Title:
Frattini supplements and Frat series

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FRATTINI SUPPLEMENTS AND FRAT SERIES

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ABSTRACT. In this study, Frattini supplement subgroup and Frattini supplemented group are defined by Frattini subgroup. By these definitions, it’s shown that finite abelian groups are Frattini supplemented and every conjugate of a Frattini supplement of a subgroup is also a Frattini supplement. A group action of a group is defined over the set of Frattini supplements of a normal subgroup of the group by conjugation and in this study new characterization of primitivity of groups has obtained in terms of Frattini supplemented groups by this action. Moreover, Frat-series of a group is defined based on Frattini supplements of normal subgroups of the group and it is shown that subgroups and factor groups of groups with Frat-series also have Frat-series under some special conditions. Furthermore, we determined a characterization of soluble groups which have Frat-series.

Keywords: Frattini subgroup, primitive group, group actions.

1. Introduction

In module theory, Rad-supplemented modules were defined as proper generalizations of supplemented modules. Over a ring with identity, a unital module $M$ is called Rad-supplemented if every submodule $N$ of $M$ has Rad-supplement in $M$, i.e. $N + K = M$ and $N \cap K \leq Rad(K)$ for some submodule $K$ of $M$, where $Rad(K)$ is the intersection of all maximal submodules of $K$. Hausen studied supplemented and amply supplemented groups in terms of nilpotency by using Frattini subgroup in [3]. We investigated the properties of these groups in a similar way with [3].

2. Preliminaries

The Frattini subgroup of an arbitrary group $G$ is defined to be the intersection of all the maximal subgroups, with the stipulation that it will equal to $G$ if
$G$ has no maximal subgroup. This subgroup, which is evidently characteristic, is written as $\text{Frat}(G)$ [1].

The Frattini subgroup has the remarkable property that it is the set of all nongenerators of the group; here an element $g$ is called a nongenerator of $G$ if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$ when $X$ is a subset of $G$ [1].

A subgroup $H$ of a group $G$ is supplemented in $G$ if there is a subgroup $K$ of $G$ such that $G = HK$. If $H \cap K = \{1\}$ then $K$ is said to be a complement of $H$ in $G$ [1].

The derived subgroup of $G$ is defined as $[G, G] = \langle [a, b] | a, b \in G \rangle$ where $[a, b] = a^{-1}b^{-1}ab$, and is written as $G'$. This study is based on the following definition.

**Definition 2.1.** Let $G$ be a group and $N \trianglelefteq G$. The subgroup $S$ of $G$ is called a Frattini supplement of $N$ in $G$ if $G = NS$ and $N \cap S \leq \text{Frat}(S)$. Clearly $G$ is Frattini supplement of Frat($G$) in $G$. If every $N \trianglelefteq G$ has a Frattini supplement in $G$, then $G$ is said to be Frattini supplemented.

**Example 2.2.** Let $G$ be a group. If $G = \text{Frat}(G)$ then $G$ is Frattini supplemented.

**Example 2.3.** $G$ itself is Frattini supplement of $1_G$ since $G = 1_G G$ and $1_G \cap G \leq \text{Frat}(G)$. So the Frattini supplement of $1_G$ is $G$.

**Example 2.4.** Let $G$ be a group in which every subgroup is normal, and $N$ be a minimal normal subgroup of $G$. Then $G$ is a Frattini supplement of $N$.

**Example 2.5.** For the generalized quaternion group $Q_{2^n} = \langle x, y | x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \rangle \ (n \geq 3)$, $\langle y \rangle$ is a Frattini supplement of the normal subgroup $\langle x \rangle$ in $Q_{2^n}$.

**Example 2.6.** Let $G$ be a finite abelian group. Then $G$ is Frattini supplemented.

**Example 2.7.** Let $G = Q_4$ be the group of Hamilton quaternions. It is easy to see that $\langle i \rangle, \langle j \rangle$, and $\langle k \rangle$ are Frattini supplement of each other and $G$ is a Frattini supplement of $\{1, -1\}$ in $G$, since every subgroup is normal.

3. Frattini supplemented groups

**Proposition 3.1.** Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $S$ is a Frattini supplement of $N$ in $G$ then $S$ is a minimal supplement of $N$ in $G$.

**Proof.** Since $S$ is a Frattini supplement of $N$ in $G$, then $G = NS$ and $N \cap S \leq \text{Frat}(S)$. Let $K$ be a Frattini supplement of $N$ in $G$ and $K \leq S$. Hence $G = NK$. Since $N \cap S \leq S$, if we intersect with $S$, we get $S = S \cap G = S \cap NK = K(N \cap S) = \langle K, N \cap S \rangle$. Finally, we have $S = K$, since $N \cap S \leq \text{Frat}(S)$. □
**Proposition 3.2.** Let $G$ be a finite group. If $G$ is a Frattini supplement of $G'$ then $G$ is nilpotent.

*Proof.* Since $G$ is a Frattini supplement of $G'$, $G = G'G$ and $G' = G \cap G' \leq \text{Frat}(G)$, then $G$ is nilpotent according to [1].

**Theorem 3.3.** Let $G$ be a group, $H$ be a finite normal subgroup of $G$ and $G = HK$ for some $K \leq G$. If $K$ is minimal, then $K$ is a Frattini supplement of $H$ in $G$.

*Proof.* Assume that $K$ is not a Frattini supplement of $H$ in $G$. Then $H \cap K \not\leq \text{Frat}(K)$ and for a maximal subgroup $M$ of $K$, $H \cap K \not\leq M$. So we have $M < (H \cap K)M \leq K$ which implies that $K = (H \cap K)M = HM \cap K$, and so $K \leq HM$. Since $G = HK$, for every $g \in G$, we have $g \in HM$. Hence $G = HM$, which is a contradiction. Therefore, $K$ is a Frattini supplement of $H$ in $G$. □

Proposition 3.1 and Theorem 3.3 could be merged for finite groups.

**Corollary 3.4.** Let $G$ be a finite group, and $N \unlhd G$. Then $S$ is a Frattini supplement of $N$ in $G$ if and only if $S$ is a minimal supplement of $N$.

**Theorem 3.5.** Let $G$ be a finite group, $N \unlhd G$, $H$ be a Frattini supplement of $N$ in $G$ and $K \leq H$. Then for $K \unlhd G$, $H/K$ is a Frattini supplement of $NK/K$.

*Proof.* Since $H$ is a Frattini supplement of $N$, $G = NH$ and $H \cap N \leq \text{Frat}(H)$. It is easy to see that $G/K = NH/K = (NK/K)(H/K)$ and $(H/K) \cap (NK/K) \leq (H \cap N)K/K \leq (\text{Frat}(H))/K$ by modular law. Then we have $(\text{Frat}(H))/K \leq \text{Frat}(H)/K$ by [1], since $G$ is finite. □

**Theorem 3.6.** Let $G$ be a finite group, $N \unlhd G$, $S$ be a Frattini supplement of $N$ in $G$ and $H$ be a subgroup of $G$ such that $S \leq H$. Then $N \cap H$ has a Frattini supplement in $H$.

*Proof.* Obviously $N \cap H \leq H$. Since $S$ is a Frattini supplement of $N$ in $G$, $G = NS$ and $S \cap N \leq \text{Frat}(S)$. So we have $G = NS$, which implies that $H = G \cap H = (NS) \cap H = (N \cap H)S$ and $(H \cap N) \cap S = H \cap (N \cap S) \leq H \cap \text{Frat}(S) \leq \text{Frat}(S)$. Hence $N \cap H$ has a Frattini supplement in $H$. □

**Theorem 3.7.** Let $G$ be an abelian group, $N, K \leq G$ and let $N$ be a Frattini supplemented group. If a Frattini supplement $X$ of $NK$ in $G$ satisfies $X \cap Y = \{1\}$ for every Frattini supplement $Y$ of the normal subgroups of $N$ then $K$ has a Frattini supplement in $G$.

*Proof.* Since $X$ is a Frattini supplement of $NK$ in $G$, $G = (NK)X$ and $NK \cap X \leq \text{Frat}(X)$. Now consider the subgroup $N \cap (KX) \leq N$. For a Frattini
supplement $Y$ of $N \cap (KX)$, $N = (N \cap (KX))Y$ and $(N \cap (KX)) \cap Y \leq \text{Frat}(Y)$. Therefore $Y \cap KX \leq \text{Frat}(Y)$. First $G = (NK)X = [(N \cap KX)Y]KX = (N \cap KXY)KX = NKX \cap KXY = G \cap KXY$. Therefore $G = K(XY)$. It is easy to see that $K \cap XY \leq [(NK) \cap X][Y \cap KX]$. Hence $K \cap XY \leq (\text{Frat}(X))(\text{Frat}(Y))$. Let $xy \in (\text{Frat}(X))(\text{Frat}(Y))$, for some $x \in \text{Frat}(X)$, $y \in \text{Frat}(Y)$ and let $XY = \langle xy, A \rangle$ for some $A \subseteq XY$. Since $G$ is an abelian group $XY = \langle xy, A \rangle = \{(xy)^n \prod_{i=1}^{n} (y_i)^{ \varepsilon_i} \mid x_i \in X, y_i \in Y, \varepsilon_i = \pm 1, n \in \mathbb{Z} \}$, where $X_i = \bigcup \{x_i\}, Y_i = \bigcup \{y_i\}, x_i$ and $y_i$ are elements of $X$ and $Y$ respectively. Hence $XY = \langle x, X_i \rangle \langle y, Y_i \rangle$. Since $X \cap Y = \{1\}$, we have $X = \langle x, X_i \rangle$ and $Y = \langle y, Y_i \rangle$. Then we have $X = \langle X_i \rangle$ and $Y = \langle Y_i \rangle$. Finally $XY = \langle X_i \rangle \langle Y_i \rangle \leq \langle A \rangle$ which implies that $XY = \langle A \rangle$. Therefore, $xy \in \text{Frat}(XY)$ and $XY$ is a Frattini supplement of $K$ in $G$.

**Proposition 3.8.** Let $G$ be a group, $N \trianglelefteq G$ and $S$ be the Frattini supplement of $N$ in $G$ and $\text{Frat}(G)$ is finite. For $K \trianglelefteq G$, if $K \leq \text{Frat}(G)$ then $S$ is a Frattini supplement of $NK$ in $G$. In particular, if $G$ has no maximal subgroup then $S$ is a Frattini supplement of $NK$ in $G$ for every $K \trianglelefteq G$.

**Proof.** $G = NS$ and $N \cap S \leq \text{Frat}(S)$ since $S$ is a Frattini supplement of $N$ in $G$, so for $K \trianglelefteq G$, $G = NKS = (NK)S$. Now we must show that $NK \cap S \leq \text{Frat}(S)$. Suppose that $NK \cap S \not\leq \text{Frat}(S)$. So $NK \cap S \not\subseteq M$, for some maximal subgroup $M$ of $S$ and then $S = \langle NK \cap S, M \rangle$. Hence $S = \langle NK \cap S, M \rangle$ implies $G = NS = N \langle NK \cap S, M \rangle \leq N \langle NK, M \rangle = \langle K, N, M \rangle$. Therefore $G = \langle K, N, M \rangle$ and then $G = NM$ since $K \leq \text{Frat}(G)$. But we have $S = M$ by minimality of $S$ from Proposition 3.1, which is a contradiction. Therefore $NK \cap S \leq \text{Frat}(S)$ and $S$ is a Frattini supplement of $NK$ in $G$. In particular, if $G$ has no maximal subgroup then $G = \text{Frat}(G)$ and obviously, $S$ is a Frattini supplement of $NK$ in $G$ for every $K \trianglelefteq G$.

**Corollary 3.9.** Let $G$ be a group, $N \trianglelefteq G$ and $S$ be the Frattini supplement of $N$ in $G$ and $\text{Frat}(G)$ is finite. If $K \leq \text{Frat}(G)$ then $K \cap S \leq \text{Frat}(S)$ for $K \trianglelefteq G$.

**Proof.** It’s obvious since $K \cap S \leq NK \cap S \leq \text{Frat}(S)$ by Proposition 3.8.

4. Primitivity for Frattini supplemented groups

If $G$ is a group and $N \trianglelefteq G$, then the Frattini supplement set of $N$ can be defined. Consider the set $\Sigma_N = \{S \leq G \mid G = NS, N \cap S \leq \text{Frat}(S)\}$. One can assume $G$ is a Frattini supplemented to ensure that $\Sigma_N \neq \emptyset$.

**Corollary 4.1.** Let $G$ be a Frattini supplemented group and $N \trianglelefteq G$. If $G \in \Sigma_N$ then $\Sigma_N = \{G\}$.

**Proof.** Since $G$ is minimal by Proposition 3.1, then $\Sigma_N = \{G\}$.
Let $G$ and $H$ be groups, $N \trianglelefteq G$ and $S$ be a Frattini supplement of $N$ in $G$. If $\varphi : G \to H$ is an isomorphism then $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in $H$. In particular, if $\sigma \in \text{Aut}(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in $G$ and if $T \in \Sigma_N$ then for every $g \in G$, $T^g \in \Sigma_N$.

Proof. Since $S$ is a Frattini supplement of $N$ in $G$, we have $G = NS$ and $N \cap S \leq \text{Frat}(S)$. If $\varphi : G \to H$ is an isomorphism and $N \trianglelefteq G$ then $\varphi(N) \trianglelefteq H$ and obviously $H = \varphi(G) = \varphi(NS) \leq \varphi(N)\varphi(S)$ which implies that $H = \varphi(N)\varphi(S)$. Now we show that $\varphi(N) \cap \varphi(S) \leq \text{Frat}(\varphi(S))$. Firstly, if $a \in \varphi(N) \cap \varphi(S)$ then $a = \varphi(n) = \varphi(s)$ for some $n \in N$ and $s \in S$. So $a = \varphi(n) = \varphi(s)$ which implies that $\varphi(n) = \varphi(s)$, and so $\varphi(n)(\varphi(s))^{-1} = 1_H$. It follows that $\varphi(n)\varphi(s^{-1}) = 1_H$, thus $\varphi(ns^{-1}) = 1_H$. Therefore, $ns^{-1} \in \text{Ker}(\varphi)$ and $n = s$, since $\varphi$ is an isomorphism and $Ker(\varphi) = 1_G$. Hence $n = s \in N \cap S$ and we have $a = \varphi(n) = \varphi(s) \in \varphi(N \cap S)$. So $\varphi(N) \cap \varphi(S) \leq \varphi(N \cap S) \leq \text{Frat}(\text{Frat}(S))$. Now we will show that $\varphi(\text{Frat}(S)) \leq \text{Frat}(\varphi(S))$. Let $\varphi(a) \in \varphi(\text{Frat}(S))$ for some $a \in \text{Frat}(S)$ and let $\varphi(S) = \langle \varphi(a), X \rangle$ for any $X \subseteq \varphi(S)$. If $X \subseteq \varphi(S)$ then $X = \varphi(A)$ for some $A \subseteq S$. Since $\varphi$ is an isomorphism $\varphi(S) = \langle \varphi(a), \varphi(A) \rangle$ and $\langle \varphi(a) \rangle \leq \langle \varphi(a), A \rangle$. Therefore, $\varphi(S) = \varphi(\langle a, A \rangle)$ and so $S = \langle a, A \rangle$. Since $a \in \text{Frat}(S)$, we have $S = \langle a \rangle$. It follows from $S = \langle a \rangle$ that $\varphi(S) = \varphi(\langle a \rangle) = \langle X \rangle$ which implies that $S = \langle X \rangle$, and so $\varphi(a) \in \text{Frat}(\varphi(S))$. Hence $\varphi(\text{Frat}(S)) \leq \text{Frat}(\varphi(S))$. Finally, $\varphi(N) \cap \varphi(S) \leq \text{Frat}(\varphi(S))$ and $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in $G$. In particular, it is obvious that if $\sigma \in \text{Aut}(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in $G$, since $\sigma$ is an isomorphism. If $T \in \Sigma_N$ then for every $\sigma \in \text{Inn}(G)$ and for every $g \in G$, $\sigma(T) = T^g$ will be a Frattini supplement of $N$ in $G$.

Example 4.3. For the group $S_3$, $\langle (12) \rangle$ is a Frattini supplement of $A_3$ in $S_3$. Anyone can easily see that every conjugate of $\langle (12) \rangle$ in $S_3$ is also a Frattini supplement of $A_3$ in $S_3$.

Let $G$ be a Frattini supplemented group and $N \trianglelefteq G$. Consider the set defined above $\Sigma_N = \{ S \leq G \mid G = NS, N \cap S \leq \text{Frat}(S) \}$. By Theorem 4.2, a group action might be defined as:

The function $G \times \Sigma_N \longrightarrow \Sigma_N, (g, S) \to S^g$, then $G$ acts on $\Sigma_N$.

Using Cayley-like representation by this action we have the function $g : \Sigma_N \to \Sigma_N, S \to S^g$ is well-defined so the morphism $\varphi : G \to \text{Sym}(\Sigma_N)$, $g \to g^{-1}$ is closed, well defined and a homomorphism. Before Theorem 4.4, consider the transitivity of $\Sigma_N$. It may not be found $g \in G$, such that $S^g = T$ for every pair of $S, T \in \Sigma_N$. Hence, let us take the subset $A_N$ of $\Sigma_N$ such that $A_N = \{ S^g \mid S \in \Sigma_N, g \in G \}$. Obviously $G$ acts transitively on $A_N$.

Theorem 4.4. Let $G$ be a Frattini supplemented group, $N \trianglelefteq G$, $G \notin \Sigma_N$ and $S$ be maximal in $G$ for every $S \in \Sigma_N$. Then $G$ acts primitively on the set $A_N$.

Proof. Since $S \in \Sigma_N$ is a maximal subgroup of $G$ and $S \leq N_G(S) \leq G$, we have $S = N_G(S)$ or $N_G(S) = G$. First, consider the case $S = N_G(S)$. For
some $g \in G \setminus S$, we have $S^g \neq S$, since $S < G$. Then $|A_N| \geq 2$. Now for
the subgroup $G_S = \{g \in G \mid S^g = S\}$, it is obvious that $G_S = N_G(S) = S$.
Therefore, $G_S$ is a maximal subgroup of $G$ and so $G$ is primitive by [2]. Now
consider the second case, when $N_G(S) = G$. Then $S \leq G$ and so, for every
g \in G$, we have $S^g = S$. Then $|A_N| = 1$. Hence $A_N$ is a trivial block for $G$ and
$G$ is primitive.

5. Frat-series

**Definition 5.1.** Let $G$ be a group and 1 = $G_0 < G_1 < \cdots < G_n = G$ be
a normal series of $G$. If $G_i$ has a Frattini supplement $S_i$ in $G_{i+1}$ for every
1 \leq i \leq n, then $G$ is said to have a Frat-series.

Let $G$ be a group which has a Frat-series, $G_{i-1}$ be a term of the series and
$S_{i-1}$ be a Frattini supplement of $G_{i-1}$ in $G_i$. If $G$ has a subgroup $H$ such that
$S_{i-1} \leq H$ for every 1 \leq i \leq n then $H$ has a Frat-series. In particular $(\{S_i\})$
has a Frat-series.

**Proof.** Let 1 = $G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of $G$. Consider
the intersection of $H$ with terms of the series in hypothesis. Obviously 1 = $H \cap G_0 < H \cap G_1 < \cdots < H \cap G_n = H$ is a normal series of $H$. Since $S_{i-1}$
is a Frattini supplement of $G_{i-1}$ in $G_i$ and $S_{i-1} \leq H$, we have $G_i \cap H =
(G_i \cap S_{i-1}) \cap H = (G_{i-1} \cap H)S_{i-1}$. Moreover, $(G_{i-1} \cap H) \cap S_{i-1} = (G_{i-1} \cap
S_{i-1}) \cap H \leq \text{Frat}S_{i-1} \cap H \leq \text{Frat}S_{i-1}$. So $H$ has a Frat-series. In particular for
$H = (\{S_i\})$ we conclude that $(\{S_i\})$ has a Frat-series.

**Theorem 5.2.** Let $G$ be a group that has a Frat-series, $G_{i-1}$ be a term of the
series and $S_{i-1}$ be a finite Frattini supplement of $G_{i-1}$ in $G_i$. If $N$ is a normal
subgroup of $G$ such that $N \trianglelefteq S_{i-1}$, then $G/N$ has a Frat-series.

**Proof.** Let 1 = $G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of $G$. One can easily
see that the series 1 = $N/N = G_0N/N \leq G_1N/N \leq ... \leq G_nN/N = G/N$
which is obtained from the Frat-series of $G$, is a normal series of $G/N$. Since
$S_{i-1}$ is a finite Frattini supplement of $G_{i-1}$ in $G_i$ for every 1 \leq i \leq n, $G/N$
has a Frat-series by Theorem 3.5.

**Theorem 5.3.** Let $G$ be a group and 1 = $G_0 < G_1 < \cdots < G_n = G$ be a Frat-
series of $G$ and $\sigma \in \text{Aut}(G)$. Then 1 = $\sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is
also a Frat-series of $G$.

**Proof.** First, we will show that 1 = $\sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is
a normal series of $G$. It is obvious that $\sigma(G_i) < \sigma(G_{i+1})$ for every $i$. Also,
it is easy to see that $\sigma(G_i) \trianglelefteq G$. Since 1 = $G_0 < G_1 < \cdots < G_n = G$
is a Frat-series of $G$, then there exists $S_i \leq G_{i+1}$ such that $G_{i+1} = G_iS_i$ and
$G_i \cap S_i \leq \text{Frat}(S_i)$ for every $i$. Furthermore, the restriction of $\sigma$ to $G_{i+1}$ is an
isomorphism from $G_{i+1}$ to $\sigma(G_{i+1})$ and $\sigma(S_i)$ is a Frattini supplement of $\sigma(G_i)$.
in \(\sigma(G_{i+1})\) by Theorem 4.2. Finally, \(1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G\) is a Frat-series of \(G\).

**Theorem 5.4.** Let \(G\) be a group, \(1 = G_0 < G_1 < \cdots < G_n = G\) be a Frat-series of \(G\), \(G_{i-1}\) be a term of the series and \(S_{i-1} \subseteq G\) be a Frattini supplement of \(G_{i-1}\) in \(G_i\) for every \(1 \leq i \leq n-1\) and \(S_{n-1} \subseteq G\) be a complement of \(G_{n-1}\) in \(G\). If \(S_{i-1}\) is a complement of \(G_{i-1}\) in \(G_i\) then \(G_{i-1}\) has a Frattini supplement in \(G_{i+1}\).

**Proof.** Since \(G_{i+1}\) and \(G_i\) are terms of the Frat-series of \(G\), we have \(G_{i+1} = G_{i-1}(S_{i-1}S_{i}),\) \(1 = G_{i-1} \cap S_{i-1} \leq \text{Frat}(S_{i-1}),\) and \(1 = G_i \cap S_i \leq \text{Frat}(S_i)\) for some \(S_{i-1}, S_i \trianglelefteq G\). Let \(a\) be an element of \(G_{i-1} \cap (S_{i-1}S_i)\). Then \(a = xy\) for some \(x \in S_{i-1}\) and \(y \in S_i\). Therefore \(a = xy\) implies that \(y = x^{-1}a \in S_{i-1}G_{i-1} = G_{i-1}S_{i-1} = G_i\) and then \(y \in G_i \cap S_i = 1\). So we have \(a = x \in G_{i-1} \cap S_{i-1} = 1\) and \(a = 1\). Hence \(1 = G_{i-1} \cap (S_{i-1}S_i) \leq \text{Frat}(S_{i-1}S_i)\).

Therefore \(S_{i-1}S_i\) is a Frattini supplement of \(G_{i-1}\) in \(G_{i+1}\).

**Theorem 5.5.** Let \(1 = G_0 < G_1 < \cdots < G_n = G\) be Frat-series of \(G\). If \(S_i \leq G_i\) for every \(0 \leq i < n\) where \(S_i\) is a Frattini supplement of \(G_i\) in \(G_{i+1}\) then \(G\) is soluble.

**Proof.** Since \(1 = G_0 < G_1 < \cdots < G_n = G\) is a Frat-series of \(G\), then \(G_{i+1} = G_iS_i\) and \(G_i \cap S_i \leq \text{Frat}(S_i)\) for every \(i\). Now, we will show that the Frat-series of \(G\) is also a derived series. Consider the element \([x,y]\) of \(S_i\).

So \([x,y] \in G_i\) and then \(S_i \leq G_i \cap S_i\) since \(S_i \leq S_i\). Hence, for every \([x,y] \in S_i\), \([x,y])G_i \cap S_i) = G_i \cap S_i\). Therefore \(x^{-1}y^{-1}xy(G_i \cap S_i) = G_i \cap S_i\) and so \(xy(G_i \cap S_i) = yx(G_i \cap S_i)\) and we obtain \(x(G_i \cap S_i)y(G_i \cap S_i) = y(G_i \cap S_i)x(G_i \cap S_i)\) for every \(x, y \in G_i\). Therefore the factor \(S_i/G_i \cap S_i\) is abelian. Since \(S_i/G_i \cap S_i \trianglelefteq G_i\), \(G_i/\text{Frat}(S_i)\) is abelian. So, \(1 = G_0 < G_1 < \cdots < G_n = G\) is a derived series of \(G\) and \(G\) is soluble.

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