ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 789–805

Title:

On boundary value problem for fractional differential equations

Author(s):

M.R. Hamidi and N. Nyamoradi

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 789–805
Online ISSN: 1735-8515

ON BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

M.R. HAMIDI AND N. NYAMORADI*

(Communicated by Asadollah Aghajani)

Abstract. In this paper, we study the existence of solutions for a fractional boundary value problem. By using critical point theory and variational methods, we give some new criteria to guarantee that the problems have at least one solution and infinitely many solutions.
Keywords: Fractional differential equations, solutions, variational methods, Morse theory, fountain Theorem.
MSC(2010): Primary: 34A08; Secondary: 35b38, 35A15.

1. Introduction

The aim of this paper is to establish the existence of solutions to the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^{\alpha}\left(\left(1+\frac{|{}_0^cD_t^{\alpha}u(t)|^2}{\sqrt{1+|{}_0^cD_t^{\alpha}u(t)|^4}}\right){}_0^cD_t^{\alpha}u(t)\right)=\lambda f(t,u(t)), \quad t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where $\alpha\in(1/2,1]$, $f\in C([0,T]\times\mathbb{R})$ and $\lambda>0$ is a parameter.

Fractional differential equations have gained importance due to their numerous applications in many phenomena in various fields of science and engineering including diffusive transport akin to diffusion, fluid flow, rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see [4, 8, 9] and the references therein. It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for fractional differential equations**;** we refer to [5, 6, 10, 14] and the references therein.

Recently, several studies have been performed for classical differential equations substituted by its fractional derivatives. In particular, in [3, 13] **the**

Article electronically published on 30 June, 2017.
Received: 13 August 2015, Accepted: 19 February 2016.
*Corresponding author.

authors are studied the following Dirichlet's boundary value problem for fractional differential equations with impulses

$$(1.2)\qquad \begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), & t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j)) = \mu I_j(u(t_j)), & j = 1,2,\ldots,n, \\ u(0) = u(T) = 0, \end{cases}$$

where $\lambda \in (0,+\infty)$ and $\mu \in (0,+\infty)$ are two parameters; the analogue case to the problem in [11], but with the fractional derivative of order $\alpha \in (1/2, 1]$ instead of the integer derivative of order $\alpha = 1$.

Motivated by the papers [3,13], in this paper, we attempt to apply the Morse theory is the critical group and Fountain **t**heorem to study the existence and multiplicity of solutions of the problem (1.1).

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our results on existence of solutions to the problem (1.1).

2. Preliminaries and reminder about fractional calculus

In this section, we present some preliminaries and lemmas that are useful **in** the proof **of** the main results. For the convenience of the reader, we also present here the necessary definitions from fractional calculus theory. We refer the reader to [5, 7, 12] or other texts on basic fractional calculus.

Definition 2.1 (Left and right Riemann-Liouville fractional derivatives [7,12])**.** Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $0 \leq \gamma < 1$ for function f denoted by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{\gamma}f(t) &= \frac{d}{dt}{}_aD_t^{\gamma-1}f(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_a^t (t-s)^{-\gamma}f(s)ds\Big), \\ {}_tD_b^{\gamma}f(t) &= -\frac{d}{dt}{}_tD_b^{\gamma-1}f(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_t^b (s-t)^{-\gamma}f(s)ds\Big), \end{aligned}$$

for all $t \in [a,b]$.

Definition 2.2 (Left and right Caputo fractional derivatives [7])**.** Let $0 < \gamma < 1$ and $f \in AC([a,b])$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$\begin{aligned} {}_a^cD_t^{\gamma}f(t) &= {}_aD_t^{\gamma-1}f'(t) = \frac{1}{\Gamma(1-\gamma)}\Big(\int_a^t (t-s)^{-\gamma}f'(s)ds\Big), \\ {}_t^cD_b^{\gamma}f(t) &= -{}_tD_b^{\gamma-1}f'(t) = -\frac{1}{\Gamma(1-\gamma)}\Big(\int_t^b (s-t)^{-\gamma}f'(s)ds\Big), \end{aligned}$$

for all $t \in [a,b]$.

Note that when $\gamma = 1$, we have ${}_a^cD_t^1 f(t) = f'(t)$, ${}_t^cD_b^1 f(t) = -f'(t)$. In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t)$, **for** $t \in [a,b]$.

Let us recall that for any fixed $t \in [0,T]$ and $1 \le r < \infty$,

$$||u||_{L^r([0,t])} = \Big(\int_0^t |u(\xi)|^r d\xi\Big)^{\frac{1}{r}}, \quad ||u||_{L^r} = \Big(\int_0^T |u(\xi)|^r d\xi\Big)^{\frac{1}{r}},$$

and $||u||_\infty = \max_{t\in[0,T]} |u(t)|$.

Definition 2.3. Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by

$$E_0^{\alpha,p} = \overline{C_0^\infty([0,T])}^{\|\cdot\|_{\alpha,p}}$$

with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.1}$$

Lemma 2.4 ([5])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive separable Banach space.*

We can now give the following useful estimates, where we have employed the equivalent norm in X^α.

Lemma 2.5 ([5])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$||u||_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} ||{}_0^cD_t^\alpha u||_{L^p}. \tag{2.2}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$||u||_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} ||{}_0^cD_t^\alpha u||_{L^p}. \tag{2.3}$$

Then, according to (2.2), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$||u||_{\alpha,p} = ||{}_0^cD_t^\alpha u||_{L^p} = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{\frac{1}{p}}, \quad \forall\, u \in E_0^{\alpha,p}. \tag{2.4}$$

In what follows, we will treat problem (1.1) in the Hilbert space $X^\alpha = E_0^{\alpha,2}$ with the corresponding norm $||u||_\alpha = ||u||_{\alpha,2}$ which we defined in (2.4).

Furthermore, we need the following lemma:

Lemma 2.6 ([5])**.** *If the sequence $\{u_n\}$ converges weakly to u in X^α, i.e., $u_n \rightharpoonup u$. Then $u_n \to u$ in $C([0,T],\mathbb{R}^N)$, i.e., $||u_n - u||_\infty \to 0$, as $n \to \infty$.*

3. **Variational setting**

Consider the energy functional $\varphi : X^\alpha \to \mathbb{R}$ associated with problem (1.1), defined by

$$\varphi(u) = \frac{1}{2}\int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}\right)dt$$
$$-\lambda \int_0^T F(t,u(t))dt, \tag{3.1}$$

for all $u \in X^\alpha$, where $F(t,u) = \int_0^u f(t,s)ds$. Clearly φ is continuously differentiable on X^α i.e., $\varphi \in C^1(X^\alpha, \mathbb{R})$, and its Gateaux derivative is

$$\varphi'(u)v = \int_0^T \left(1 + \frac{|{}_0^cD_t^\alpha u(t)|^2}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}\right) {}_0^cD_t^\alpha u(t) {}_0^cD_t^\alpha v(t)dt$$
$$-\lambda \int_0^T f(t,u(t))v(t)dt, \tag{3.2}$$

for every $u, v \in X^\alpha$. It is clear that the critical points of φ are weak solutions of the problem (1.1).

We collect some basic notions and theorems to prove our main results. Let X be a Banach space with the norm $\|\cdot\|$. The main concept in Morse theory is the critical group $C_q(\varphi, u)$ for a C^1-functional $\varphi : X \to \mathbb{R}$ at an isolated critical point u with $\varphi(u) = c$ **which** is defined by

$$C_q(\varphi, u) := H_q(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}), \quad q \in \mathbb{N},$$

where U is any neighborhood of u, H_q is the singular relative homology with coefficients in an Abelian group G and $\varphi^c = \varphi^c((-\infty, c])$.

Also, we say that φ satisfies the (C) condition, if any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ is bounded and $(1+\|u_n\|)\|\varphi'(u_n)\|_{X^*} \to 0$ has a convergent subsequence; such a sequence is then called a (C) sequence. If φ satisfies the (C) condition and the critical values of φ are bonded from below by some $\nu > -\infty$, then the critical groups at infinity **as** introduced by Bartsch and Li [2] **is defined by**

$$C_q(\varphi, \infty) := H_q(X, \varphi^\nu). \tag{3.3}$$

In applications, we use critical groups to distinguish critical points, and use Morse inequalities to find unknown critical points.

Theorem 3.1 ([2])**.** *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the (C) condition and φ has only finitely many critical points, then*

(i) *if for some $q \in \mathbb{N}$ we have $C_q(\varphi, \infty) \neq 0$, then φ has a critical point u with $C_q(\varphi, u) \neq 0$;*
(ii) *let 0 be an isolated critical point of $\varphi(u)$. If for some $q \in \mathbb{N}$, we have $C_q(\varphi, 0) \neq C_q(\varphi, \infty)$, then $\varphi(u)$ has a nonzero critical point.*

Let $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, where X_k, for each $k \in \mathbb{N}$, **is a** finite-dimensional subspace of X, **and** assume that $Y_k = \bigoplus_{j=0}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. The functional Φ is said to satisfy the Palais-Smale condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset X$ such that $\{\Phi(u_j)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'(u_j) \to 0$ as $j \to +\infty$ has a convergent subsequence.

Let us recall, for reader's convenience, a critical point result as follow:

Theorem 3.2 ([1, 15])**.** *Suppose that the functional $\Phi \in C^1(X, \mathbb{R})$ is even. If, for every $k \in \mathbb{N}$, there exist $\varrho_k > r_k > 0$ such that*

(F1) $a_k := \max_{u\in Y_k, \|u\|=\varrho_k} \Phi(u) \leq 0$,

(F2) $b_k := \inf_{u\in Z_k, \|u\|=r_k} \Phi(u) \to +\infty$ *as* $k \to \infty$,

(F3) Φ *satisfies the Palais-Smale condition,*

t*hen Φ possesses an unbounded sequence of critical values.*

Now, we state our main results.

Theorem 3.3. *If $\lambda > 0$, and f satisfies the condition*

(H1) *$f \in C([0,T] \times \mathbb{R}; \mathbb{R})$ with $f(t,0) = 0$ and there is a constant $C > 0$ such that*

$$|f(t,u)| \leq C(1 + |u|^{q-1}), \quad \forall\, (t,u) \in [0,T] \times \mathbb{R}, \quad 1 < q < 2. \tag{3.4}$$

Then, the problem (1.1) *has a solution.*

Theorem 3.4. *Suppose that the following conditions hold:*

(H1*) *$f \in C([0,T] \times \mathbb{R}; \mathbb{R})$ with $f(t,0) = 0$ and there is a constant $C > 0$ such that*

$$|f(t,u)| \leq C(1 + |u|^{q-1}), \quad \forall\, (t,u) \in [0,T] \times \mathbb{R}, \quad 1 < q < +\infty; \tag{3.5}$$

(H2) *$\lim_{|u|\to+\infty} \frac{F(t,u)}{|u|^2} = +\infty$ uniformly on $[0,T]$ and there exists $\Pi > 0$ such that $F(t,u) \geq -\Pi|u|^2$ for $(t,u) \in [0,T] \times \mathbb{R}$;*

(H3) *There exists $\theta > 2$ such that $\theta(f(t,u)u - 2F(t,u)) \geq f(t,su)su - 2F(t,su) \geq 0$ for all $(t,u) \in [0,T] \times \mathbb{R}$ and $s \in [0,1]$;*

(H4) *There exists $\delta > 0$ such that $F(t,u) \leq 0$ for all $t \in [0,T]$, $|u| < \delta$.*

Then, for any $\lambda > 0$ the problem (1.1) *has at least one nontrivial solution.*

Theorem 3.5. *Suppose that the following conditions hold:*

(H1**) *$f \in C([0,T] \times \mathbb{R}; \mathbb{R})$ with $f(t,0) = 0$ and there is a constant $C > 0$ such that*

$$|f(t,u)| \leq C(1 + |t|^{q-1}), \quad \forall\, (t,u) \in [0,T] \times \mathbb{R}, \quad 2 < q < +\infty; \tag{3.6}$$

(H3*) *There exist two constants $M > 0$ and $\theta_1 > 2$ such that*

$$0 < \theta_1 F(t,u) \leq uf(t,u), \quad |u| \geq M, t \in [0,T];$$

(H5) $f(t,-u) = -f(t,u)$, $t \in [0,T]$, $u \in \mathbb{R}$.

Then, for any $0 < \lambda < \frac{q}{2}$ *the problem* (1.1) *has infinitely many solutions.*

Proof of Theorem 3.3. From (2.2) and (3.4), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T \Big(|{}_0^cD_t^\alpha u(t)|^2 + \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}\Big)dt - \lambda\int_0^T F(t,u(t))dt \\
&\geq \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt - \lambda C\int_0^T |u(t)|^q dt - \lambda C_0 \\
&\geq \frac{1}{2}\|u\|^2 - \lambda C\Big(\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)^q \|u\|^q - \lambda C_0 \to \infty, \quad \text{as } \|u\| \to \infty.
\end{aligned}$$

As φ is weakly lower semicontinuous, φ has a minimum point u in X^α and u is a weak solution of (1.1)**, this** completes the proof. □

Proof of Theorem 3.4. We **divide** the proof **in to** the following two steps.

Step 1. We will prove that the functional φ satisfies **the** (C) condition. To this end, let $\{u_n\}$ be a (C) sequence of φ, that is,

$$\{\varphi(u_n)\} \quad \text{is bounded and} \quad (1+\|u_n\|)\|\varphi'(u_n)\|_{X^*} \to 0.$$

We first claim that the sequence $\{u_n\}$ is bounded in X^α. If $\{u_n\}$ is unbounded, by passing to a subsequence if necessary, we may assume that for some $c \in \mathbb{R}$, $\|u_n\| \to +\infty$ and

$$\varphi(u_n) \to c, \qquad \varphi'(u_n)u_n \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Hence, one can get

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\frac{1}{2}\int_0^T \left(\sqrt{1+|{}_0^cD_t^\alpha u_n(t)|^4} - \frac{|{}_0^cD_t^\alpha u_n(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u_n(t)|^4}}\right)dt \right.\\
&\qquad \left. -\lambda\int_0^T F(t,u_n(t))dt + \frac{\lambda}{2}\int_0^T f(t,u_n(t))u_n(t)dt\right\} \\
&\qquad = \lim_{n\to\infty}\left[\varphi(u_n) - \frac{1}{2}\varphi'(u_n)u_n\right] = c.
\end{aligned}$$

Let $w_n = \frac{u_n}{\|u_n\|}$, we may assume that

$$\begin{cases}
w_n \rightharpoonup w, & \text{weakly in } X^\alpha, \\
w_n \to w, & \text{strongly in } L^p([0,T]), \quad 1 < p < +\infty, \\
w_n \to w, & \text{a.e. in } [0,T].
\end{cases}$$

Let $\Theta := \{t \in [0,T]: \ w(t) \neq 0\}$, then

$$\lim_{n\to\infty} w_n(t) = \lim_{n\to\infty}\frac{u_n(t)}{\|u_n\|} = w(t) \neq 0 \quad \text{in } \Theta,$$

which implies that $|u_n(t)| \to +\infty$ as $n \to \infty$. By the condition (H2), one has

$$\frac{F(t, u_n(t))}{|u_n(t)|^2} \to +\infty, \quad t \in \Theta \quad \text{as} \;\; n \to \infty,$$

which shows that

$$\frac{F(t, u_n(t))}{|u_n(t)|^2} |w_n(t)|^2 \to +\infty, \quad t \in \Theta \quad \text{as} \;\; n \to \infty.$$

Note that the Lebesgue measure of Θ is positive, using the Fatou's Lemma, we have

$$\int_{\{t \in [0,T]:\, w(t) \neq 0\}} \frac{F(t, u_n(t))}{|u_n(t)|^2} |w_n(t)|^2 dt \to +\infty, \quad \text{as} \;\; n \to \infty. \tag{3.8}$$

On the other hand, from (3.7), by $\varphi(u_n) \to c$, we obtain

$$c = \varphi(u_n) + o(1). \tag{3.9}$$

owing to (3.9), we have that

$$\begin{aligned} c &= \varphi(u_n) + o(1) \\ &= \frac{1}{2} \int_0^T \left(|{}_0^c D_t^\alpha u_n(t)|^2 + \sqrt{1 + |{}_0^c D_t^\alpha u_n(t)|^4} \right) dt \\ &\qquad\qquad -\lambda \int_0^T F(t, u_n(t)) dt + o(1) \\ &\leq \frac{1}{2} \int_0^T \left(|{}_0^c D_t^\alpha u_n(t)|^2 + 1 + |{}_0^c D_t^\alpha u_n(t)|^2 \right) dt \\ &\qquad\qquad -\lambda \int_0^T F(t, u_n(t)) dt + o(1) \\ &\leq \|u_n\|^2 + \frac{T}{2} - \lambda \int_0^T F(t, u_n(t)) dt + o(1), \end{aligned} \tag{3.10}$$

Thus

$$\frac{1}{\lambda} \|u_n\|^2 + \frac{T}{2\lambda} - \frac{c}{\lambda} + o(1) \;\; \geq \;\; \int_0^T F(t, u_n(t)) dt. \tag{3.11}$$

It follows from (3.11) and condition (H2) that

$$\begin{aligned}\frac{1}{\lambda}+\frac{T}{2\lambda\|u_n\|^2}-\frac{c}{\lambda\|u_n\|^2}+\frac{o(1)}{\|u_n\|^2} &\geq \int_0^T\frac{F(t,u_n(t))}{\|u_n\|^2}dt\\ &= \left(\int_{w\neq 0}+\int_{w=0}\right)\frac{F(t,u_n(t))}{|u_n(t)|^2}|w_n(t)|^2dt\\ &\geq \int_{w\neq 0}\frac{F(t,u_n(t))}{|u_n(t)|^2}|w_n(t)|^2dt\\ &\quad -\Pi\int_{w=0}|w_n(t)|^2dt, \end{aligned} \tag{3.12}$$

which contradicts (3.8). This shows that $|\Theta|=0$ and then $w(t)=0$ a.e. in $[0,T]$. Since $\varphi(tu_n)$ is continuous in $t\in[0,1]$, for each n there exists $t_n\in[0,1]$ $(n=1,2,\ldots)$, such that

$$\varphi(t_nu_n):=\max_{t\in[0,1]}\varphi(tu_n). \tag{3.13}$$

For any $k>1$, set $v_n=\sqrt{k}w_n$. Fix k, since $w_n\to 0$ in $L^p([0,T])$ for all $1<p<+\infty$ and $w_n(t)\to 0$ a.e. $t\in[0,T]$ as $n\to+\infty$, owing to the condition (H1) and **the** Lebsgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^T F(t,v_n(t))dt=0. \tag{3.14}$$

Recall that $\|u_n\|\to+\infty$ as $n\to\infty$. Thus, we have $\frac{\sqrt{k}}{\|u_n\|}\in(0,1)$ for n large enough. So, by (3.14), we can get

$$\begin{aligned}\varphi(t_nu_n) &\geq \varphi(v_n)\\ &= \frac{1}{2}\int_0^T\left(|{}_0^cD_t^\alpha v_n(t)|^2+\sqrt{1+|{}_0^cD_t^\alpha v_n(t)|^4}\right)dt-\lambda\int_0^T F(t,v_n(t))dt\\ &\geq \int_0^T|{}_0^cD_t^\alpha v_n(t)|^2dt-\lambda\int_0^T F(t,v_n(t))dt\\ &= k\|w_n\|^2-\lambda\int_0^T F(t,v_n(t))dt\\ &= k \end{aligned} \tag{3.15}$$

for any n large enough. From (3.15) and letting $n\to+\infty$, we have $\varphi(t_nu_n)\to+\infty$. Also, by (3.7), one can get $\varphi'(t_nu_n)t_nu_n\to 0$. Hence using (3.8) and

(H3), for all n large enough, we obtain

$$\begin{aligned}\frac{1}{\theta}\varphi(t_nu_n) &= \frac{1}{\theta}\left(\varphi(t_nu_n)-\frac{1}{2}\varphi'(t_nu_n)t_nu_n\right)\\ &\leq \frac{1}{2}\int_0^T\left(\sqrt{1+|{}_0^cD_t^\alpha t_nu_n(t)|^4}-\frac{|{}_0^cD_t^\alpha t_nu_n(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha t_nu_n(t)|^4}}\right)dt\\ &\quad +\frac{\lambda}{2}\int_0^T\frac{f(t,t_nu_n(t))t_nu_n(t)-2F(t,t_nu_n(t))}{\theta}dt\\ &\leq \frac{1}{2}\int_0^T\left(\sqrt{1+|{}_0^cD_t^\alpha u_n(t)|^4}-\frac{|{}_0^cD_t^\alpha u_n(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u_n(t)|^4}}\right)dt\\ &\quad +\frac{\lambda}{2}\int_0^T\Big(f(t,u_n(t))u_n(t)-2F(t,u_n(t))\Big)dt\to c,\end{aligned}\tag{3.16}$$

as $n\to+\infty$.
This contradicts $\varphi(t_nu_n)\to+\infty$. Therefore we have proved that $\{u_n\}$ is bounded in X^α, so by passing to a subsequence if necessary, we may assume that

$$\begin{cases} u_n\rightharpoonup u, & \text{weakly in } X^\alpha,\\ u_n\to u, & \text{strongly in } L^p([0,T]),\quad 1<p<+\infty.\end{cases}\tag{3.17}$$

Now, we **c**onsider the operator $J':X^\alpha\to(X^\alpha)^*$ which is defined by

$$\langle J'(u),v\rangle=\int_0^T\left(1+\frac{|{}_0^cD_t^\alpha u(t)|^2}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}\right){}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)dt,\tag{3.18}$$

for all $u,v\in X^\alpha$.

We note that the linear operator J' is strongly monotone, that is

$$\langle J'(u)-J'(v),u-v\rangle\geq c\|u-v\|^2,\quad\forall\, u,v\in X^\alpha.$$

In fact, then, for all $u, v \in X^\alpha$, we have

$$\begin{aligned}
&\langle J'(u)-J'(v), u-v\rangle \\
&= \int_0^T \left(1+\frac{|{}_0^cD_t^\alpha u(t)|^2}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}\right){}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha(u(t)-v(t))dt \\
&- \int_0^T \left(1+\frac{|{}_0^cD_t^\alpha v(t)|^2}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}\right){}_0^cD_t^\alpha v(t){}_0^cD_t^\alpha(u(t)-v(t))dt \\
\geq\ & \int_0^T {}_0^cD_t^\alpha(u(t)-v(t)){}_0^cD_t^\alpha(u(t)-v(t))dt \\
&+ \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt - \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^3|{}_0^cD_t^\alpha v(t)|}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt \\
&+ \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt - \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^3|{}_0^cD_t^\alpha u(t)|}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt, \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt &= \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}\frac{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt \\
&= \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{1+|{}_0^cD_t^\alpha u(t)|^4}\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt \\
&= \int_0^T \left(1-\frac{1}{1+|{}_0^cD_t^\alpha u(t)|^4}\right)\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt \\
&\geq \int_0^T \left(1-\frac{1}{1+c_0}\right)\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt, \\
&\geq C_0\int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt,
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt \\
&\qquad\qquad + \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt \\
\geq\ & C_1\Big(\int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt \\
&+ \int_0^T \sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}dt\Big), \qquad (3.20)
\end{aligned}$$

for $|{}_0^cD_t^\alpha u(t)| \geq c_0 > 0$, **and** $|{}_0^cD_t^\alpha v(t)| \geq c_0' > 0$.
Also, one has

$$\begin{aligned}
&-\int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^3|{}_0^cD_t^\alpha v(t)|}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt - \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^3|{}_0^cD_t^\alpha u(t)|}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt \\
&\quad\geq -\frac{1}{2}\int_0^T \left(\frac{|{}_0^cD_t^\alpha u(t)|^3}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}} + |{}_0^cD_t^\alpha v(t)|^2\right)dt \\
&\qquad -\frac{1}{2}\int_0^T \left(\frac{|{}_0^cD_t^\alpha v(t)|^3}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}} + |{}_0^cD_t^\alpha u(t)|\right)dt. \qquad (3.21)
\end{aligned}$$

Then, by (3.20) and (3.21), we have

$$\begin{aligned}
&+\int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt - \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^3|{}_0^cD_t^\alpha v(t)|}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt \\
&+\int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt - \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^3|{}_0^cD_t^\alpha u(t)|}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt \\
&\quad\geq \int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt + \int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt \\
&\qquad -\frac{1}{2}\int_0^T \left(\frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}} + |{}_0^cD_t^\alpha v(t)|\right)dt \\
&\qquad -\frac{1}{2}\int_0^T \left(\frac{|{}_0^cD_t^\alpha v(t)|^3}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}} + |{}_0^cD_t^\alpha u(t)|\right)dt \\
&\quad\geq \frac{1}{2}\int_0^T \frac{|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}}dt + \frac{1}{2}\int_0^T \frac{|{}_0^cD_t^\alpha v(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}}dt \\
&\qquad -\frac{1}{2}\int_0^T |{}_0^cD_t^\alpha v(t)|dt - \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|dt \\
&\quad\geq \frac{C_1}{2}\Big(\int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt + \int_0^T \sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}dt\Big) \\
&\qquad -\frac{1}{2}\int_0^T \sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}dt - \frac{1}{2}\int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt \\
&\quad\geq \frac{C_1}{2}\Big(\int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt + \int_0^T \sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}dt \\
&\qquad -\int_0^T \sqrt{1+|{}_0^cD_t^\alpha v(t)|^4}dt - \int_0^T \sqrt{1+|{}_0^cD_t^\alpha u(t)|^4}dt\Big). \qquad (3.22)
\end{aligned}$$

Therefore, it follows from (3.19) and (3.22) that

(3.23) $$\langle J'(u) - J'(v), u - v\rangle \geq \|u - v\|^2 > \frac{1}{2}\|u - v\|^2.$$

Moreover, the operator J' possesses the property of type $(\mathrm{S})_+$, that is

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to+\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \leq 0 \quad \text{implies} \quad u_n \to u.$$

Clearly, the strong monotonicity property implies that J' satisfies $(\mathrm{S})_+$.

Consequently, it suffices to prove the following fact

(3.24) $$\limsup_{n\to+\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \leq 0.$$

Indeed, from (H1*) and the Hölder inequality, we get

$$\begin{aligned}\left|\int_0^T f(t,u_n)(u_n - u)dt\right| &\leq \int_0^T |f(t,u_n)||u_n - u|dt \\ &\leq C\int_0^T |1 + |u_n|^{q-1}||u_n - u|dt \\ &\leq 2C\|1 + |u_n|^{q-1}\|_{L^{q'}}\|u_n - u\|_{L^q} \to 0,\end{aligned}$$

as, $n \to +\infty$, which implies that

(3.25) $$\lim_{n\to+\infty}\int_0^T f(t,u_n)(u_n - u)dt = 0.$$

On the other hand, by (3.7), one can get $\langle\varphi'(u_n), u_n\rangle = o(1)$, which implies that

(3.26) $$\langle\varphi'(u_n), u_n - u\rangle \to 0, \quad \text{as } n \to +\infty.$$

Combining (3.25) and (3.26), we have

$$\langle J'(u_n), u_n - u\rangle = \lambda\int_0^T f(t,u_n)(u_n - u)dt + \langle\varphi'(u_n), u_n - u\rangle \to 0,$$

as, $n \to +\infty$. Therefore

$$\begin{aligned}\limsup_{n\to+\infty}\langle J'(u_n) - J'(u), u_n - u\rangle &\leq \limsup_{n\to+\infty}\langle J'(u_n), u_n - u\rangle \\ &\quad - \liminf_{n\to+\infty}\langle J'(u), u_n - u\rangle \leq 0.\end{aligned}$$

Thus, (3.24) holds. Since J' is of type $(\mathrm{S})_+$, so we obtain $u_n \to u$ in X^α. Hence, the functional φ satisfies **the** (C) condition on X^α.

Step 2. We claim that $C_q(\varphi, \infty) = 0$.

To this end, we will prove that the functional φ satisfies **the** (C) **condition**, and then we can **obtain** the value of $C_q(\varphi,\infty)$. Suppose that S **is** the boundary of the unit disc in X^α, i.e., $S=\{u\in X^\alpha;\ \|u\|=1\}$. Thus for $u\in S$, it follows from the condition (H2) and (3.1) that

$$\begin{aligned}\varphi(xu) &= \frac{1}{2}\int_0^T \Big(|{}_0^cD_t^\alpha xu(t)|^2+\sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}\Big)dt-\lambda\int_0^T F(t,xu(t))dt\\ &\le \frac{1}{2}\int_0^T \Big(|{}_0^cD_t^\alpha xu(t)|^2+1+|{}_0^cD_t^\alpha xu(t)|^2\Big)dt-\lambda\int_0^T F(t,xu(t))dt\\ &\le x^2-\lambda C_2x^2\int_0^T|u(t)|dt+C_3\to-\infty,\quad \text{as } t\to+\infty, \end{aligned}\tag{3.27}$$

where C_2 and C_3 are positive constants. Choose

$$a<\min\left\{\inf_{\|u\|\le 1}\varphi(u),0\right\}.\tag{3.28}$$

So for any $u\in S$, there exists $x_0>1$ such that $\varphi(x_0u)\le a$. Thus it follows from (H3) that

$$f(t,u)u-2F(t,u)\ge 0,\quad \forall\ (t,u)\in[0,T]\times\mathbb{R}.\tag{3.29}$$

Now, if

$$\begin{aligned}\varphi(xu) &= \frac{1}{2}\int_0^T \Big(|{}_0^cD_t^\alpha xu(t)|^2+\sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}\Big)dt\\ &\qquad -\lambda\int_0^T F(t,xu(t))dt\le a,\end{aligned}\tag{3.30}$$

then combined with (3.29) yields that

$$\begin{aligned}\frac{d}{dx}\varphi(xu) &= \int_0^T\Big(x|{}_0^cD_t^\alpha u(t)|^2+\frac{x^3|{}_0^cD_t^\alpha u(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}}\Big)dt\\ &\qquad -\lambda\int_0^T u(t)f(t,xu(t))dt\\ &= \frac{1}{x}\Big(\int_0^T\Big(|{}_0^cD_t^\alpha xu(t)|^2+\frac{|{}_0^cD_t^\alpha xu(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}}\Big)dt\\ &\qquad -\lambda\int_0^T xu(t)f(t,xu(t))dt\Big)\\ &\le \frac{1}{x}\Big(\int_0^T\Big(|{}_0^cD_t^\alpha xu(t)|^2+\frac{1+|{}_0^cD_t^\alpha xu(t)|^4}{\sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}}\Big)dt\\ &\qquad -\lambda\int_0^T xu(t)f(t,xu(t))dt\Big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x}\Big(\int_0^T \Big(|{}_0^cD_t^\alpha xu(t)|^2 + \sqrt{1+|{}_0^cD_t^\alpha xu(t)|^4}\Big)dt \\
&\qquad\qquad -\lambda\int_0^T xu(t)f(t,xu(t))dt\Big) \\
&\le \frac{1}{x}\Big(2a+\lambda\int_0^T 2F(t,xu(t))dt \\
&\qquad\qquad -\lambda\int_0^T xu(t)f(t,xu(t))dt\Big) \\
&= \frac{1}{x}\left(2a-\lambda\int_0^T (xu(t)f(t,xu(t))-2F(t,xu(t)))dt\right) \\
&\le \frac{1}{x}2a<0. \qquad (3.31)
\end{aligned}
$$

Hence, by the implicit function theorem, there exists a unique $\chi \in C(S,\mathbb{R})$ such that $\varphi(\chi(u)u) = a$.

It follows from **a** similar argument **as** in [16] **that we can** construct a strong deformation retract from X^α to φ^a and the function χ that

$$C_q(\varphi,\infty) = H_q(X^\alpha,\varphi^a) \cong H_q(X^\alpha, X^\alpha \setminus \{0\}) \cong 0.$$

Owing to condition (H4), the function 0 is a local minimizer of φ. Thus, $C_q(\varphi,0)=\delta_{q,0}\mathbb{Z}$. Therefore**,** using Theorem 3.1, the problem (1.1) has at least one nontrivial solution in X^α. This complete**s** the proof. □

Proof of Theorem 3.5. Since X^α is a separable Banach space, then there are $\{e_j\}\subset X^\alpha$ and $\{e_j^*\}\subset (X^\alpha)^*$ such that $X^\alpha = \overline{\text{span}\{e_j : j=1,2,\ldots\}}$, $(X^\alpha)^* = \overline{\text{span}\{e_j^* : j=1,2,\ldots\}}$ and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i=j,\\ 0, & i\neq j.\end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$, **and** $Z_k = \bigoplus_{j=k}^\infty X_j$. Also, we know that $\Phi \in C^1(X^\alpha,\mathbb{R})$ is even. Let us prove that the functionals Φ satisfy the required conditions in Theorem 3.2.

We firstly verify the condition (F2) in Theorem 3.2. Let

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_{L^q},$$

then $\beta_k \to 0$ as $k\to+\infty$. Obviously, $0<\beta_{k+1}\le\beta_k$, so we assume that $\beta_k \to \overline{\beta}\ge 0$, **as** $k\to+\infty$. For every $k\ge 0$, there exists $u_k\in Z_k$ such that $\|u_k\|=1$ and $\|u_k\|_{L^q} > \frac{\beta_k}{2}$. Then, up to a subsequence, we may assume that $u_k \rightharpoonup u$ weakly in X^α. Noticing that Z_k is a closed subspace of X^α, by Mazur's theorem, we have $u\in Z_k$, for all $k>\widetilde{n}$. Consequently, we get

$u \in \bigcap_{k=\tilde{n}}^{\infty} Z_k = \{0\}$, which implies $u_k \rightharpoonup 0$ weakly in X^α. So, we have $u_k \to 0$ in $L^q(\mathbb{R}, \mathbb{R}^n)$. Thus we have proved that $\overline{\beta} = 0$. Hence we get $\beta_k \to 0$.

We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that

$$b_k = \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty, \quad \text{as } k \to \infty,$$
$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0, \quad \text{as } k \to \infty. \tag{3.32}$$

In view of (H1*), for any $u \in Z_k$, $\|u\| = r_k = (qC\beta_k^q)^{\frac{1}{2-q}}$, one can get

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + \sqrt{1 + |{}_0^cD_t^\alpha u(t)|^4}\right)dt - \lambda \int_0^T F(t, u(t))dt \\
&\geq \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt - \lambda C \int_0^T |u(t)|^q dt - \lambda c_1 \\
&= \frac{1}{2}\|u\|^2 - \lambda C \|u\|_{L^q}^q - \lambda c_1 \\
&\geq \frac{1}{2}\|u\|^2 - \lambda C \beta_k^q \|u\|^q - \lambda c_1 \\
&= \frac{1}{2}(qC\beta_k^q)^{\frac{2}{2-q}} - \lambda C \beta_k^q (qC\beta_k^q)^{\frac{q}{2-q}} - \lambda c_1 \\
&= \left(\frac{1}{2} - \frac{\lambda}{q}\right)(qC\beta_k^q)^{\frac{2}{2-q}} - \lambda c_1 \to \infty, \quad \text{as } k \to \infty
\end{aligned} \tag{3.33}$$

because $0 < \lambda < \frac{q}{2}$, $2 < q$ and $\beta_k \to 0$.

We now verify the condition (F1) in Theorem 3.2. Since $\dim Y_k < \infty$ and all norms of a finite-dimensional normed space are equivalent, there exists a constant $M_0 > 0$ such that

$$\|u\| \leq M_0 \|u\|_{L^{\theta_1}}, \quad \forall\, u \in Y_k. \tag{3.34}$$

(H3*) implies that

$$F(t, u) \geq c_2 |u|^{\theta_1} - c_3. \tag{3.35}$$

Then, for any $u \in Y_k$, in view of (3.34) and (3.35), one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + \sqrt{1 + |{}_0^cD_t^\alpha u(t)|^4}\right)dt - \lambda \int_0^T F(t, u(t))dt \\
&= \frac{1}{2}\int_0^T \left(|{}_0^cD_t^\alpha u(t)|^2 + 1 + |{}_0^cD_t^\alpha u(t)|^2\right)dt - \lambda c_2 \int_0^T |u(t)|^{\theta_1} dt - \lambda c_3 T \\
&\leq \|u\|^2 - \lambda c_2 \|u\|_{L^{\theta_1}}^{\theta_1} - \lambda c_3 T + \frac{T}{2} \\
&\leq \|u\|^2 - \frac{\lambda c_2}{M_0^{\theta_1}} \|u\|^{\theta_1} - \lambda c_3 T + \frac{T}{2}.
\end{aligned}$$

Hence, we can choose $\|u\| = \rho_k$ large enough ($\rho_k > r_k > 0$) such that

$$a_k = \max_{u \in Y_k, \|u\|=\rho_k} \varphi(u) \le 0.$$

Finally, we prove that φ satisfies the Palais-Smale condition. Let $\{u_n\}_{n\in\mathbb{N}} \subset X^\alpha$ be a Palais-Smale sequence, that is, $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. Then there exists a constant $M_1 > 0$ such that

$$|\varphi(u_n)| \le M_1, \quad \|\varphi'(u_n)\|_{(X^\alpha)^*} \le M_1 \tag{3.36}$$

for every $n \in \mathbb{N}$, where $(X^\alpha)^*$ is the dual space of X^α.

We now prove that $\{u_n\}$ is bounded in X^α.

$$\begin{aligned} M_1 \ge \varphi(u_n) &= \frac{1}{2}\int_0^T \left(|{}_0^cD_t^\alpha u_n(t)|^2 + \sqrt{1 + |{}_0^cD_t^\alpha u_n(t)|^4}\right)dt \\ &\quad -\lambda \int_0^T F(t, u_n(t))dt \\ &\ge \frac{1}{2}\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2 dt - \lambda \int_0^T \frac{u_n(t)}{\theta_1} f(t, u_n(t))dt - c \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta_1}\right)\int_0^T |{}_0^cD_t^\alpha u_n(t)|^2 dt \\ &\quad + \frac{1}{\theta_1}\int_0^T \left[|{}_0^cD_t^\alpha u_n(t)|^2 - \lambda u_n(t) f(t, u_n(t))\right] dt - c \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta_1}\right)\|u_n\|^2 \\ &\quad + \frac{1}{\theta_1}\left[\varphi'(u_n)u_n - \int_0^T + \frac{|{}_0^cD_t^\alpha u_n(t)|^2}{\sqrt{1 + |{}_0^cD_t^\alpha u_n(t)|^4}} dt\right] - c \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta_1}\right)\|u_n\|^2 + \frac{1}{\theta_1}\|\varphi'(u_n)\|\|u_n\| - c. \end{aligned}$$

So, $\{u_n\}$ is bounded, and by similar methods **as** in the proof of Theorem 3.4, $\{u_n\}$ has a convergent subsequence. So φ satisfies the Palais-Smale condition. Therefore using Theorem 3.2, the problem (1.1) has infinite many solutions in X^α. The proof is completed. □

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and suggestions.

References

[1] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.* **20** (1993), no. 10, 1205–1216.

[2] T. Bartsch and S.J. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, *Nonlinear Anal.* **28** (1997), no. 3, 419–441.
[3] G. Bonanno, R. Rodriguez-Löpez and S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.* **17** (2014), no. 3, 717–744.
[4] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal.* **33** (1998), no. 2, 181–186.
[5] F. Jiao and Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.* **62** (2011), no. 3, 1181–1199.
[6] F. Jiao and Y. Zhou, Existence results for fractional boundary value problem via critical point theory, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22** (2012), no. 4, 1–17.
[7] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Elsevier Science, Amsterdam, 2006.
[8] A.A. Kilbas and J.J. Trujillo, Differential equations of fractional order: methods, results and problems I, *Appl. Anal.* **78** (2001), no. 1-2, 153–192.
[9] A.A. Kilbas and J.J. Trujillo, Differential equations of fractional order: methods, results and problems II, *Appl. Anal.* **81** (2002), no. 2, 435–493.
[10] Y.N. Li, H.R. Sun and Q.G. Zhang, Existence of solutions to fractional boundary-value problems with a parameter, *Electron. J. Differential Equations* **2013** (2013), no. 141, 12 pages.
[11] J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.* **10** (2009), no. 2, 680–690.
[12] I. Podlubny, Fractional Differential Equations, Academic Press, NewYork, 1999.
[13] R. Rodriguez-Löpez and S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.* **17** (2014), no. 4, 1016–1038.
[14] C. Torres, Existence of solutions for fractional Hamiltonian systems, *Electron. J. Differential Equations* **2013** (2013), no 259, 12 pages.
[15] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
[16] Z.Q. Wang, On a superlinear elliptic equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8** (1991), no. 1, 43–57.

(Mohammad Rassol Hamidi) Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.
E-mail address: mohammadrassol.hamidi@yahoo.com

(Nemat Nyamoradi) Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.
E-mail address: nyamoradi@razi.ac.ir; neamat80@yahoo.com.