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FRATTINI SUPPLEMENTS AND FRAT SERIES

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Abstract. In this study, Frattini supplement subgroup and Frattini supplemented group are defined by Frattini subgroup. By these definitions, it's shown that finite abelian groups are Frattini supplemented and every conjugate of a Frattini supplement of a subgroup is also a Frattini supplement. A group action of a group is defined over the set of Frattini supplements of a normal subgroup of the group by conjugation and in this study new characterization of primitivity of groups has obtained in terms of Frattini supplemented groups by this action. Moreover, Frat-series of a group is defined based on Frattini supplements of normal subgroups of the group and it is shown that subgroups and factor groups of groups with Frat-series also have Frat-series under some special conditions. Furthermore, we determined a characterization of soluble groups which have Frat-series.
Keywords: Frattini subgroup, primitive group, group actions.
MSC(2010): Primary: 20D25; Secondary: 20B15, 58E40.

1. Introduction

In module theory, *Rad-supplemented* modules were defined as proper generalizations of supplemented modules. Over a ring with identity, a unitial module M is called *Rad-supplemented* if every submodule N of M has Rad-supplement in M, i.e. $N + K = M$ and $N \cap K \leq Rad(K)$ for some submodule K of M, where $Rad(K)$ is the intersection of all maximal submodules of K. Hausen studied supplemented and amply supplemented groups in terms of nilpotency by using Frattini subgroup in [3]. We investigated the properties of these groups in a similar way with [3].

2. Preliminaries

The *Frattini subgroup* of an arbitrary group G is defined to be the intersection of all the maximal subgroups, with the stipulation that it will equal to G if

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G has no maximal subgroup. This subgroup, which is evidently characteristic, is written as $Frat(G)$ [1].

The Frattini subgroup has the remarkable property that it is the set of all nongenerators of the group; here an element g is called a *nongenerator* of G if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$ when X is a subset of G [1].

A subgroup H of a group G is *supplemented* in G if there is a subgroup K of G such that $G = HK$. If $H \cap K = \{1\}$ then K is said to be a *complement* of H in G [1].

The derived subgroup of G is defined as $[G, G] = \langle [a, b] | a, b \in G \rangle$ where $[a, b] = a^{-1}b^{-1}ab$, and is written as G' .

This study is based on the following definition.

Definition 2.1. Let G be a group and $N \trianglelefteq G$. The subgroup S of G is called a *Frattini supplement* of N in G if $G = NS$ and $N \cap S \leq Frat(S)$. Clearly G is *Frattini supplement* of $Frat(G)$ in G. If every $N \trianglelefteq G$ has a Frattini supplement in G, then G is said to be *Frattini supplemented*.

Example 2.2. Let G be a group. If $G = Frat(G)$ then G is Frattini supplemented.

Example 2.3. G itself is Frattini supplement of 1_G since $G = 1_G G$ and $1_G \cap G \leq Frat(G)$. So the Frattini supplement of 1_G is G.

Example 2.4. Let G be a group in which every subgroup is normal, and N be a minimal normal subgroup of G. Then G is a Frattini supplement of N.

Example 2.5. For the generalized quaternion group $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \rangle$ $(n \geq 3)$, $\langle y \rangle$ is a Frattini supplement of the normal subgroup $\langle x \rangle$ in Q_{2^n}.

Example 2.6. Let G be a finite abelian group. Then G is Frattini supplemented.

Example 2.7. Let $G = Q_8$ be the group of Hamilton quaternions. It is easy to see that $\langle i \rangle, \langle j \rangle$, and $\langle k \rangle$ are Frattini supplement of each other and G is a Frattini supplement of $\{1, -1\}$ in G, since every subgroup is normal.

3. Frattini supplemented groups

Proposition 3.1. *Let G be a finite group and N be a normal subgroup of G. If S is a Frattini supplement of N in G then S is a minimal supplement of N in G.*

Proof. Since S is a Frattini supplement of N in G, then $G = NS$ and $N \cap S \leq Frat(S)$. Let K be a Frattini supplement of N in G and $K \leq S$. Hence $G = NK$. Since $N \cap S \trianglelefteq S$, if we intersect with S, we get $S = S \cap G = S \cap NK = K(N \cap S) = \langle K, N \cap S \rangle$. Finally, we have $S = K$, since $N \cap S \leq Frat(S)$. $\square$

Proposition 3.2. *Let G be a finite group. If G is a Frattini supplement of G' then G is nilpotent.*

Proof. Since G is a Frattini supplement of G', $G = G'G$ and $G' = G \cap G' \leq Frat(G)$, then G is nilpotent according to [1]. □

Theorem 3.3. *Let G be a group, H be a finite normal subgroup of G and $G = HK$ for some $K \leq G$. If K is minimal, then K is a Frattini supplement of H in G.*

Proof. Assume that K is not a Frattini supplement of H in G. Then $H \cap K \nleq Frat(K)$ and for a maximal subgroup M of K, $H \cap K \nleq M$. So we have $M < (H \cap K)M \leq K$ which implies that $K = (H \cap K)M = HM \cap K$, and so $K \leq HM$. Since $G = HK$, for every $g \in G$, we have $g \in HM$. Hence $G = HM$, which is a contradiction. Therefore, K is a Frattini supplement of H in G. □

Proposition 3.1 and Theorem 3.3 could be merged for finite groups.

Corollary 3.4. *Let G be a finite group, and $N \trianglelefteq G$. Then S is a Frattini supplement of N in G if and only if S is a minimal supplement of N.*

Theorem 3.5. *Let G be a finite group, $N \trianglelefteq G$, H be a Frattini supplement of N in G and $K \leq H$. Then for $K \trianglelefteq G$, H/K is a Frattini supplement of NK/K.*

Proof. Since H is a Frattini supplement of N, $G = NH$ and $H \cap N \leq Frat(H)$. It is easy to see that $G/K = NH/K = (NK/K)(H/K)$ and $(H/K) \cap (NK/K) \leq (H \cap N)K/K \leq (Frat(H))K/K$ by modular law. Then we have $(Frat(H))K/K$
$\leq Frat(H/K)$ by [1], since G is finite. □

Theorem 3.6. *Let G be a finite group, $N \trianglelefteq G$, S be a Frattini supplement of N in G and H be a subgroup of G such that $S \leq H$. Then $N \cap H$ has a Frattini supplement in H.*

Proof. Obviously $N \cap H \trianglelefteq H$. Since S is a Frattini supplement of N in G, $G = NS$ and $S \cap N \leq FratS$. So we have $G = NS$, which implies that $H = G \cap H = (NS) \cap H = (N \cap H)S$ and $(H \cap N) \cap S = H \cap (N \cap S) \leq H \cap Frat(S) \leq Frat(S)$. Hence $N \cap H$ has a Frattini supplement in H. □

Theorem 3.7. *Let G be an abelian group $N, K \leq G$ and let N be a Frattini supplemented group. If a Frattini supplement X of NK in G satisfies $X \cap Y = \{1\}$ for every Frattini supplement Y of the normal subgroups of N then K has a Frattini supplement in G.*

Proof. Since X is a Frattini supplement of NK in G, $G = (NK)X$ and $NK \cap X \leq Frat(X)$. Now consider the subgroup $N \cap (KX) \leq N$. For a Frattini supplement Y of $N\cap(KX)$, $N=(N\cap(KX))Y$ and $(N\cap(KX))\cap Y\leq Frat(Y)$. Therefore $Y\cap KX\leq Frat(Y)$. First $G=(NK)X=[(N\cap KX)Y]KX=(N\cap KXY)KX=NKX\cap KXY=G\cap KXY$. Therefore $G=K(XY)$. It is easy to see that $K\cap XY\leq[(NK)\cap X][Y\cap KX]$. Hence $K\cap XY\leq[(NK)\cap X][Y\cap KX]\leq(Frat(X))(Frat(Y))$. Let $xy\in(Frat(X))(Frat(Y))$, for some $x\in Frat(X)$, $y\in Frat(Y)$ and let $XY=\langle xy,A\rangle$ for some $A\subseteq XY$. Since G is an abelian group $XY=\langle xy,A\rangle=\{(xy)^n\prod(x_iy_i)^{\varepsilon_i}\mid x_i\in X,\, y_i\in Y,\, \varepsilon_i=\pm1,\, n\in\mathbb{Z}\}=\{(x^n\prod x_i^{\varepsilon_i})(y^n\prod y_i^{\varepsilon_i})\mid x_i\in X, y_i\in Y, \varepsilon_i=\pm1, n\in\mathbb{Z}\}\leq\langle x,X_i\rangle\langle y,Y_i\rangle$, where $X_i=\bigcup\{x_i\}$, $Y_i=\bigcup\{y_i\}$, x_i and y_i are elements of X and Y respectively. Hence $XY=\langle x,X_i\rangle\langle y,Y_i\rangle$. Since $X\cap Y=\{1\}$, we have $X=\langle x,X_i\rangle$ and $Y=\langle y,Y_i\rangle$. Then we have $X=\langle X_i\rangle$ and $Y=\langle Y_i\rangle$. Finally $XY=\langle X_i\rangle\langle Y_i\rangle\leq\langle A\rangle$ which implies that $XY=\langle A\rangle$. Therefore, $xy\in Frat(XY)$ and XY is a Frattini supplement of K in G. □

Proposition 3.8. *Let G be a group, $N\trianglelefteq G$ and S be the Frattini supplement of N in G and $Frat(G)$ is finite. For $K\trianglelefteq G$, if $K\leq Frat(G)$ then S is a Frattini supplement of NKin G. In particular, if G has no maximal subgroup then S is a Frattini supplement of NK in G for every $K\trianglelefteq G$.*

Proof. $G=NS$ and $N\cap S\leq Frat(S)$ since S is a Frattini supplement of N in G , so for $K\trianglelefteq G$, $G=NKS=(NK)S$. Now we must show that $NK\cap S\leq Frat(S)$. Suppose that $NK\cap S\nleq Frat(S)$. So $NK\cap S\nleq M$, for some maximal subgroup M of S and then $S=\langle NK\cap S,M\rangle$. Hence $S=\langle NK\cap S,M\rangle$ implies $G=NS=N\ \langle NK\cap S,M\rangle\leq N\langle NK,M\rangle=\langle K,N,M\rangle$. Therefore $G=\langle K,N,M\rangle$ and then $G=NM$ since $K\leq Frat(G)$. But we have $S=M$ by minimality of S from Proposition 3.1, which is a contradiction. Therefore $NK\cap S\leq Frat(S)$ and S is a Frattini supplement of NK in G. In particular, if G has no maximal subgroup then $G=Frat(G)$ and obviously, S is a Frattini supplement of NK in G for every $K\trianglelefteq G$. □

Corollary 3.9. *Let G be a group, $N\trianglelefteq G$ and S be the Frattini supplement of N in G and $Frat(G)$ is finite. If $K\leq Frat(G)$ then $K\cap S\leq Frat(S)$ for $K\trianglelefteq G$.*

Proof. It's obvious since $K\cap S\leq NK\cap S\leq Frat(S)$ by Proposition 3.8. □

4. Primitivity for Frattini supplemented groups

If G is a group and $N\trianglelefteq G$, then the Frattini supplement set of N can be defined. Consider the set $\Sigma_N=\{S\leq G\mid G=NS, N\cap S\leq Frat(S)\}$. One can assume G is a Frattini supplemented to ensure that $\Sigma_N\neq\emptyset$.

Corollary 4.1. *Let G be a Frattini supplemented group and $N\trianglelefteq G$. If $G\in\Sigma_N$ then $\Sigma_N=\{G\}$.*

Proof. Since G is minimal by Proposition 3.1, then $\Sigma_N=\{G\}$. □

Theorem 4.2. *Let G and H be groups, $N \trianglelefteq G$ and S be a Frattini supplement of N in G. If $\varphi : G \to H$ is an isomorphism then $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in H. In particular, if $\sigma \in Aut(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in G and if $T \in \Sigma_N$ then for every $g \in G$, $T^g \in \Sigma_N$.*

Proof. Since S is a Frattini supplement of N in G, we have $G = NS$ and $N \cap S \leq Frat(S)$. If $\varphi : G \to H$ is an isomorphism and $N \trianglelefteq G$ then $\varphi(N) \trianglelefteq H$ and obviously $H = \varphi(G) = \varphi(NS) \leq \varphi(N)\varphi(S)$ which implies that $H = \varphi(N)\varphi(S)$. Now we show that $\varphi(N) \cap \varphi(S) \leq Frat(\varphi(S))$. Firstly, if $a \in \varphi(N) \cap \varphi(S)$ then $a = \varphi(n) = \varphi(s)$ for some $n \in N$, and $s \in S$. So $a = \varphi(n) = \varphi(s)$ which implies that $\varphi(n) = \varphi(s)$, and so $\varphi(n)(\varphi(s))^{-1} = 1_H$. It follows that $\varphi(n)\varphi(s^{-1}) = 1_H$, thus $\varphi(ns^{-1}) = 1_H$. Therefore, $ns^{-1} \in Ker(\varphi)$ and $n = s$, since φ is an isomorphism and $Ker(\varphi) = 1_G$. Hence $n = s \in N \cap S$ and we have $a = \varphi(n) = \varphi(s) \in \varphi(N \cap S)$. So $\varphi(N) \cap \varphi(S) \leq \varphi(N \cap S) \leq \varphi(Frat(S))$. Now we will show that $\varphi(Frat(S)) \leq Frat(\varphi(S))$. Let $\varphi(a) \in \varphi(Frat(S))$ for some $a \in Frat(S)$ and let $\varphi(S) = \langle \varphi(a), X \rangle$ for any $X \subseteq \varphi(S)$. If $X \subseteq \varphi(S)$ then $X = \varphi(A)$ for some $A \subseteq S$. Since φ is an isomorphism $\varphi(S) = \langle \varphi(a), \varphi(A) \rangle \leq \varphi(\langle a, A \rangle)$. Therefore, $\varphi(S) = \varphi(\langle a, A \rangle)$ and so $S = \langle a, A \rangle$. Since $a \in Frat(S)$, we have $S = \langle A \rangle$. It follows from $S = \langle A \rangle$ that $\varphi(S) = \varphi(\langle A \rangle) \leq \langle \varphi(A) \rangle = \langle X \rangle$ which implies that $S = \langle X \rangle$, and so $\varphi(a) \in Frat(\varphi(S))$. Hence $\varphi(Frat(S)) \leq Frat(\varphi(S))$. Finally, $\varphi(N) \cap \varphi(S) \leq Frat(\varphi(S))$ and $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in G. In particular, it is obvious that if $\sigma \in Aut(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in G, since σ is an isomorphism. If $T \in \Sigma_N$ then for every $\sigma \in Inn(G)$ and for every $g \in G$, $\sigma(T) = T^g$ will be a Frattini supplement of N in G. ☐

Example 4.3. For the group S_3, $\langle (12) \rangle$ is a Frattini supplement of A_3 in S_3. Anyone can easily see that every conjugate of $\langle (12) \rangle$ in S_3 is also a Frattini supplement of A_3 in S_3.

Let G be a Frattini supplemented group and $N \trianglelefteq G$. Consider the set defined above $\Sigma_N = \{S \leq G \mid G = NS, N \cap S \leq Frat(S)\}$. By Theorem 4.2, a group action might be defined as:

The function $G \times \Sigma_N \longrightarrow \Sigma_N$, $(g, S) \to S^g$, then G acts on Σ_N.

Using Cayley-like representation by this action we have the function $g : \Sigma_N \to \Sigma_N$, $S \to S^g$ is well-defined so the morphism $\varphi : G \to Sym(\Sigma_N)$, $g \to g^{-1}$ is closed, well defined and a homomorphism. Before Theorem 4.4, consider the transitivity of Σ_N. It may not be found $g \in G$, such that $S^g = T$ for every pair of $S, T \in \Sigma_N$. Hence, let us take the subset A_N of Σ_N such that $A_N = \{S^g | S \in \Sigma_N, g \in G\}$. Obviously G acts transitively on A_N.

Theorem 4.4. *Let G be a Frattini supplemented group, $N \trianglelefteq G$, $G \notin \Sigma_N$ and S be maximal in G for every $S \in \Sigma_N$. Then G acts primitively on the set A_N.*

Proof. Since $S \in \Sigma_N$ is a maximal subgroup of G and $S \leq N_G(S) \leq G$, we have $S = N_G(S)$ or $N_G(S) = G$. First, consider the case $S = N_G(S)$. For some $g \in G \setminus S$, we have $S^g \neq S$, since $S < G$. Then $|A_N| \geq 2$. Now for the subgroup $G_S = \{g \in G \mid S^g = S\}$, it is obvious that $G_S = N_G(S) = S$. Therefore, G_S is a maximal subgroup of G and so G is primitive by [2]. Now consider the second case, when $N_G(S) = G$. Then $S \trianglelefteq G$ and so, for every $g \in G$, we have $S^g = S$. Then $|A_N| = 1$. Hence A_N is a trivial block for G and G is primitive. □

5. **Frat-series**

Definition 5.1. Let G be a group and $1 = G_0 < G_1 < \cdots < G_n = G$ be a normal series of G. If G_i has a Frattini supplement S_i in G_{i+1} for every $1 \leq i \leq n$, then G is said to have a *Frat-series*.

Let G be a group which has a Frat- series, G_{i-1} be a term of the series and S_{i-1} be a Frattini supplement of G_{i-1} in G_i. If G has a subgroup H such that $S_{i-1} \leq H$ for every $1 \leq i \leq n$ then H has a Frat-series. In particular $\langle\{S_i\}\rangle$ has a Frat-series.

Proof. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G. Consider the intersection of H with terms of the series in hypothesis. Obviously $1 = H \cap G_0 < H \cap G_1 < \cdots < H \cap G_n = H$ is a normal series of H. Since S_{i-1} is a Frattini supplement of G_{i-1} in G_i and $S_{i-1} \leq H$, we have $G_i \cap H = (G_{i-1}S_{i-1}) \cap H = (G_{i-1} \cap H)S_{i-1}$. Moreover, $(G_{i-1} \cap H) \cap S_{i-1} = (G_{i-1} \cap S_{i-1}) \cap H \leq FratS_{i-1} \cap H \leq FratS_{i-1}$. So H has a Frat-series. In particular for $H = \langle\{S_i\}\rangle$ we conclude that $\langle\{S_i\}\rangle$ has a Frat-series. □

Theorem 5.2. *Let G be a group that has a Frat-series, G_{i-1} be a term of the series and S_{i-1} be a finite Frattini supplement of G_{i-1} in G_i. If N is a normal subgroup of G such that $N \trianglelefteq S_{i-1}$, then G/N has a Frat-series.*

Proof. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G. One can easily see that the series $1 = N/N = G_0N/N \leq G_1N/N \leq ... \leq G_nN/N = G/N$ which is obtained from the Frat-series of G, is a normal series of G/N. Since S_{i-1} is a finite Frattini supplement of G_{i-1} in G_i for every $1 \leq i \leq n$, G/N has a Frat-series by Theorem 3.5. □

Theorem 5.3. *Let G be a group and $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G and $\sigma \in Aut(G)$. Then $1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is also a Frat-series of G.*

Proof. First, we will show that $1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is a normal series of G. It is obvious that $\sigma(G_i) < \sigma(G_{i+1})$ for every i. Also, it is easy to see that $\sigma(G_i) \trianglelefteq G$. Since $1 = G_0 < G_1 < \cdots < G_n = G$ is a Frat-series of G, then there exists $S_i \leq G_{i+1}$ such that $G_{i+1} = G_iS_i$ and $G_i \cap S_i \leq Frat(S_i)$ for every i. Furthermore, the restriction of σ to G_{i+1} is an isomorphism from G_{i+1} to $\sigma(G_{i+1})$ and $\sigma(S_i)$ is a Frattini supplement of $\sigma(G_i)$

in $\sigma(G_{i+1})$ by Theorem 4.2. Finally, $1=\sigma(G_0)<\sigma(G_1)<\cdots<\sigma(G_n)=G$ is a Frat-series of G. □

Theorem 5.4. *Let G be a group, $1=G_0<G_1<\cdots<G_n=G$ be a Frat-series of G, G_{i-1}be a term of the series and $S_{i-1}\trianglelefteq G$ be a Frattini supplement of G_{i-1} in G_i for every $1\leq i\leq n-1$ and $S_{n-1}\trianglelefteq G$ be a complement of G_{n-1} in G. If S_{i-1} is a complement of G_{i-1} in G_i then G_{i-1} has a Frattini supplement in G_{i+1}.*

Proof. Since G_{i+1} and G_i are terms of the Frat-series of G, we have $G_{i+1}=G_{i-1}(S_{i-1}S_i)$, $1=G_{i-1}\cap S_{i-1}\leq Frat(S_{i-1})$, and $1=G_i\cap S_i\leq Frat(S_i)$ for some $S_{i-1},S_i\trianglelefteq G$. Let a be an element of $G_{i-1}\cap(S_{i-1}S_i)$. Then $a=xy$ for some $x\in S_{i-1}$ and $y\in S_i$. Therefore $a=xy$ implies that $y=x^{-1}a\in S_{i-1}G_{i-1}=G_{i-1}S_{i-1}=G_i$ and then $y\in G_i\cap S_i=1$. So we have $a=x\in G_{i-1}\cap S_{i-1}=1$ and $a=1$. Hence $1=G_{i-1}\cap(S_{i-1}S_i)\leq Frat(S_{i-1}S_i)$. Therefore $S_{i-1}S_i$ is a Frattini supplement of G_{i-1} in G_{i+1}. □

Theorem 5.5. *Let $1=G_0<G_1<\cdots<G_n=G$ be Frat-series of G. If $S_i^{'}\leq G_i$ for every $0\leq i<n$ where S_i is a Frattini supplement of G_i in G_{i+1} then G is soluble.*

Proof. Since $1=G_0<G_1<\cdots<G_n=G$ is a Frat-series of G, then $G_{i+1}=G_iS_i$ and $G_i\cap S_i\leq Frat(S_i)$ for every i. Now, we will show that the Frat-series of G is also a derived series. Consider the element $[x,y]$ of $S_i^{'}$. So $[x,y]\in G_i$ and then $S_i^{'}\leq G_i\cap S_i$ since $S_i^{'}\leq S_i$. Hence, for every $[x,y]\in S_i^{'}$, $[x,y](G_i\cap S_i)=G_i\cap S_i$. Therefore $x^{-1}y^{-1}xy(G_i\cap S_i)=G_i\cap S_i$ and so $xy(G_i\cap S_i)=yx(G_i\cap S_i)$ and we obtain $x(G_i\cap S_i)y(G_i\cap S_i)=y(G_i\cap S_i)x(G_i\cap S_i)$ for every $x,y\in G_i$. Therefore the factor $S_i/G_i\cap S_i$ is abelian. Since $S_i/G_i\cap S_i\simeq G_iS_i/G_i=G_{i+1}/G_i$, we have G_{i+1}/G_i is abelian. So, $1=G_0<G_1<\cdots<G_n=G$ is a derived series of G and G is soluble. □

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