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UNMIXED r-PARTITE GRAPHS

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Abstract. Unmixed bipartite graphs have been characterized by Ravindra and Villarreal independently. Our aim in this paper is to characterize unmixed r-partite graphs under a certain condition, which is a generalization of Villarreal's theorem on bipartite graphs. Also, we give some examples and counterexamples in relevance to this subject.
Keywords: r-partite graph, well-covered, unmixed, perfect matching, clique.
MSC(2010): Primary: 05E40; Secondary: 05C69, 05C75.

1. Introduction

In the sequel, we use [4] as a reference for terminology and notation on graph theory.

Let G be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A subset C of $V(G)$ is said to be a vertex cover of G if every edge of G, is adjacent with some vertices in C. A vertex cover C is called minimal, if there is no proper subset of C which is a vertex cover. A graph is called unmixed, if all minimal vertex covers of G have the same number of elements. A subset H of $V(G)$ is said to be independent, if G has not any edge $\{x, y\}$ such that $\{x, y\} \subseteq H$. A maximal independent set of G, is an independent set I of G, such that for every $H \supsetneqq I$, H is not an independent set of G. Notice that C is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent set. A graph G is called well-covered if all the maximal independent sets of G have the same cardinality. Therefore, a graph is unmixed if and only if it is well-covered. The minimum cardinality of all minimal vertex covers of G is called the covering number of G, and the maximum cardinality of all maximal independent sets of G is called the independence number of G. For determining the independence number see [6]. For relation between unmixedness of a graph and other graph properties see [1,5,9,12].

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Well-covered graphs were introduced by Plummer. See [7] for a survey on well-covered graphs and properties of them. For an integer $r \geq 2$, a graph G is said to be r-partite, if $V(G)$ can be partitioned into r disjoint parts such that for every $\{x, y\} \in E(G)$, x and y do not lie in the same part. If $r = 2, 3$, G is said to be bipartite and tripartite, respectively. Let G be an r-partite graph. For a vertex $v \in V(G)$, let $N(v)$ be the set of all vertices $u \in V(G)$ where $\{u, v\}$ is an edge of G. Let G be a bipartite graph, and let $e = \{u, v\}$ be an edge of G. Then G_e is the subgraph induced on $N(u) \cup N(v)$. If G is connected, the distance between x and y where $x, y \in V(G)$, denoted by $d(x, y)$, is the length of the shortest path between x and y. A set $M \subseteq E(G)$ is said to be a matching of G, if for any two $\{x, y\}, \{x', y'\} \in M$, $\{x, y\} \cap \{x', y'\} = \emptyset$. A matching M of G is called perfect if for every $v \in V(G)$, there exists an edge $\{x, y\} \in M$ such that $v \in \{x, y\}$. A clique in G is a set Q of vertices such that for every $x, y \in Q$, if $x \neq y$, x, y lie in an edge. An r-clique is a clique of size r.

Unmixed bipartite graphs have already been characterized by Ravindra and Villarreal in a combinatorial way independently [8,11]. Also these graphs have been characterize by an algebraic method [10].

In 1977, Ravindra gave the following criteria for unmixedness of bipartite graphs.

Theorem 1.1 ([8])**.** *Let G be a connected bipartite graph. Then G is unmixed if and only if G contains a perfect matching F such that for every edge $e = \{x, y\} \in F$, the induced subgraph G_e is a complete bipartite graph.*

Villarreal in 2007, gave the following characterization of unmixed bipartite graphs.

Theorem 1.2 ([11, Theorem 1.1])**.** *Let G be a bipartite graph without isolated vertices. Then G is unmixed if and only if there is a bipartition $V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\}$ of G such that: (a) $\{x_i, y_i\} \in E(G)$, for all i, and (b) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E(G)$, and i, j, k are distinct, then $\{x_i, y_k\} \in E(G)$.*

H. Haghighi in [3] gives the following characterization of unmixed tripartite graphs under certain conditions.

Theorem 1.3 ([3, Theorem 3.2])**.** *Let G be a tripartite graph which satisfies the condition $(*)$. Then the graph G is unmixed if and only if the following conditions hold:*

(1) *If $\{u_i, x_q\}, \{v_j, y_q\}, \{w_k, z_q\} \in E(G)$, where no two vertices of $\{x_q, y_q, z_q\}$ lie in one of the tree parts of $V(G)$ and i, j, k, q are distinct, then the set $\{u_i, v_j, w_k\}$ contains an edge of G.*

(2) *If $\{r, x_q\}, \{s, y_q\}, \{t, z_q\}$ are edges of G, where r and s belong to one of the three parts of $V(G)$ and t belongs to another part, then the set $\{r, s, t\}$ contains an edge of G (Here r and s may be equal.)*

In the above theorem, he has considered condition $(*)$ as:
being a tripartite graph with partitions

$$U = \{u_1, \ldots u_n\}, V = \{v_1, \ldots v_n\}, W = \{w_1, \ldots w_n\},$$

in which $\{u_i, v_i\}, \{u_i, w_i\}, \{v_i, w_i\} \in E(G)$, for all $i = 1, \ldots, n$.

Also, to simplify the notations, he has used $\{x_i, y_i, z_i\}$ and $\{r_i, s_i, t_i\}$ as two permutations of $\{u_i, v_i, w_i\}$.

We give a characterization of unmixed r-partite graphs under certain condition which we name it $(*)$. (See Theorem 2.3.)

In both Theorems 1.1 and 1.2 in an unmixed connected bipartite graph, there is a perfect matching, with cardinality equal to the cardinality of a minimal vertex cover, i.e. $\frac{|V(G)|}{2}$. An unmixed graph with n vertices such that its independence number is $\frac{n}{2}$, is said to be very well-covered. The unmixed connected bipartite graphs are contained in the class of very well-covered graphs. A characterization of very well-covered graphs is given in [2].

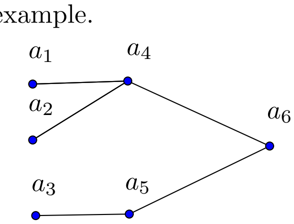
2. A generalization

By the following proposition, bipartition in connected bipartite graphs is unique.

Proposition 2.1. *Let G be a connected bipartite graph with bipartition $\{A, B\}$, and let $\{X, Y\}$ be any bipartition of G. Then $\{A, B\} = \{X, Y\}$.*

Proof. Let $x \in A$ be an arbitrary vertex of G. Then $x \in X$ or $x \in Y$. Without loss of generality let x be in X. Let $a \in A$. Then $d(x, a)$ is even. Then a and x are in the same part (of partition $\{X, Y\}$). Then $A \subseteq X$, and by the same argument we have $X \subseteq A$. Therefore $A = X$, and then $\{A, B\} = \{X, Y\}$. □

The above fact for bipartite graphs, is not true in case of tripartite graphs, as shown in the following example.



In the above graph there are two different tripartitions:

$$\{\{a_1, a_2, a_3\}, \{a_4, a_5\}, \{a_6\}\},$$

and

$$\{\{a_1, a_2\}, \{a_4, a_5\}, \{a_3, a_6\}\}.$$

A natural question refers to find criteria which characterize a special class of unmixed r-partite ($r \geq 2$) graphs.

In the above two characterizations of bipartite graphs, having a perfect matching is essential in both proofs. This motivates us to impose the following condition.
We say a graph G satisfies the condition $(*)$ for an integer $r \geq 2$, if G can be partitioned to r parts $V_i = \{x_{1i}, \ldots, x_{ni}\}$, $(1 \leq i \leq r)$, such that for all $1 \leq j \leq n$, $\{x_{j1}, \ldots, x_{jr}\}$ is a clique.

Lemma 2.2. *Let G be a graph which satisfies $(*)$ for $r \geq 2$. If G is unmixed, then every minimal vertex cover of G contains $(r-1)n$ vertices. Moreover the independence number of G is $n = \frac{|V(G)|}{r}$*

Proof. Let C be a minimal vertex cover of G. Since for every $1 \leq j \leq n$, the vertices $x_{j1}, \ldots, x_{jr}$ are in a clique, C must contain at least $r-1$ vertices in $\{x_{j1}, \ldots, x_{jr}\}$. Therefore C contains at least $(r-1)n$ vertices. By hypothesis $\bigcup_{i=1}^{r-1} V_i$ is a minimal vertex cover with $(r-1)n$ vertices, and G is unmixed. Then every minimal vertex cover of G contains exactly $(r-1)n$ elements. The last claim can be concluded from the fact that the complement of a minimal vertex cover, is an independent set. □

Now we are ready for the main theorem. By the notation $x \sim y$, we mean the vertices x and y are adjacent.

Theorem 2.3. *Let G be an r-partite graph which satisfies the condition $(*)$ for r. Then G is unmixed if and only if the following condition hold:*
For every $1 \leq q \leq n$, if there is a set $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ such that

$$x_{k_1 s_1} \sim x_{q1}, \ldots, x_{k_r s_r} \sim x_{qr},$$

then the set $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ is not independent.

Proof. Let G be an arbitrary r-partite graph which satisfies condition $(*)$ for r.

Let G be unmixed. We prove that the mentioned condition holds. Assume the contrary. Let

$$x_{k_1 s_1} \sim x_{q1}, \ldots, x_{k_r s_r} \sim x_{qr},$$

but the set $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ is independent. Then there is a maximal independent set M, such that M contains this set. Since M is maximal, $C = V(G) \backslash M$ is a minimal vertex cover of G. Since the set $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ is contained in M, then its elements are not in C, and since C is a cover of G, then all vertices x_{qi}, $(1 \leq i \leq r)$ are in C. But by Lemma 2.2, every minimal vertex cover, contains $n-1$ vertices of clique q th, a contradiction.

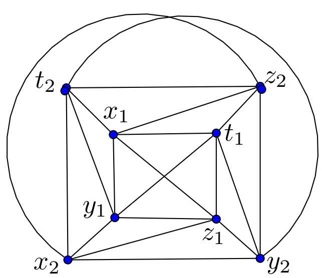
Conversely, let the condition hold. We have to prove that G is unmixed. We show that all minimal vertex covers of G, intersect the set $\{x_{q1}, \ldots, x_{qr}\}$ in exactly $r-1$ elements (for every $1 \leq q \leq n$). Let C be a minimal vertex cover and q be arbitrary. Since C is a vertex cover and $\{x_{q1}, \ldots, x_{qr}\}$ is a clique, then C intersects this set at least in $r-1$ elements. Let the contrary. Let the cardinality of $C \cap \{x_{q1}, \ldots, x_{qr}\}$ be r. Attending to minimality of C, for every $1 \leq i \leq r$, $N(x_{qi})$ contains at least one element, distinct from the elements of $\{x_{q1}, \ldots, x_{qr}\} \backslash \{x_{qi}\}$, which is not in C, because we can not remove x_{qi} from cover. Let this element be $x_{k_i s_i}$ where $s_i \neq i$ and $k_i \neq q$. Then $x_{k_i s_i} \notin C$ and $\{x_{k_i s_i}, x_{qi}\}$ is in $E(G)$. There are at least two elements i and j such that $1 \leq i < j \leq r$ and $s_i \neq s_j$, because x_{qi} can not choose its adjacent vertex from the part i. Therefore the set $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ contains at least two elements. Then by hypothesis, at least two elements, say a and b of $\{x_{k_1 s_1}, \ldots, x_{k_r s_r}\}$ are adjacent by an edge. Now C is a cover but a and b are not in C, a contradiction. □

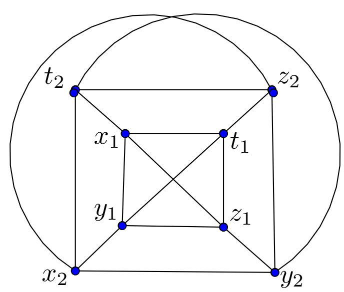
Remark 2.4. Villarreal's theorem (Theorem 1.2) for bipartite graphs, and also Haghighi's theorem (Theorem 1.3) for tripartite graphs, are special cases of Theorem 2.3 (where $r = 2$, and $r = 3$).

3. **Examples and counterexamples**

In this section, we give examples of two classes of unmixed graphs, and an example which shows that it is not necessary that an unmixed r-partite graph satisfies condition $(*)$.

Example 3.1. By Theorem 2.3, the following 4-partite graphs are unmixed.





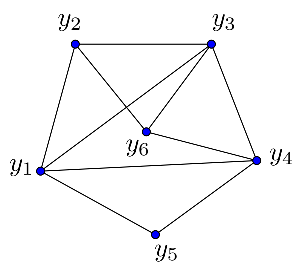
In each of the above graphs, there are two complete graphs of order 4 and some edges between them.

For $r > 4$, also $r = 3$, using two complete graphs of order r, we can construct r-partite unmixed graphs which are natural generalization of the above graphs.

Example 3.2. For every n, $n \geq 3$, the complete graph K_n, is an n-partite graph which satisfies condition $(*)$. By Theorem 2.3, K_n is unmixed.

Theorem 2.3 does not characterize all unmixed r-partite graphs. More precisely, condition $(*)$ is not valid for all unmixed graphs. In the following, we give an example of an unmixed r-partite graph which does not satisfy condition $(*)$.

Example 3.3. The following graph is a 4-partite graph with partition $\{y_1\}$, $\{y_2, y_4\}$, $\{y_3\}$, and $\{y_5, y_6\}$. This graph does not satisfy condition $(*)$ because 6 is not a multiple of 4.



We show that this graph is unmixed. Let C be an arbitrary minimal vertex cover of G. We show that C is of size 4.

Since C is a cover, it selects at least one element of $\{y_4, y_6\}$. Now we consider the following cases:
Case 1: $y_6 \in C$ and $y_4 \notin C$. In this case, since C is a vertex cover, $y_1, y_3, y_5 \in C$. Now $\{y_1, y_3, y_5, y_6\}$ is a vertex cover of G, and since C is minimal, $C = \{y_1, y_3, y_5, y_6\}$.
Case 2: $y_4 \in C$ and $y_6 \notin C$. In this case, $y_2, y_3 \in C$, and at least one vertex of $\{y_1, y_5\}$ and by minimality, only one is in C. Now since $\{y_2, y_3, y_4, y_i\}$, where $i \in \{1, 5\}$ is one of two vertices y_1 and y_5, is a cover of G, by minimality of C, $C = \{y_2, y_3, y_4, y_i\}$.
Case 3: $y_4, y_6 \in C$. In this case, at least one of two vertices y_1, y_5 and by minimality of C, only one is in C. Now if $y_5 \in C$, y_3 should be in C (because the edge $\{y_1, y_3\}$ should be covered). Also $y_2 \in C$ (because the edge $\{y_1, y_2\}$ should be covered). Now $\{y_2, y_3, y_5, y_4, y_6\}$ is a cover, and since C is minimal, $C = \{y_2, y_3, y_5, y_4, y_6\}$, which is a contradiction because y_6 can be removed. If $y_1 \in C$, at least one of y_2 and y_3, and by minimality only one, is in C. Now since $\{y_1, y_4, y_6, y_j\}$, where $j \in \{2, 3\}$ is one of two vertices y_2 and y_3, is a vertex cover, by minimality of C, $C = \{y_1, y_4, y_6, y_j\}$.

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