [9]

$$b(r) = \frac{2ra(r)^2 + a'(r)}{r - 2a(r)r^3}.$$

Take a look at the special case: when $a(r) = -2$,

(4.2) $Q(r, s) = -2 + \frac{8s^2}{1 + 4r^2}.$

We put

(4.3) $T := P - sP_s.$

Then (4.1b) is equivalent to

(4.4) $\frac{1}{2r}T_r + \frac{1}{s}\left(\frac{1}{2} - \Psi Q\right)T_s = -QT.$

The characteristic equation of the quasi-linear PDE (4.4) is

(4.5) $\frac{dr}{\frac{1}{2r}} = \frac{ds}{\frac{1}{s}\left(\frac{1}{2} - \Psi Q\right)} = \frac{dT}{-QT}.$

It follows that

$$\frac{(1 + 4r^2)\Psi}{1 + 4\Psi} = c_1, \qquad \frac{T}{\sqrt{\frac{1 + 4r^2}{1 + 4\Psi}}} = c_2$$

are independent integrals of (4.5). Hence the solution of (4.4) is

$$T = f\left(\frac{(1 + 4r^2)\Psi}{1 + 4\Psi}\right)\sqrt{\frac{1 + 4r^2}{1 + 4\Psi}},$$

where f is any continuously differentiable function. Let us consider the special
solution of (4.4) in the form $T = c\sqrt{\frac{1+4r^2}{1+4\Psi}}$ where c is a constant. In this case

(4.6) $P = g(r)s + c\rho, \quad P_s = g(r) + c\rho_s$

where we have used (4.3) and

(4.7) $\rho := \sqrt{\frac{1 + 4\Psi}{1 + 4r^2}}.$

Now we determine $g(r)$ and c in (4.6) using our necessary condition (3.3). By
direct calculations one obtains

(4.8) $\rho_s = -\frac{4s}{\rho\mu}, \quad \rho_r = \frac{16rs^2}{\rho\mu^2}, \quad \mu := 1 + 4r^2.$

Thus

$$(4.9a) \qquad P_r = g's + \frac{16crs^2}{\rho\mu^2}, \quad P - sP_s = \frac{c}{\rho},$$

$$(4.9b) \qquad 2P - sP_s = gs + \frac{2c(\mu - 2s^2)}{\rho\mu}, \quad sP_{ss} = -\frac{4cs}{\rho^3\mu},$$

$$(4.9c) \qquad P + sP_s = 2gs + \frac{c(\mu - 8s^2)}{\rho\mu}, \quad 2P_r - sP_{rs} = g's - \frac{64cr}{(\rho\mu)^3}s^4.$$

Using (4.2), we obtain

$$(4.10a) \qquad Q_s = \frac{16s}{\mu}, \quad Q_{ss} = \frac{16}{\mu}, \quad Q_s - sQ_{ss} = 0,$$

$$(4.10b) \qquad 2Q - sQ_s = -4, \quad 2Q_r - sQ_{rs} = 0.$$

Using (4.9) and (4.10), we compute the terms in (3.3) as follows.

$$(4.11) \qquad A = g's^2 - \frac{32r}{\mu}s^2 - \frac{32r^2s^2}{\mu}g + 2g\frac{crs}{\rho} + 2rc^2 + (I),$$

where
(4.12)
$$(I) := \frac{4crs}{\mu\rho^3} + \frac{16crs}{\mu\rho^3}\Psi - \frac{64cr^3s^3}{\mu^2\rho^3} - \frac{32cr^3s}{\mu}\rho = \frac{4crs}{\mu^2\rho}\left[\mu(1 - 4r^2) + 16r^2s^2\right].$$

Plugging (4.12) into (4.11) yields
(4.13)
$$A = \frac{2crs}{\mu^2\rho}\left[2\mu(1 - 4r^2) + \mu^2 g + 32r^2s^2\right] + 2rc^2 + \left[g' - \frac{32r}{\mu}(1 + r^2 g)\right]s^2.$$

By using (4.10), we have

$$(4.14) \qquad B = 4rsQ(1 + sP) - s^2P_r - r(1 - 2Q\Psi)(P + sP_s) = (II) + (III),$$

where
(4.15a)
$$(II) := 4rsQ(1 + gs^2) - g's^3 - 2rg(1 - 2Q\Psi)s = 4rsQ - 2r\mu gs + \frac{32r^3}{\mu}gs^3 - g's^3,$$

(4.15b)
$$(III) := 4crs^2\rho Q - \frac{16cr}{\mu^2\rho}s^4 - cr(1 - 2Q\Psi)\frac{\mu - 8s^2}{\mu\rho} = -\frac{cr\rho}{\mu}(\mu^2 - 16r^2s^2).$$

Substituting (4.15) into (4.14), we have

$$(4.16) \qquad B = -\frac{cr\rho}{\mu}(\mu^2 - 16r^2s^2) - 2r(4 + \mu g)s + \left[\frac{32r}{\mu}(1 + r^2 g) - g'\right]s^3.$$

By using (4.6), (4.9b) and (4.10b), we obtain

$$\Phi = \frac{2P - sP_s - 4s}{1 + 4\Psi + sP}$$

$$(4.17) \qquad = \frac{gs + \frac{2c(\mu - 2s^2)}{\mu\rho} - 4s}{\mu - 4s^2 + s(gs + c\rho)} = \frac{\mu\rho(g-4)s + 2c(\mu - 2s^2)}{\mu^2\rho + \mu\rho(g-4)s^2 + c\mu\rho^2 s}.$$

By (4.13), (4.16) and (4.17), (3.3) holds if and only if

$$(4.18) \qquad \frac{c}{\mu\rho}X_2 s^2 + X_1 s + \frac{cr}{\rho}X_0 = 0,$$

where

$$(4.19a) \qquad\qquad\qquad X_0 := 2c^2 - 8 - 3\mu g,$$

$$(4.19b) \qquad\qquad\qquad X_1 := \mu g' + 4c^2 rg - 2r\mu g^2,$$

$$(4.19c) \qquad\qquad X_2 := 8r(4 - c^2) + 8r(1 + 6r^2)g + 2r\mu g^2 - \mu g'.$$

From this together with (4.7) and (4.18) we obtain the following

$$(4.20) \qquad\qquad X_j = X_j(r) = 0, \quad j = 0, 1, 2.$$

It follows that

$$0 = X_1 + X_2 = 4r\left[c^2 + 2(1 + 6r^2)\right]g + 8r(4 - c^2),$$

where we have used (4.19b) and (4.19c). Note that $r > 0$. Hence we have

$$(4.21) \qquad\qquad\qquad g = \frac{2(c^2 - 4)}{c^2 + 2(1 + 6r^2)}.$$

By (4.19a) and (4.20), we have $g = \frac{2(c^2 - 4)}{3\mu}$. Combine this with (4.21) we obtain

$$(4.22) \qquad\qquad\qquad c = \pm 2, \quad c = \pm 1.$$

Take $c = \pm 2$. Then $g = 0$. In this case, we have

$$P^2 - \frac{1}{r}(sP_r + rP_s) + 2Q[1 + sP + (r^2 - s^2)P_s]$$

$$= c^2 - 4 - \mu g + 2c\rho gs + \left[g^2 + \frac{16r^2}{\mu}g - \frac{1}{r}g' - \frac{4}{\mu}(c^2 - 4)\right] = 0,$$

where we have used (4.2), (4.6) and (2.4b). It is impossible to solve ϕ by using the equation (2.4b). Now we are going to solve ϕ by using another approach.

From (4.6), (4.9b) and (4.10b), we obtain

$$(4.23) \qquad P = 2\rho, \quad 2P - sP_s = \pm\frac{4(\mu - 2s^2)}{\mu\rho}, \quad 2Q - sQ_s = -4.$$

Substituting (4.23) into (3.4a) yields

$$
\begin{aligned}
(\log \phi)_s &= \frac{\pm 4(\mu - 2s^2) - 4s\mu\rho}{\mu\rho\left[1 + 4(r^2 - s^2) \pm 2s\rho\right]} \\
&= \frac{(-4s \pm 2\rho)\rho \pm 2}{\rho\left[\mu - 4s^2 \pm 2s\rho\right]} = \frac{-4s \pm 2\rho}{\rho(\mu\rho \pm 2s)} \pm \frac{2}{\rho^2(\mu\rho \pm 2s)},
\end{aligned}
$$
(4.24)

where we have used the fact $\rho^2\mu = \mu - 4s^2$. It follows that

$$
\begin{aligned}
\phi &= e^{\log \phi} \\
&= t_1(r)e^{2\int \frac{(-2s \pm \rho)\rho \pm 1}{\rho^2(\mu\rho \pm 2s)}\, ds} \\
&= t_2(r)\frac{(\mu\rho \pm 2s)^2}{\rho} = t(r)\frac{(\mu\rho \pm 2s)^2}{\mu^2\rho}.
\end{aligned}
$$
(4.25)

In particular,

$$
\log \phi(r,\, 0) = \log \left[t(r)\rho(r,\, 0)\right] = \log t(r)
$$
(4.26)

where we have made use of (4.7). Using (4.2), (4.6), (4.10a) and (3.5), we have

$$
Q = -2\rho^2, \quad P_s = \mp \frac{8s}{\mu\rho}, \quad Q_s = \frac{16s}{\mu}, \quad E = \rho(\rho\mu \pm 2s).
$$
(4.27)

Together with the first equation of (4.23) we have the following:
(4.28)
$$
P_s + 2P^2 + 2sPQ + sQ_s + 2\Psi(PQ_s - QP_s) = \mp\frac{8s}{\mu\rho} + 8\rho^2 \mp 8s\rho^3 + \frac{16}{\mu}s^2 \pm 2\Psi\frac{16\rho}{\mu}s.
$$

Plugging (4.28) and the last equation of (4.27) into (3.4b) yields

$$
(\log \phi)_r = \frac{2r}{\rho(\mu\rho \pm 2s)}\left(\mp\frac{4s}{\mu\rho} + 4\rho^2 \mp 4s\rho^3 + \frac{8}{\mu}s^2 \pm \Psi\frac{16\rho}{\mu}s\right).
$$

Together with (4.26) we obtain

$$
(\log \mu)' = \frac{8r}{\mu} = (\log \phi)_r|_{s=0} = \left[\log \phi(r,\, 0)\right]' = \left[t(r)\right]'.
$$

It follows that $\log \frac{t(r)}{\mu} = \lambda = $ constant. We may assume that $\lambda = 0$. Hence we have

$$
t(r) = \mu = 1 + 4r^2.
$$

Plugging this into (4.25) yields $\phi(r,\, s) = \frac{(\mu\rho \pm 2s)^2}{\mu\rho}$. Define ξ by

$$
\xi(x,\, y) := |y|\rho\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right) = \sqrt{\frac{(1 + 4|x|^2)|y|^2 - 4\langle x,\, y\rangle^2}{1 + 4|x|^2}}.
$$
(4.29)

Then

$$
F(x,\, y) = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right) = \frac{\left[\xi(1 + 4|x|^2) \pm 2\langle x,\, y\rangle\right]^2}{\xi(1 + 4|x|^2)}.
$$
(4.30)

By straightforward calculations, we have

$$\frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + \Psi\phi_{ss}} = -2\frac{1 + 4\Psi}{1 + 4r^2},$$

$$\frac{r\phi_s + s\phi_r}{2r\phi} + \frac{\Psi}{r^3\phi}(s\phi + \Psi\phi_s) = \pm 2\sqrt{\frac{1 + 4\Psi}{1 + 4r^2}},$$

and

$$\phi - s\phi_s > 0, \quad \phi - s\phi_s + \Psi\phi_{ss} > 0,$$

that is, $\phi = \phi(r, s)$ satisfies (2.4b) (with $K = 0$), (4.1a), (4.1b), (3.1) and (3.2). By Theorem 4.1, (4.30) is of constant curvature $K = 0$. Furthermore, (4.2) tells us F is locally projectively flat. Thus we complete the proof of Theorem 1.2.

In (4.22), taking $c = \pm 1$, we have the following:

Theorem 4.2. *The following spherically symmetric Finsler metrics are locally projectively flat*

$$(4.31) \qquad F(x, y) = \sqrt{|y|^2 \pm \frac{4\langle x, y\rangle\xi}{1 + 4|x|^2} - \frac{16|x|^2\langle x, y\rangle^2}{(1 + 4|x|^2)^2}},$$

where ξ is given in (4.29). Moreover, F is of constant flag curvature $K = -1$.

In [7, Theorem 7.3], the author claims that on a convex domain $\mathcal{U} \subset \mathbb{R}^n$, a spherically symmetric Finsler metric F is locally projectively flat with constant flag curvature $K = -1$ if and only if F is given by

$$(4.32) \qquad F = \frac{1}{2}\left[\Theta_c(x, y) - \epsilon\Theta_c(\epsilon x, y)\right], \qquad \epsilon < 1$$

where

$$\Theta_c = \frac{\sqrt{c|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} \pm \langle x, y\rangle}{c - |x|^2}.$$

Actually, we have proved (4.31) is also a locally projectively flat spherically symmetric Finsler metric with constant flag curvature $K = -1$ which differs from the Finsler metric (4.32).

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