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ALMOST VALUATION RINGS

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Abstract. The aim of this paper is to generalize the notion of almost valuation domains to arbitrary commutative rings. Also, we consider relations between almost valuation rings and pseudo-almost valuation rings. We prove that the class of almost valuation rings is properly contained in the class of pseudo-almost valuation rings. Among the properties of almost valuation rings, we show that a quasilocal ring R with regular maximal ideal M is a pseudo-almost valuation ring if and only if $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$. Furthermore, we show that pseudo-almost valuation rings are precisely the pullbacks of almost valuation rings.

Keywords: Strongly prime ideal, almost valuation domain, Pseudo-almost valuation ring.
MSC(2010): Primary: 13A18; Secondary: 13G05, 13F30, 13F05, 13A15.

1. **Introduction**

All rings considered in this paper are commutative with identity elements and all modules are unital. In [12], Hedstrom and Houston introduced a class of integral domains which is closely related to the class of valuation domains. An integral domain R with quotient field K is called a *pseudo-valuation domain* (PVD) when each prime ideal P of R is a *strongly prime ideal*, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is presented by [8].

In [10], the study of pseudo-valuation domains was generalized to arbitrary commutative rings (possibly with nonzero zero-divisors) in the following way: A prime ideal P of a ring R is said to be *strongly prime* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring* (PVR) if each prime ideal of R is strongly prime. A pseudo-valuation ring is necessarily quasi-local ([10, Lemma 1(b)]). Also, an integral

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domain is a pseudo-valuation ring if and only if it is a pseudo-valuation domain, see [1, Proposition 3.1], [2, Proposition 4.2] and [6, Proposition 3]. For additional characterizations of PVRs, see [3] and [7]. Badawi and Houston in [11] gave another generalization of PVDs. Anderson and Zaffrullah [5], introduced and studied the notion of an almost valuation domain which is another generalization of valuation domain. An integral domain R is called an *almost valuation domain* (AVD) if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in R$ or $x^{-n} \in R$. In [9], Badawi introduced a new class of integral domains closely related to AVD's, that is, the class of pseudo-almost valuation domains. A prime ideal P of an integral domain R is called a *pseudo-strongly prime ideal* if whenever $x, y \in K$ and $xyP \subseteq P$, there is an integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If each prime ideal of R is a pseudo-strongly prime ideal, then R is called a *pseudo-almost valuation domain* (PAVD). The same author showed that the class of AVD's is properly contained in the class of PAVD's and that PAVD's are precisely the pullbacks of AVD's. In [14], the generalization of the pseudo-almost valuation domains to arbitrary commutative rings (possibly with nonzero zero-divisors) is considered as follows. A prime ideal P of a ring R is said to be a *pseudo-strongly prime ideal* if for every $a, b \in R$, there is an integer $m \geq 1$ such that either $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$. A ring R is called a *pseudo-almost valuation ring* (PAVR) if each maximal ideal of R is pseudo-strongly prime. A pseudo-almost valuation ring is necessarily quasi-local. Also, an integral domain R is a pseudo-almost valuation ring if and only if R is a pseudo-almost valuation domain ([14, Proposition 2.7]).

In this paper, we generalize the notion of almost valuation domains as defined in [5] to arbitrary commutative rings. We show that if R is an almost valuation ring, then R is a pseudo-almost valuation ring. Also, we consider relations between almost valuation rings and pseudo-almost valuation rings.

This paper is organized as follows. In the second section, we introduce the class of almost valuation rings. We prove that every almost valuation ring is a quasi-local ring with linearly ordered prime ideals. Also, we show that every almost valuation ring is a pseudo-almost valuation ring. Furthermore, if R is an almost valuation ring, then the localization R at each prime ideal P is an almost valuation ring.

In the third section, we study almost valuation overrings of PAVRs. We show that if a ring R contains a regular principal pseudo-strongly prime ideal P, then R is an almost valuation ring with maximal ideal P. Also, we show that pseudo-almost valuation rings are precisely the pullbacks in the category of commutative rings (with 1) of diagrams of the form

$$\begin{array}{c} V \\ \downarrow \\ H \to F. \end{array}$$

Here V is an almost valuation ring having maximal ideal $Rad_V(M)$ for some ideal M of V, $F = V/M$, the vertical map is the canonical surjection, H is a field contained in F, and the horizontal map is inclusion. Also, we prove that if P is a regular pseudo-strongly prime ideal of R, then $(P : P)$ is an almost valuation ring. Therefore, using [14] and the results of this paper, we have the following implications, none of which is reversible:

$$\begin{tikzcd}
 & \text{valuation ring} \arrow[dl] \arrow[dr] & \\
\text{almost valuation ring} \arrow[dr] & & \text{pseudo-valuation ring} \arrow[dl] \\
 & \text{pseudo-almost valuation ring} \arrow[d] & \\
 & \text{quasi-local ring with linearly ordered prime ideals} &
\end{tikzcd}$$

We close the introduction by the following result on pseudo-almost valuation rings.

Proposition 1.1. *Let R be a PAVR with maximal ideal M. Then the nilradical N of R is a pseudo-strongly prime ideal.*

Proof. Let $a, b \in R$ such that $a^nR \not\subseteq b^nR$ for every $n \geq 1$. Since M is a pseudo-strongly prime ideal, there exists an $n \geq 1$ such that $b^nM \subseteq a^nM$. Thus, we get $b^nN \subseteq b^nM \subseteq a^nM$. Hence, for every $c \in N$, there is $d \in M$ such that $b^nc = a^nd$. Since N is a prime ideal of R, by [14, Corollary 3.3] we get, $a \in N$ or $d \in N$. If $a \in N$, then there is an $m \geq 1$ such that $a^m = 0$, and so $a^mR \subseteq b^mR$, which is a contradiction. Thus $d \in N$, and consequently, we have $b^nN \subseteq a^nN$. Therefore N is a pseudo-strongly prime ideal. $\square$

Corollary 1.2. *Let R be a ring with nonzero zero-divisors such that $dimR = 1$. Then R is a PAVR if and only if every prime ideal of R is pseudo-strongly prime.*

2. Almost valuation rings

We recall from [5] that an integral domain R with quotient field K is said to be an *almost valuation domain* if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that $x^n \in R$ or $x^{-n} \in R$. In this section, we generalize this concept to arbitrary commutative rings. Also, a commutative ring R is called a *valuation ring* if for every $a, b \in R$, $aR \subseteq bR$ or $bR \subseteq aR$. The concept of an almost valuation ring was introduced by F. Khoshayand in her Ph.D thesis as follows.

Definition 2.1. A commutative ring R is called an *almost valuation ring* if for every $a, b \in R$, there is an integer $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$.

The definition of almost valuation ring was also given independently by Mahdou et al., [15, Definition 1.1(1)]. It is clear that every valuation ring is an almost valuation ring.

Proposition 2.2. *Let R be an almost valuation ring. Then the prime ideals of R are linearly ordered. In particular, R is quasi-local.*

Proof. Suppose that P and Q are two distinct prime ideals of R such that $P \nsubseteq Q$, then there is a $p \in P \backslash Q$. Let $q \in Q$. Since R is an almost valuation ring, there is an $n \geq 1$ such that $p^n R \subseteq q^n R$ or $q^n R \subseteq p^n R$. If $p^n R \subseteq q^n R$, then $p \in Q$ because $q \in Q$, which is a contradiction. Hence $q^n R \subseteq p^n R$, and so $q \in P$. Therefore $Q \subseteq P$. □

Let R be an almost valuation ring and I be an ideal of R. It is easily shown that R/I is an almost valuation ring. In particular, if D is an almost valuation domain and I is a non-prime ideal of D then R/I is an almost valuation ring with nonzero zero-divisors.

Example 2.3. Let R be a ring in which every element is either a unit or nilpotent. Then R is an almost valuation ring. Also, every Artinian quasi-local ring is an almost valuation ring. In particular, for each prime integer p and for every $n \geq 1$, $\mathbf{Z}_{p^n}$ is an almost valuation ring.

Proposition 2.4. *Every almost valuation ring is a PAVR.*

Proof. Let R be an almost valuation ring and M be the maximal ideal of R. Suppose that $a, b \in R$ and $b^n R \nsubseteq a^n R$ for every $n \geq 1$. Since R is an almost valuation ring, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$. Thus for every $d \in M$, there is an $r \in R$ such that $a^n d = b^n r$. If $r \notin M$, then r is a unit, and so $b^n R \subseteq a^n R$, which is a contradiction. Hence $r \in M$, and so $a^n M \subseteq b^n M$. Therefore, using [14, Proposition 3.2], M is a pseudo-strongly prime ideal of R, and consequently R is a PAVR. □

Example 2.5. Let F be a field, and $R_\infty = F[X_1, ..., X_n, ...]$ and let I denote the ideal of R_∞ generated by $\{X_i^i \mid i \in \mathbf{N}\}$. If $P = (X_1, X_2, ...)/(X_1, X_2^2, ...)$, then every element of P is nilpotent. Also, if $g \in R_\infty/I$ such that $g \notin P$, then $g = a_0 + f$ for some $a_0 \in F$ and $f \in P$. Since f is nilpotent and a_0 is a unit, g is a unit. Thus R_∞/I is a ring in which every element is either a unit or nilpotent, and so R_∞/I is an almost valuation ring by Example 2.3, and it follows from Proposition 2.4 that R_∞/I is a PAVR with unique prime ideal P. Let $x = (1 + X_1) + I$. Then $P \nsubseteq x(R_\infty/I)$ and $x(R_\infty/I) \nsubseteq P$, and so R_∞/I is not a pseudo-valuation ring. Therefore, R_∞/I is a PAVR with unique prime ideal $(X_1, X_2, ...)/(X_1, X_2^2, ...)$ that is not a pseudo valuation ring. Furthermore, R_∞/I is an almost valuation ring which is not a valuation ring.

The following example gives a PAVR that is not an almost valuation ring. This example uses the idealization construction $R(+)B$ arising from a ring R

and an R-module B as in [13, Chapter VI]. We recall that if R is an integral domain and B is an R-module, then B is said to be *divisible* if for every nonzero element $r \in R$ and $b \in B$, there exists an $f \in B$ such that $rf = b$.

Example 2.6. Let F be a finite field and $K = F(X)$ be the quotient field of $F[X]$. Set $D = F + KY^2 + Y^4K[[Y]]$. Then D is a pseudo-almost valuation domain that is not an almost valuation domain by [9, Example 3.5]. If B is a divisible R-module, then $R = D(+)B$ is a PAVR by [14, Proposition 3.15]. Since D is not AVD, R is not an almost valuation ring by [15, Theorem 2.1(1)].

Theorem 2.7. *Let R be an almost valuation ring with maximal ideal M and S be a multiplicatively closed subset of R such that $S \cap M \neq \emptyset$. Then R_S is an almost valuation ring.*

Proof. Let $x, y \in R_S$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and $s, t \in S$. Suppose that $x^n R_S \nsubseteq y^n R_S$ for every $n \geq 1$. If $(at)^n R \subseteq (bs)^n R$ for some $n \geq 1$, then there exists $r \in R$ such that $(at)^n = (bs)^n r$. It follows that $(a/s)^n = (r/1)(b/t)^n$ in R_S. Hence $x^n R_S \subseteq y^n R_S$, which is a contradiction. Since R is an AVR, R is a PAVR by Proposition 2.4, and so M is a pseudo-strongly prime ideal of R. Therefore, there is an integer $n \geq 1$ such that $(bs)^n M \subseteq (at)^n M$. Since $S \cap M \neq \emptyset$, there exists $u \in S \cap M$. Consequently, there is $c \in M$ such that $(bs)^n u = (at)^n c$. Hence $(b/t)^n = (a/s)^n (c/u)$ in R_S, and so $y^n R_S \subseteq x^n R_S$. Therefore, R_S is an almost valuation ring. □

Corollary 2.8. *Let R be an almost valuation ring and P be a prime ideal of R. Then R_P is an almost valuation ring.*

Proof. If P is the maximal ideal of R, then $R_P = R$ is an AVR. Suppose that P is a non-maximal prime ideal of R. Then we get, $M \cap (R \setminus P) \neq \emptyset$, and so R is an AVR by Theorem 2.7. □

The following result is an analog of Hedstrom and Houston [12, Proposition 2.6], and Badawi [9, Corollary 4.2].

Corollary 2.9. *Let R be a pseudo-almost valuation ring and P be a non-maximal prime ideal of R. Then R_P is an almost valuation ring.*

Proposition 2.10. *Let R be a ring and P be a pseudo-strongly prime ideal of R. Then for each prime ideal Q of R such that $Q \subset P$, R_Q is an almost valuation ring.*

Proof. Since P is pseudo-strongly prime ideal, R_P is a PAVR by [14, Proposition 3.9]. We now assume that Q is a prime ideal of R such that $Q \subset P$. Then QR_P is a non-maximal prime ideal of R_P, and so $R_Q = (R_P)_{QR_P}$ is an AVR by Corollary 2.9. □

3. **Almost valuation overrings**

Let R be a ring and S be the set of all regular elements of R. Then the ring of fractions $T = R_S$ is called *the total quotient ring* of R. As usual, we say that a ring A is an *overring* of R if $R \subseteq A \subseteq T$. For every ideal I of R, the subset $(I : I) = \{ x \in T \mid xI \subseteq I \}$ is an overring of R.

Proposition 3.1. *Let R be an almost valuation ring and A be an overring of R. Then A is an almost valuation ring.*

Proof. Let $x, y \in A$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and regular elements $s, t \in R$. Since R is an AVR, there exists an integer $n \geq 1$ such that $(at)^n R \subseteq (bs)^n R$ or $(bs)^n R \subseteq (at)^n R$. It follows that there is an $r \in R$ such that $(at)^n = (bs)^n r$ or $(bs)^n = (at)^n r$, and so $x^n = (a/s)^n = (b/t)^n (r/1) = y^n (r/1)$ or $y^n = x^n (r/1)$. Therefore, $x^n A \subseteq y^n A$ or $y^n A \subseteq x^n A$ which leads to the fact that A is an AVR. □

The following result corresponds to [10, Theorem 8] and [9, Theorem 2.15].

Theorem 3.2. *Let R be a quasi-local ring with maximal ideal M. If M is a regular ideal, then R is a PAVR if and only if $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$, where*

$$Rad_V(M) = \{ x \in V \mid x^n \in M \text{ for some } n \geq 1 \}.$$

Proof. Let R be a pseudo-almost valuation ring and $x, y \in V$. Then $x = a/b$ and $y = c/d$ for some $a, c \in R$ and regular elements $b, d \in R$. Suppose that $x^n V \nsubseteq y^n V$ for every $n \geq 1$. Since M is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $(ad)^n R \subseteq (bc)^n R$ or $(bc)^n M \subseteq (ad)^n M$. If $(ad)^n R \subseteq (bc)^n R$, then $(a/b)^n R \subseteq (c/d)^n R$, and so $x^n V \subseteq y^n V$, which is a contradiction. Hence $(bc)^n M \subseteq (ad)^n M$ for some $n \geq 1$. Let $e \in M$ be a regular element of R. Then there is an $f \in M$ such that $ec^n b^n = f a^n d^n$. Thus, we have $(c/d)^n = (f/e)(a/b)^n$. Since R is a pseudo-almost valuation ring, there is an $m \geq 1$ such that $e^m R \subseteq f^m R$ or $f^m M \subseteq e^m M$. It is clear that if $f^m M \subseteq e^m M$, then $(f/e)^m \in V$. Thus, we get $y^{mn} = (c/d)^{mn} = (f/e)^m (a/b)^{mn} \in x^{mn} V$, and so $y^{mn} V \subseteq x^{mn} V$. We now assume that $e^m R \subseteq f^m R$ for some $m \geq 1$. Then f is a regular element and $(e/f)^m \in R \subseteq V$ which leads to $x^{mn} = (a/b)^{mn} = (e/f)^m (c/d)^{mn} \in y^{mn} V$. Hence $x^{mn} V \subseteq y^{mn} V$. Therefore V is an almost valuation ring.

Now, suppose that $x \in V$ is a non-unit element of V. Then $x = a/b$ for some $a, b \in R$. Since R is a pseudo-almost valuation ring with maximal ideal M, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n M \subseteq a^n M$. If $b^n M \subseteq a^n M$ for some $n \geq 1$ and $d \in M$ is a regular element, then there exists a $c \in M$ such that $db^n = ca^n$. Since d and b are regular, a is regular as well. Hence x is a unit of V, which is a contradiction. Therefore, one easily obtains that $a^n R \subseteq b^n R$ for some $n \geq 1$. Thus $x^n = a^n / b^n \in R$ is a non-unit of R because

x is a non-unit of V. It follows that $x^n \in M$, and so $x \in Rad_V(M)$. Hence $Rad_V(M)$ is the maximal ideal of V.

Conversely, suppose that $V = (M : M)$ is an almost valuation ring with maximal ideal $Rad_V(M)$ and $a, b \in R$. Thus there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Now assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is a $(c/d) \in V$ such that $a^n = (c/d)b^n$. If $(c/d) \in Rad_V(M)$, then there is an $m \geq 1$ such that $(c/d)^m \in M$. Thus, we get $a^{mn} = (c/d)^m b^{mn} \in b^{mn}M$, and so $a^{mn}R \subseteq b^{mn}R$. If $(c/d) \notin Rad_V(M)$, then c/d is a unit of V, and so $dM = cM$ which leads to $da^nM = cb^nM = db^nM$. It follows that $a^nM = b^nM$ because d is a regular element. Therefore R is a pseudo-almost valuation ring. □

It was shown in [9, Proposition 2.16] that if an integral domain R admits a nonzero principal pseudo-strongly prime ideal P, then R is an almost valuation domain with maximal ideal P. We have the following result:

Proposition 3.3. *Let R be a ring and P a regular principal ideal of R. If P is a pseudo-strongly prime ideal of R, then R is an almost valuation ring with maximal ideal P.*

Proof. Suppose that $P = (p)$ for some regular prime element $p \in R$. If P is a non-maximal ideal of R, then there is a non-unit $r \in R\backslash P$. Based on [14, Proposition 2.4], let n be the least positive integer such that $p^n = r^nd$ for some $d \in P$. It follows that $d = ps$ for some $s \in R$ because $d \in P$. If $n = 1$, then we get $p = rd$ which leads to $p = psr$. Since p is regular, r is a unit, which is a contradiction. Hence $n > 1$, and we have $p^{n-1} = r^{n-1}(rs)$, which is a contradiction to our choice of n. Thus P is a maximal ideal of R. Hence R is a PAVR, and so $(P : P)$ is an almost valuation ring by Theorem 3.2. Since $P = (p)$, we have $(P : P) = R$. □

We recall that an overring V of R is said to be a *root extension* of R if for every $x \in V$, there is an integer $n \geq 1$ such that $x^n \in R$. The following theorem corresponds to [9, Theorem 2.17].

Theorem 3.4. *Let R be a quasi-local ring with maximal ideal M. Suppose that V is an almost valuation overring of R such that M is an ideal of V and $Rad_V(M)$ is the maximal ideal of V. Then R is an almost valuation ring if and only if V is a root extension of R.*

Proof. If $V = R$, then there is nothing to prove. Hence assume that $V \neq R$ and R is an almost valuation ring. Let $x \in V\backslash R$. Then $x = a/b$ for some $a, b \in R$, where b is a regular element. If $x \in Rad_V(M)$, then $x^n \in M \subset R$ for some $n \geq 1$. Now, assume that $x \notin Rad_V(M)$. Since $Rad_V(M)$ is the maximal ideal of V, x is a unit of V, and so a is a regular element of R. Since R is an almost valuation ring, there is an $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$. If $a^nR \subseteq b^nR$, then $x^n \in R$, and consequently if $b^nR \subseteq a^nR$, then $x^{-n} \in R$.

If $x^{-n} \in M$, then $x^{-1} \in Rad_V(M)$, which is a contradiction. Hence x^{-n} is a unit of R, and so $x^n \in R$. Therefore V is a root extension of R.

Conversely, suppose that V is a root extension of R and $a, b \in R$. Since V is an almost valuation ring, there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is a $y \in V$ such that $a^n = b^ny$. Since V is a root extension of R, there is an $m \geq 1$ such that $y^m \in R$, and so $a^{mn} = b^{mn}y^m \in b^{mn}R$. Hence, we get $a^{mn}R \subseteq b^{mn}R$, and consequently R is an almost valuation ring. □

It is known ([4, Proposition 2.6]) that a PVD is a pullback of a valuation domain. Badawi shows that a pseudo-almost valuation domain is a pullback of an almost valuation domain, see [9, Theorem 2.19]. Our next result shows how to construct a PAVR as a pullback of an AVR.

Theorem 3.5. *Let V be an almost valuation ring with nonzero maximal ideal N and $F = V/M$, where M is an ideal of V such that $\sqrt{M} = N$. Let $\alpha : V \longrightarrow F$ be the canonical epimorphism and H be a subring of F. If H is a field and $R = \alpha^{-1}(H)$, then the pullback $R = V \times_F H$ is a PAVR with maximal ideal M. In particular, if H is properly contained in F and V is not a root extension of R, then R is a PAVR which is not an almost valuation ring.*

Proof. In view of the construction stated in the hypothesis, it is well known that M is an ideal of ring R such that $R/M \simeq H$, and so M is a maximal ideal of R. Let $a, b \in R$. Since V is an almost valuation ring, there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there is an $x \in V$ such that $a^n = b^nx$. If $x \in N$, then $x^m \in M \subset R$ for some m, and so $a^{mn}R \subseteq b^{mn}R$. If $x \notin N$, then x is a unit of V which leads to $b^n = a^nx^{-1}$. Since M is an ideal of V, we obtain that $b^nM = a^nx^{-1}M \subseteq a^nM$. Hence M is a pseudo-strongly prime ideal of R, and it follows that R is a PAVR. The remaining is clear from Theorem 3.4. □

In the light of Theorems 3.2 and 3.5, we have the following corollary which is an analog of Anderson and Dobbs [4, Proposition 2.6], and Badawi [9, Corollary 2.21].

Corollary 3.6. *Pseudo-almost valuation rings are precisely the pullbacks in the category of commutative rings (with 1) of diagrams of the form*

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ H & \rightarrow & F, \end{array}$$

where V is an almost valuation ring having maximal ideal $Rad_V(M)$ for some ideal M of V, $F = V/M$, the vertical map is the canonical surjection, H is a field contained in F, and the horizontal map is inclusion.

The following result is an analog of Badawi [9, Theorem 4.1].

Theorem 3.7. *Let R be a PAVR with maximal ideal M. Suppose that V is an overring of R such that $1/e \in V$ for some regular element $e \in M$. Then V is an almost valuation ring.*

Proof. Let $x, y \in V$ such that $x^nV \nsubseteq y^nV$ for every $n \geq 1$. Then $x = a/b$ and $y = c/d$ for some $a, c \in R$ and regular elements $b, d \in R$. If $(ad)^nR \subseteq (bc)^nR$ for some $n \geq 1$, then it is clear that $x^nV \subseteq y^nV$, which is a contradiction. Since M is a pseudo-strongly prime ideal of R, there is an $n \geq 1$ such that $(bc)^nM \subseteq (ad)^nM$. We now assume that $e \in M$ is a regular element of R such that $1/e \in V$. Thus $e(bc)^n \in (ad)^nM \subseteq (ad)^nV$, and consequently $(bc)^n \in (1/e)(ad)^nV \subseteq (ad)^nV$. Therefore, $y^nV \subseteq x^nV$. □

The following result is an analog of Anderson ([1, Proposition 4.2 and 4.3]) and Badawi ([9, Theorem 4.3]).

Theorem 3.8. *Let P be a regular pseudo-strongly prime ideal of R. Then $(P : P)$ is an almost valuation ring.*

Proof. Set $V = (P : P)$. Let $x, y \in V$ such that $x^nV \nsubseteq y^nV$ for every $n \geq 1$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and regular elements $s, t \in R$. If $(at)^nR \subseteq (bs)^nR$ for some $n \geq 1$, then $x^nV \subseteq y^nV$, which is a contradiction. Since P is pseudo-strongly prime, there is an $n \geq 1$ such that $(bs)^nP \subseteq (at)^nP$. Since P is regular, there exists a regular element $e \in P$. Therefore, there exists an element $p \in P$ such that $(bs)^ne = (at)^np$. Thus $(b/t)^n = (a/s)^n(p/e)$. Also, there is an $m \geq 1$ such that $e^mR \subseteq p^mR$ or $p^mP \subseteq e^mP$ because P is a pseudo-strongly prime ideal of R. If $e^mR \subseteq p^mR$, then we have $e^m = p^mr$ for some $r \in R$. Thus p is regular and $(e/p)^m = r \in R \subseteq V$. Hence, $x^{mn} = (a/s)^{mn} = (b/t)^{mn}(e/p)^m \in y^{mn}V$, and we conclude that $x^{mn}V \subseteq y^{mn}V$, which is a contradiction. Therefore, $p^mP \subseteq e^mP$ for some $n \geq 1$ which leads to $(p/e)^mP \subseteq P$. It follows that $(p/e)^m \in V$, and so $y^{mn} = (b/t)^{mn} = (a/s)^{mn}(p/e)^m \in x^{mn}V$. Thus $y^{mn}V \subseteq x^{mn}V$ which shows that V is an almost valuation ring. □

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