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**W-convergence of the proximal point algorithm in complete CAT(0) metric spaces**

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## W-CONVERGENCE OF THE PROXIMAL POINT ALGORITHM IN COMPLETE CAT(0) METRIC SPACES

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**ABSTRACT.** In this paper, we generalize the proximal point algorithm to complete CAT(0) spaces and show that the sequence generated by the proximal point algorithm  $w$ -converges to a zero of the maximal monotone operator. Also, we prove that if  $f : X \rightarrow ]-\infty, +\infty]$  is a proper, convex and lower semicontinuous function on the complete CAT(0) space  $X$ , then the proximal point algorithm  $w$ -converges to a zero of the subdifferential of  $f$ , i.e., a minimizer of  $f$ . Some strong convergence results (convergence in metric) are also presented with additional assumptions on the monotone operator and the convex function  $f$ .

**Keywords:** Hadamard space, maximal monotone operator, proximal point algorithm,  $w$ -convergence, subdifferential.

**MSC(2010):** Primary: 47H05; Secondary: 47J05, 47J20.

### 1. Introduction

One of the most important parts in nonlinear and convex analysis is monotone operator theory. It has an essential role in convex analysis, optimization, variational inequalities, semigroup theory and evolution equations. A zero of a maximal monotone operator is a solution of variational inequality associated to the monotone operator also an equilibrium point of an evolution equation governed by the monotone operator as well as a solution of a minimization problem for a convex function when the monotone operator is the Fenchel-Moreu subdifferential of the convex function. Therefore existence and approximation of a zero of a maximal monotone operator is the center of consideration of many recent researchers. The most popular method for approximation of a zero of a maximal monotone operator is the proximal point algorithm, which was introduced by Martinet [25] and Rockafellar [27]. Rockafellar [27] showed the weak convergence of the sequence generated by the proximal point algorithm to a

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zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [15] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper, and lower semicontinuous function. For some generalizations in Hilbert spaces, see [5, 7, 11, 15, 18].

In this paper, we consider the proximal point algorithm in nonlinear version of Hilbert spaces (i.e., complete CAT(0) spaces) and define maximal monotone operators on CAT(0) spaces using the duality theory introduced in [2]. Our results extend the previous results in Hilbert spaces as well as the recent results on Hadamard manifolds (see for example [23] and references therein) to complete CAT(0) spaces. Also, it extends a recent work of Bačák [3] to general maximal monotone operators. The paper is organized as follows.

In Section 2 we give some preliminaries. In Section 3 we study some properties of the monotone operators and convex lower semicontinuous functions that we need them in the sequel. Section 4 is devoted to the proximal point algorithm in Hadamard spaces, in this section, we prove that the proximal point algorithm  $w$ -converges to a zero of the maximal monotone operator in Hadamard spaces. In Section 4, we approximate a minimizer of a proper, convex, lower semicontinuous real-valued function on a Hadamard space by the proximal point algorithm.

## 2. Preliminaries

Let  $(X, d)$  be a metric space and  $x, y \in X$ . A geodesic path joining  $x$  to  $y$  is an isometry  $c : [0, d(x, y)] \rightarrow X$  such that  $c(0) = x, c(d(x, y)) = y$ . The image of a geodesic path joining  $x$  to  $y$  is called a geodesic segment between  $x$  and  $y$ . The metric space  $(X, d)$  is said to be a geodesic space if every two points of  $X$  are joined by a geodesic, and  $X$  is said to be uniquely geodesic if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ .

A geodesic space  $(X, d)$  is a CAT(0) space if it satisfies the following inequality which is called the *CN-inequality*.

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2),$$

where  $x, y_0, y_1, y_2$  belong to  $X$  and  $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$ . A complete CAT(0) space is called a *Hadamard* space. It is known that a CAT(0) space is a uniquely geodesic space. For other equivalent definitions of the *CN-inequality* and its basic properties, see [6, 9, 14, 17]. Some examples of CAT(0) spaces are pre-Hilbert spaces (see [6]),  $\mathbb{R}$ -trees (see [21]), Euclidean buildings (see [8]), the complex Hilbert ball with a hyperbolic metric (see [13]), Hadamard manifolds and many others.

For all  $x$  and  $y$  belong to a CAT(0) space  $X$ , we write  $(1-t)x \oplus ty$  for the unique point  $z$  in the geodesic segment joining from  $x$  to  $y$  such

that  $d(z, x) = td(x, y)$  and  $d(z, y) = (1 - t)d(x, y)$ . Set  $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$ , a subset  $C$  of  $X$  is called convex if  $[x, y] \subseteq C$  for all  $x, y \in C$ .

The notion of  $\Delta$ -convergence in complete CAT(0) spaces was introduced by Lim [24] as follows. Let  $(x_n)$  be a bounded sequence in complete CAT(0) space  $(X, d)$  and  $x \in X$ . Set  $r(x, (x_n)) = \limsup_{n \rightarrow \infty} d(x, x_n)$ . The asymptotic radius of  $(x_n)$  is given by  $r((x_n)) = \inf\{r(x, (x_n)) : x \in X\}$  and the asymptotic center of  $(x_n)$  is the set  $A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}$ . It is known that in a complete CAT(0) space,  $A((x_n))$  consists exactly one point (see [22]). A sequence  $(x_n)$  in the complete CAT(0) space  $(X, d)$  is said  $\Delta$ -convergent to  $x \in X$  if  $A((x_{n_k})) = \{x\}$  for every subsequence  $(x_{n_k})$  of  $(x_n)$ . The concept of  $\Delta$ -convergence has been studied by many authors (see e.g., [10, 12]).

Berg and Nikolaev [4] introduced the concept of *quasilinearization* for a CAT(0) space  $X$ . They denoted a pair  $(a, b) \in X \times X$  by  $\vec{ab}$  and called it a *vector*. Then the quasilinearization map  $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$  is defined by

$$\langle \vec{ab}, \vec{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad (a, b, c, d \in X).$$

It can be easily verified that  $\langle \vec{ab}, \vec{ab} \rangle = d^2(a, b)$ ,  $\langle \vec{ba}, \vec{cd} \rangle = -\langle \vec{ab}, \vec{cd} \rangle$  and  $\langle \vec{ab}, \vec{cd} \rangle = \langle \vec{ae}, \vec{cd} \rangle + \langle \vec{eb}, \vec{cd} \rangle$  are satisfied for all  $a, b, c, d, e \in X$ . Also, we can formally add compatible vectors, more precisely  $\vec{ac} + \vec{cb} = \vec{ab}$ , for all  $a, b, c \in X$ . We say that  $X$  satisfies the Cauchy-Schwarz inequality if

$$\langle \vec{ab}, \vec{cd} \rangle \leq d(a, b)d(c, d) \quad (a, b, c, d \in X).$$

It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality (see [4, Corollary 3]).

Ahmadi Kakavandi and Amini [2] introduced the concept of dual space of a complete CAT(0) space  $X$ , based on a work of Berg and Nikolaev [4], as follows. Consider the map  $\Theta : \mathbb{R} \times X \times X \rightarrow C(X, \mathbb{R})$  defined by

$$\Theta(t, a, b)(x) = t\langle \vec{ab}, \vec{ax} \rangle \quad (t \in \mathbb{R}, a, b, x \in X),$$

where  $C(X, \mathbb{R})$  is the space of all continuous real-valued functions on  $X$ . Then the Cauchy-Schwarz inequality implies that  $\Theta(t, a, b)$  is a Lipschitz function with Lipschitz semi-norm  $L(\Theta(t, a, b)) = |t|d(a, b)$  ( $t \in \mathbb{R}, a, b \in X$ ), where  $L(\varphi) = \sup\{\frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y\}$  is the Lipschitz semi-norm for any function  $\varphi : X \rightarrow \mathbb{R}$ . A pseudometric  $D$  on  $\mathbb{R} \times X \times X$  is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)) \quad (t, s \in \mathbb{R}, a, b, c, d \in X).$$

For a Hadamard space  $(X, d)$ , the pseudometric space  $(\mathbb{R} \times X \times X, D)$  can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions  $(Lip(X, \mathbb{R}), L)$ . It is obtained ([2, Lemma 2.1]) that  $D((t, a, b), (s, c, d)) = 0$  if and only if  $t\langle \vec{ab}, \vec{xy} \rangle = s\langle \vec{cd}, \vec{xy} \rangle$ , for all  $x, y \in X$ . Thus,  $D$  can impose an equivalent relation on  $\mathbb{R} \times X \times X$ , where the equivalence class of  $(t, a, b)$  is

$$[t\vec{ab}] = \{s\vec{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The set  $X^* = \{[\vec{tab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$  is a metric space with metric  $D([\vec{tab}], [\vec{sca}]) := D((t, a, b), (s, c, d))$ , which is called the dual space of  $(X, d)$ . It is clear that  $[\vec{aa}] = [\vec{bb}]$  for all  $a, b \in X$ . For a fix  $o \in X$ , we write  $\mathbf{0} = [\vec{oo}]$  as the zero of the dual space. In [2], it is shown that the dual of a closed and convex subset of Hilbert space  $H$  with nonempty interior is  $H$  and  $t(b - a) \equiv [\vec{tab}]$  for all  $t \in \mathbb{R}$ ,  $a, b \in H$ . Note that  $X^*$  acts on  $X \times X$  by

$$\langle x^*, \vec{xy} \rangle = t \langle \vec{ab}, \vec{xy} \rangle \quad (x^* = [\vec{tab}] \in X, x, y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \vec{xy} \rangle := \alpha \langle x^*, \vec{xy} \rangle + \beta \langle y^*, \vec{xy} \rangle$$

for all  $\alpha, \beta \in \mathbb{R}$ ,  $x, y \in X$ ,  $x^*, y^* \in X^*$ . Introducing of a dual for a CAT(0) space leads to a concept of weak convergence with respect to the dual space, which is named *w-convergence* in [2]. In [2], authors also showed that *w-convergence* is stronger than  $\Delta$ -convergence. Ahmadi Kakavandi [1] presented an equivalent definition of *w-convergence* in complete CAT(0) spaces without using dual space as follows.

**Definition 2.1** ([1]). A sequence  $(x_n)$  in the complete CAT(0) space  $(X, d)$  *w-converges* to  $x \in X$  if and only if  $\lim_{n \rightarrow \infty} \langle \vec{xx}_n, \vec{xy} \rangle = 0$ , for all  $y \in X$ .

*w-convergence* is equivalent to the weak convergence in a Hilbert space  $H$ , since the inner product  $(\cdot, \cdot)$  in a Hilbert space  $H$  leads to

$$2 \langle \vec{xz}, \vec{xy} \rangle = d^2(x, y) + d^2(z, x) - d^2(z, y) = 2(x - z, x - y).$$

It is obvious that convergence in the metric implies *w-convergence*. Furthermore, it was shown (see [2]) that *w-convergence* implies  $\Delta$ -convergence but the converse does not hold, see [1]. However, Ahmadi Kakavandi [1] proved that  $(x_n)$  is  $\Delta$ -convergent to  $x \in X$  if and only if  $\limsup_{n \rightarrow \infty} \langle \vec{xx}_n, \vec{xy} \rangle \leq 0$ , for all  $y \in X$ . It is known that every bounded sequence in a Hadamard space  $X$  has a  $\Delta$ -convergent subsequence. This is not true for *w-convergence*, see [1, Example 4.7]. We say that a Hadamard space  $X$  satisfies the condition *Q* if every bounded sequence in  $X$  has a *w-convergent* subsequence. A Hadamard space  $(X, d)$  is said to satisfy the (*S*) property if for any  $(x, y) \in X \times X$  there exists a point  $y_x \in X$  such that  $[\vec{xy}] = [\vec{y_xx}]$ . Hilbert spaces and symmetric Hadamard manifolds satisfy the (*S*) property, see [1, Definition 2.7]. It follows from [1, Lemma 2.8] that, if a Hadamard space  $(X, d)$  satisfies the (*S*) property, then it satisfies the condition *Q*; because every bounded sequence in a Hadamard space  $(X, d)$  has a  $\Delta$ -convergent subsequence. Also, the proper Hadamard spaces satisfy the condition *Q*, see [1, Propositions 4.3 and 4.4]. In the sequel, we denote *w-convergence* by  $\rightharpoonup$  and strong convergence by  $\rightarrow$ .

For  $X = H$ , where  $H$  is a Hilbert space, the multi-valued operator  $A : D(A) \subset H \rightarrow 2^H$  with  $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$  is called monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall x, y \in D(A), \quad \forall x^* \in Ax, \quad \forall y^* \in Ay.$$

The multi-valued monotone operator  $A : H \rightarrow 2^H$  is maximal if there exists no monotone operator  $B : H \rightarrow 2^H$  such that  $\text{gra}(B)$  properly contains  $\text{gra}(A)$ . It is well-known that maximality of monotone operators is equivalent to surjectivity of  $I + A$ , where  $I$  is the identity operator (see [26]). The proximal point algorithm introduced by Rockafellar [27] is defined as follows:

$$(2.1) \quad x_{n-1} - x_n \in \lambda_n A(x_n), \quad x_0 \in H,$$

where  $(\lambda_n)$  is a sequence of positive real numbers. In fact, Rockafellar [27] proved that the sequence generated by the proximal point algorithm is weakly convergent to a zero of the maximal monotone operator  $A$  provided that  $\lambda_n \geq \lambda > 0$ , for all  $n \geq 1$ . Conditions on the control sequence  $\lambda_n$  was improved by Brézis and Lions [7]. In this paper, we introduce proximal point algorithm in Hadamard spaces and prove  $w$ -convergence of the proximal point algorithm to a zero of the operator. Our results extend the previous results of the proximal algorithm to complete CAT(0) spaces.

### 3. Maximal monotone operators

Let  $X$  be a Hadamard space with dual  $X^*$  and  $A : X \rightarrow 2^{X^*}$  be a multi-valued operator with domain  $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$ , range  $\mathbb{R}(A) := \bigcup_{x \in X} Ax$ ,  $A^{-1}(x^*) := \{x \in X : x^* \in Ax\}$  and graph  $\text{gra}(A) := \{(x, x^*) \in X \times X^* : x \in \mathbb{D}(A), x^* \in Ax\}$ .

**Definition 3.1.** Let  $X$  be a Hadamard space with dual space  $X^*$ . The multi-valued operator  $A : X \rightarrow 2^{X^*}$  with domain  $\mathbb{D}(A) := \{x \in X : A(x) \neq \emptyset\}$ , is

- (i) monotone if for all  $x, y \in \mathbb{D}(A)$ ,  $x \neq y$ ,  $x^* \in Ax, y^* \in Ay$ ,

$$\langle x^* - y^*, \overrightarrow{y\hat{x}} \rangle \geq 0,$$

- (ii) strictly monotone if for all  $x, y \in \mathbb{D}(A)$ ,  $x \neq y$ ,  $x^* \in Ax, y^* \in Ay$ ,

$$\langle x^* - y^*, \overrightarrow{y\hat{x}} \rangle > 0,$$

- (iii)  $\alpha$ -strongly monotone for  $\alpha > 0$  if for all  $x, y \in \mathbb{D}(A)$ ,  $x \neq y$ ,  $x^* \in Ax, y^* \in Ay$ ,

$$\langle x^* - y^*, \overrightarrow{y\hat{x}} \rangle \geq \alpha d^2(x, y).$$

It is clear that every  $\alpha$ -strongly monotone operator, for  $\alpha > 0$ , is strictly monotone and every strictly monotone operator is monotone.

**Definition 3.2.** Let  $X$  be a Hadamard space with dual  $X^*$ . The multi-valued monotone operator  $A : X \rightarrow 2^{X^*}$  is maximal if there exists no monotone operator  $B : X \rightarrow 2^{X^*}$  such that  $\text{gra}(B)$  properly contains  $\text{gra}(A)$ , (i.e., for any  $(y, y^*) \in X \times X^*$ , the inequality  $\langle x^* - y^*, \overrightarrow{y\hat{x}} \rangle \geq 0$  for all  $(x, x^*) \in \text{gra}(A)$ , implies that  $y \in D(A)$  and  $y^* \in A(y)$ ).

**Theorem 3.3** ([16]). *Let  $X$  be a Hadamard space with dual  $X^*$  and  $A : X \rightarrow 2^{X^*}$  be a multi-valued maximal monotone operator. Suppose  $(x_n, x_n^*) \in \text{gra}(A)$  for all  $n \in \mathbb{N}$ , where  $(x_n)$  is a bounded sequence in  $X$  which is  $w$ -convergent to  $x \in X$  and  $(x_n^*) \subset X^*$  converges to  $x^* \in X^*$  in metric  $D$ , then  $x^* \in Ax$ .*

We say that a subset  $C$  of Hadamard space  $X$  is  $w$ -sequentially closed if for any sequence  $(x_n) \subset C$  that  $x_n \rightarrow x$ , we have  $x \in C$ . It is clear that every  $w$ -sequentially closed subset of  $X$  is closed. By Theorem 3.3, it is easy to verify that if  $A : X \rightarrow 2^{X^*}$  is a multi-valued maximal monotone operator, then  $A^{-1}(x^*)$  is a  $w$ -sequentially closed subset of the Hadamard space  $X$ , for any  $x^* \in X^*$ . One of the most important examples of monotone operators are subdifferentials of convex, proper and lower semicontinuous functions. Therefore, in the following, we study convex and lower semicontinuous functions and their subdifferentials in Hadamard spaces. Let  $(X, d)$  be a Hadamard space. The sub-level set of the function  $f : X \rightarrow ]-\infty, +\infty]$  at  $a \in \mathbb{R}$  is  $f^a := \{x \in X : f(x) \leq a\}$ . It is known that if  $f$  is a convex and lower semicontinuous function then the sub-level set  $f^a$  is closed and convex for all  $a \in \mathbb{R}$ . We say that a function  $f : X \rightarrow ]-\infty, +\infty]$  is  $w$ -sequentially lower semicontinuous if and only if

$$f(x) \leq \liminf_n f(x_n)$$

for each sequence  $(x_n)$  with  $x_n \rightarrow x$ .

**Proposition 3.4.** ([22, Proposition 5.2]). *If a sequence  $(x_n)$  in a Hadamard space  $(X, d)$  is  $\Delta$ -convergent to  $x \in X$ , then*

$$x \in \bigcap_{k=1}^{\infty} \overline{\text{conv}}\{x_k, x_{k+1}, \dots\},$$

where  $\overline{\text{conv}}(A) := \bigcap \{B : A \subset B \text{ and } B \text{ is closed and convex}\}$  for any  $A \subset X$ .

The following corollary is a result of Proposition 3.4 and [2, Proposition 2.5].

**Corollary 3.5.** *Every closed and convex subset of a Hadamard space is  $w$ -sequentially closed.*

**Proposition 3.6.** *Let  $(X, d)$  be a Hadamard space. A function  $f : X \rightarrow ]-\infty, +\infty]$  is  $w$ -sequentially lower semicontinuous if and only if for every  $a \in \mathbb{R}$ ,  $f^a$  is a  $w$ -sequentially closed subset in  $X$ .*

*Proof.* The only if part is clear. For the if part, let for every  $a \in \mathbb{R}$ ,  $f^a$  be a  $w$ -sequentially closed and  $(x_n)$  be a sequence in  $X$  such that  $x_n \rightarrow x$ . Set  $\mu = \liminf_n f(x_n)$ . If  $\mu = \infty$ , the proof is completed. Let  $\mu < \infty$ , there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $\lim_k f(x_{n_k}) = \mu$ . For an arbitrary  $\varepsilon > 0$ , there exists  $N > 0$  such that for any  $k > N$ ,  $f(x_{n_k}) \leq \mu + \varepsilon$ . Therefore, we get  $x_{n_k} \in f^{\mu+\varepsilon}$  for all  $k > N$ . Thus, by assumptions, for every arbitrary  $\varepsilon > 0$ , we have  $x \in f^{\mu+\varepsilon}$  and so by letting  $\varepsilon \rightarrow 0$  we get  $x \in f^\mu$ . Hence,  $f(x) \leq \mu = \liminf_n f(x_n)$ . This completes the proof.  $\square$

By Corollary 3.5 and Proposition 3.6, the following corollary is obtained.

**Corollary 3.7.** *Let  $(X, d)$  be a Hadamard space. Every lower semicontinuous and convex function  $f : X \rightarrow ]-\infty, +\infty]$  is  $w$ -sequentially lower semicontinuous.*

In [2], subdifferential of a proper function on a Hadamard space  $X$  was defined as follows.

**Definition 3.8** ([2]). Let  $X$  be a Hadamard space with dual  $X^*$  and  $f : X \rightarrow ]-\infty, +\infty]$  be a proper function with efficient domain  $D(f) := \{x : f(x) < +\infty\}$ . The subdifferential of  $f$  is the multi-valued function  $\partial f : X \rightarrow 2^{X^*}$  defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle \quad (z \in X)\},$$

when  $x \in D(f)$ , and  $\partial f(x) := \emptyset$ , otherwise.

The following theorem was proved in [2]. The proof is given for sake of completeness.

**Theorem 3.9** ([2, Theorem 4.2]). *Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper, lower semicontinuous and convex function on a Hadamard space  $X$  with dual  $X^*$ , then*

- (i)  $f$  attains its minimum at  $x \in X$  if and only if  $\mathbf{0} \in \partial f(x)$ .
- (ii)  $\partial f : X \rightarrow 2^{X^*}$  is a monotone operator.
- (iii) for any  $y \in X$  and  $\alpha > 0$ , there exists a unique point  $x \in X$  such that  $[\alpha \overrightarrow{xy}] \in \partial f(x)$ .

*Proof.* We check property (iii). Let  $y \in X$  and  $\alpha > 0$  be fixed, and set  $g(x) = f(x) + \frac{\alpha}{2}d^2(x, y)$ . Similar to [2, Theorem 4.2], there exists a point  $x \in X$  such that  $[\alpha \overrightarrow{xy}] \in \partial f(x)$ . To prove uniqueness, let there exist  $x, z \in X$  such that  $[\alpha \overrightarrow{xy}] \in \partial f(x)$  and  $[\alpha \overrightarrow{zy}] \in \partial f(z)$ . Then using part (ii), we have

$$0 \leq 2\langle [\alpha \overrightarrow{xy}] - [\alpha \overrightarrow{zy}], \overrightarrow{zx} \rangle = 2\alpha \langle \overrightarrow{xy}, \overrightarrow{zx} \rangle - 2\alpha \langle \overrightarrow{zy}, \overrightarrow{zx} \rangle = -2\alpha d^2(x, z),$$

which implies  $x = z$ . □

#### 4. Proximal point algorithm

Let  $X$  be a Hadamard space with dual  $X^*$ . The problem of finding a zero of the monotone operator  $A : X \rightarrow 2^{X^*}$  can be formulated as follows.

$$(4.1) \quad \text{Find } x \in X, \text{ such that } \mathbf{0} \in A(x),$$

where  $\mathbf{0}$  is the zero element of the dual space  $X^*$ . We say that  $A$  satisfies the *range condition* if for every  $y \in X$  and every  $\alpha > 0$ , there exists a point  $x \in X$  such that  $[\alpha \overrightarrow{xy}] \in Ax$ . It is known that if  $A$  is a maximal monotone operator on a Hilbert space  $H$  then  $R(I + \lambda A) = H$  for all  $\lambda > 0$ , where  $I$  denotes the identity operator. Thus, every maximal monotone operator  $A$  on a Hilbert space satisfies the range condition. Theorem 3.9 shows that  $\partial f$  satisfies the range condition, whenever  $f$  is a proper, lower semicontinuous and convex function on a Hadamard space  $X$ . However, we do not know if every maximal



monotone operator  $A : X \rightarrow 2^{X^*}$ , where  $X$  is a Hadamard space, satisfies the range condition.

**Lemma 4.1.** *If  $A$  is a monotone operator on a Hadamard space  $X$  that satisfies the range condition, then for every  $y \in X$  and every  $\alpha > 0$ , there exists a unique point  $x \in X$  such that  $[\alpha \overrightarrow{xy}] \in Ax$ .*

*Proof.* If there exists  $x, z \in X$  such that  $[\alpha \overrightarrow{xy}] \in Ax$  and  $[\alpha \overrightarrow{zy}] \in Az$  then, using monotonicity of  $A$ , we have

$$\begin{aligned} 0 &\leq 2\langle [\alpha \overrightarrow{xy}] - [\alpha \overrightarrow{zy}], \overrightarrow{zx} \rangle \\ &= 2\alpha \langle \overrightarrow{xy}, \overrightarrow{zx} \rangle - 2\alpha \langle \overrightarrow{zy}, \overrightarrow{zx} \rangle \\ &= \alpha(d^2(y, z) - d^2(x, z) - d^2(y, x)) - \alpha(d^2(x, z) + d^2(y, z) - d^2(y, x)) \\ &= -2\alpha d^2(x, z), \end{aligned}$$

which implies that  $x = z$ .  $\square$

Let  $A : X \rightarrow 2^{X^*}$  be a multi-valued monotone operator on the Hadamard space  $X$  with dual  $X^*$  which satisfies the range condition and let  $(\lambda_n)$  be a sequence of positive real numbers. The proximal point algorithm for a monotone operator  $A$  on a Hadamard space  $X$  is the sequence generated by

$$(4.2) \quad \begin{cases} [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}] \in Ax_n, \\ x_0 \in X. \end{cases}$$

Note that Lemma 4.1 implies the proximal point algorithm (4.2) is well-defined and also (4.2) is in accordance with the proximal point algorithm (2.1) in a Hilbert space.

In the following, we prove  $w$ -convergence of the sequence generated by the proximal point algorithm (4.2) to an element of  $A^{-1}(\mathbf{0})$ , where  $\mathbf{0}$  is the zero of the dual space  $X^*$ . To this end, we need the following lemma that is a generalization of Opial lemma in CAT(0) spaces.

**Lemma 4.2.** *Let  $X$  be a Hadamard space that satisfies the condition Q, and let  $(x_n)$  be a sequence in  $X$  such that there exists a nonempty subset  $F$  of  $X$  verifying:*

- (1) *For every  $z \in F$ ,  $\lim_n d(x_n, z)$  exists.*
- (2) *If subsequence  $(x_{n_j})$  of  $(x_n)$  is  $w$ -convergent to  $x \in X$ , then  $x \in F$ .  
Then, there exists  $p \in F$  such that  $(x_n)$   $w$ -converges to  $p$  in  $X$ .*

*Proof.* Suppose  $x, y \in X$  and there exist subsequences  $(x_{n_j})$  and  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_j} \rightarrow x$  and  $x_{n_k} \rightarrow y$ . So, we have  $\langle \overrightarrow{x x_{n_j}}, \overrightarrow{xy} \rangle \rightarrow 0$  and  $\langle \overrightarrow{y x_{n_k}}, \overrightarrow{yx} \rangle \rightarrow 0$ . Using (2), we have  $x, y \in F$ . By (1), set

$$l_1 = \lim_n d(x_n, x) \quad \text{and} \quad l_2 = \lim_n d(x_n, y).$$

We also have

$$\begin{aligned} 2\langle \overrightarrow{xx_{n_j}}, \overrightarrow{xy} \rangle &= d^2(x, x_{n_j}) - d^2(y, x_{n_j}) + d^2(x, y), \\ 2\langle \overrightarrow{yx_{n_k}}, \overrightarrow{yx} \rangle &= d^2(y, x_{n_k}) - d^2(x, x_{n_k}) + d^2(x, y). \end{aligned}$$

Therefore, by letting  $j \rightarrow \infty$  and  $k \rightarrow \infty$ , we get  $d^2(x, y) = l_1 - l_2 = -d^2(x, y)$ , and consequently  $x = y$ . Hence,  $(x_n)$  is  $w$ -convergent to  $x \in F$ .  $\square$

In the following theorem, a solution of the problem 4.1 for a maximal monotone operator  $A$  is approximated.

**Theorem 4.3.** *Let  $X$  be a Hadamard space with dual  $X^*$ . Suppose that  $X$  satisfies the condition  $Q$ , and that  $A : X \rightarrow 2^{X^*}$  is a multi-valued maximal monotone operator which satisfies the range condition and  $A^{-1}(\mathbf{0}) \neq \emptyset$ , where  $\mathbf{0}$  is the zero of the dual space. Let  $(\lambda_n)$  be a sequence of positive real numbers such that  $\sum_{n=1}^{\infty} \lambda_n^2 = \infty$ . Then the sequence generated by the proximal point algorithm (4.2)  $w$ -converges to a point  $p \in A^{-1}(\mathbf{0})$ .*

*Proof.* Let  $x \in A^{-1}(\mathbf{0})$ . By monotonicity of  $A$ , we have

$$\begin{aligned} 0 &\leq 2\langle [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}] - \mathbf{0}, \overrightarrow{xx_n} \rangle \\ &= \frac{2}{\lambda_n} \langle \overrightarrow{x_n x_{n-1}}, \overrightarrow{xx_n} \rangle \\ &= \frac{1}{\lambda_n} (d^2(x, x_{n-1}) - d^2(x, x_n) - d^2(x_n, x_{n-1})), \end{aligned}$$

which implies

$$(4.3) \quad d^2(x, x_n) + d^2(x_n, x_{n-1}) \leq d^2(x, x_{n-1}), \quad \forall x \in A^{-1}(\mathbf{0}).$$

Thus,  $(d(x, x_n))$  is convergent for all  $x \in A^{-1}(\mathbf{0})$ . Hence  $(x_n)$  is bounded. By monotonicity of  $A$  and (4.2), for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} 0 &\leq \langle [\frac{1}{\lambda_{n-1}} \overrightarrow{x_{n-1} x_{n-2}}] - [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \overrightarrow{x_n x_{n-1}} \rangle \\ &= \frac{1}{\lambda_{n-1}} \langle \overrightarrow{x_{n-1} x_{n-2}}, \overrightarrow{x_n x_{n-1}} \rangle - \frac{1}{\lambda_n} \langle \overrightarrow{x_n x_{n-1}}, \overrightarrow{x_n x_{n-1}} \rangle \\ &\leq \frac{1}{\lambda_{n-1}} d(x_{n-1}, x_n) d(x_{n-1}, x_{n-2}) - \frac{1}{\lambda_n} d^2(x_{n-1}, x_n), \end{aligned}$$

which implies

$$\frac{1}{\lambda_n} d(x_{n-1}, x_n) \leq \frac{1}{\lambda_{n-1}} d(x_{n-1}, x_{n-2}), \quad \forall n \in \mathbb{N}.$$

Hence, we get

$$L(\Theta(\frac{1}{\lambda_n}, x_n, x_{n-1})) \leq L(\Theta(\frac{1}{\lambda_{n-1}}, x_{n-1}, x_{n-2})), \quad \forall n \in \mathbb{N},$$

that is

$$D([\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \mathbf{0}) \leq D([\frac{1}{\lambda_{n-1}} \overrightarrow{x_{n-1} x_{n-2}}], \mathbf{0}), \quad \forall n \in \mathbb{N}.$$

Therefore, for all  $k \geq n$ , we have

$$(4.4) \quad D^2([\frac{1}{\lambda_k} \overrightarrow{x_k x_{k-1}}], \mathbf{0}) \leq D^2([\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \mathbf{0}).$$

On the other hand, using (4.3), we get

$$\lambda_n^2 (L(\Theta(\frac{1}{\lambda_n}, x_n, x_{n-1})))^2 \leq d^2(x, x_{n-1}) - d^2(x, x_n), \quad \forall x \in A^{-1}(\mathbf{0}),$$

that is

$$\lambda_n^2 D^2([\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \mathbf{0}) \leq d^2(x, x_{n-1}) - d^2(x, x_n), \quad \forall x \in A^{-1}(\mathbf{0}).$$

Hence, using (4.4), for all  $k \geq n$ , we obtain

$$\lambda_n^2 D^2([\frac{1}{\lambda_k} \overrightarrow{x_k x_{k-1}}], \mathbf{0}) \leq d^2(x, x_{n-1}) - d^2(x, x_n), \quad \forall x \in A^{-1}(\mathbf{0}).$$

Summing up from  $n = 1$  to  $k$ , we obtain

$$\begin{aligned} \sum_{n=1}^k \lambda_n^2 D^2([\frac{1}{\lambda_k} \overrightarrow{x_k x_{k-1}}], \mathbf{0}) &\leq \sum_{n=1}^k (d^2(x, x_{n-1}) - d^2(x, x_n)) \\ &= d^2(x, x_0) - d^2(x, x_k), \end{aligned}$$

which implies

$$D^2([\frac{1}{\lambda_k} \overrightarrow{x_k x_{k-1}}], \mathbf{0}) \leq \frac{d^2(x, x_0)}{\sum_{n=1}^k \lambda_n^2}.$$

Hence, using the assumptions,  $[\frac{1}{\lambda_k} \overrightarrow{x_k x_{k-1}}]$  converges to  $\mathbf{0} \in X^*$  in metric  $D$  as  $k \rightarrow \infty$ . Now, by condition  $Q$ , suppose  $(x_{n_j})$  is a subsequence of the sequence  $(x_n)$  such that  $x_{n_j} \rightarrow q$ . Then  $[\frac{1}{\lambda_{n_j}} \overrightarrow{x_{n_j} x_{n_j-1}}]$  converges to  $\mathbf{0} \in X^*$  in metric  $D$  as  $j \rightarrow \infty$ . Theorem 3.3 implies that  $\mathbf{0} \in Aq$  and so  $q \in A^{-1}(\mathbf{0})$ . Therefore, we proved that

(1) for every  $x \in A^{-1}(\mathbf{0})$ ,  $\lim_n d(x_n, x)$  exists,

(2) if subsequence  $(x_{n_j})$  of  $(x_n)$  is  $w$ -convergent to  $q \in X$ , then  $q \in A^{-1}(\mathbf{0})$ .

Hence, Lemma 4.2 completes the proof.  $\square$

In the following two theorems, with some additional assumptions on the operator  $A$ , we prove the strong convergence of the proximal point algorithm (4.2) to a point of  $A^{-1}(\mathbf{0})$ .

**Theorem 4.4.** *Let  $X$  be a Hadamard space with dual  $X^*$  and let  $A : X \rightarrow 2^{X^*}$  be a  $\alpha$ -strongly monotone operator which satisfies the range condition and  $A^{-1}(\mathbf{0}) \neq \emptyset$ , where  $\mathbf{0} \in X^*$  is the zero of dual space. Suppose  $(\lambda_n)$  is a sequence of positive real numbers such that  $\sum_{n=1}^{\infty} \lambda_n = \infty$ . Then the sequence generated by the proximal point algorithm (4.2) converges strongly to a single element  $p \in A^{-1}(\mathbf{0})$ .*

*Proof.* Let  $x \in A^{-1}(\mathbf{0})$ . By the strong monotonicity of  $A$ , we have

$$\begin{aligned} \alpha d^2(x_n, x) &\leq 2 \langle [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}] - \mathbf{0}, \overrightarrow{x x_n} \rangle \\ &= \frac{2}{\lambda_n} \langle \overrightarrow{x_n x_{n-1}}, \overrightarrow{x x_n} \rangle \\ &= \frac{1}{\lambda_n} (d^2(x, x_{n-1}) - d^2(x, x_n) - d^2(x_n, x_{n-1})), \end{aligned}$$

which implies

$$(4.5) \quad \alpha \lambda_n d^2(x, x_n) \leq d^2(x, x_{n-1}) - d^2(x, x_n), \quad \forall x \in A^{-1}(\mathbf{0}).$$

Summing up from  $n = 1$  to  $n = k$  and letting  $k \rightarrow +\infty$ , we get

$$\sum_{n=1}^{+\infty} \lambda_n d^2(x, x_n) < +\infty,$$

which, by the assumptions, implies  $\liminf_{n \rightarrow +\infty} d(x_n, x) = 0$ . Since by (4.5),  $\lim_n d(x_n, x)$  exists, we have  $x_n \rightarrow x$  as  $n \rightarrow +\infty$ .  $\square$

We say that the multi-valued operator  $A : X \rightarrow 2^{X^*}$  satisfies *condition I* if there exists  $y \in X$  and a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$  and  $f(r) > 0$  for all  $r > 0$  such that for all  $x \in X$  and all  $x^* \in Ax$ ,

$$f(d(x, A^{-1}(\mathbf{0}))) \leq |\langle x^*, \overrightarrow{x y} \rangle|,$$

where  $d(x, A^{-1}(\mathbf{0})) = \inf\{d(x, z) : z \in A^{-1}(\mathbf{0})\}$ .

**Example 4.5.** Let  $X$  be a Hadamard space and  $o \in X$ . Define  $A : X \rightarrow 2^{X^*}$  with  $Ax = \{[\alpha \overrightarrow{o x}] : \alpha \in \mathbb{R}\}$ . Then  $o \in A^{-1}(\mathbf{0})$  and for all  $x \in X$  and  $[\alpha \overrightarrow{o x}] \in Ax$  we have

$$\begin{aligned} |\langle [\alpha \overrightarrow{o x}], \overrightarrow{x o} \rangle| &= |\alpha| d^2(x, o) \\ &\geq |\alpha| \inf\{d^2(x, u) : u \in A^{-1}(\mathbf{0})\} \\ &= |\alpha| d^2(x, A^{-1}(\mathbf{0})), \end{aligned}$$

which implies  $A$  satisfies the condition I with  $f(r) = |\alpha| r^2$  and  $y = o$ .

**Example 4.6.** Suppose  $H = \mathbb{R}^2$  with the Euclidean norm and with polar coordinates  $(r, \theta)$ . Set  $C = \{(r, \theta) : r \in [0, 1] \text{ and } \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}]\}$ . Define  $T : C \rightarrow C$  by  $T((r, \theta)) = (r, -\frac{\pi}{2})$  for all  $(r, \theta) \in C$ .  $T$  is nonexpansive because;

$$\begin{aligned} d(T(r_1, \theta_1), T(r_2, \theta_2)) &= |r_2 - r_1| \\ &\leq \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)} \\ &= d((r_1, \theta_1), (r_2, \theta_2)). \end{aligned}$$

We also have,  $Fix(T) = \{(r, -\frac{\pi}{2}) : r \in [0, 1]\}$ , where  $Fix(T)$  is the set of fixed points of  $T$ . Clearly, for  $x = (r, \theta) \in C$ , we get

$$\begin{aligned} d(x, Tx) &= d((r, \theta), (r, -\frac{\pi}{2})) \\ &\geq \inf\{d((r, \theta), (r, -\frac{\pi}{2})) : r \in [0, 1]\} \\ &= d((r, \theta), Fix(T)) \\ &= d(x, Fix(T)). \end{aligned}$$

Now, set  $Ax = \{\overrightarrow{[(Tx)x]}\} = \{x - Tx\}$ , then  $A^{-1}(\mathbf{0}) = Fix(T)$  and  $A$  is a maximal monotone operator that satisfies the range condition. Let  $p \in Fix(T)$ , then  $p \in A^{-1}(\mathbf{0})$ . For all  $x \in C$ , using nonexpansiveness of  $T$ , we have

$$\begin{aligned} 2|\langle Ax, \overrightarrow{x\hat{p}} \rangle| &= |2\langle \overrightarrow{(Tx)x}, \overrightarrow{x\hat{p}} \rangle| \\ &= d^2(x, Tx) + d^2(x, p) - d^2(Tx, p) \\ &\geq d^2(x, Tx) \\ &\geq d^2(x, Fix(T)) \\ &= d^2(x, A^{-1}(\mathbf{0})). \end{aligned}$$

Hence,  $A$  satisfies the condition  $I$  with  $f(r) = \frac{1}{2}r^2$  and  $y = p$ .

*Remark 4.7.* Note that if  $X$  is a Hadamard space and  $T : X \rightarrow X$  be a nonexpansive mapping, then the operator  $Ax = \{\overrightarrow{[(Tx)x]}\}$  is a monotone operator (see [19, proposition 4.2]), and if  $X$  is a flat Hadamard space then  $A$  satisfies the range condition (see [20, Proposition 6.4]). Also, in [20], it is shown that in spite of Hilbert spaces, in the case of Hadamard spaces, there are examples of nonexpansive mappings  $T$  for which the operator  $A$  is not maximal monotone.

**Theorem 4.8.** *Let  $X$  be a Hadamard space with dual  $X^*$ , and let  $A : X \rightarrow 2^{X^*}$  be a multi-valued maximal monotone operator which satisfies the range condition, the condition  $I$ , and  $A^{-1}(\mathbf{0}) \neq \emptyset$ . Suppose  $(\lambda_n)$  is a sequence of positive real numbers such that  $\liminf_n \lambda_n > 0$ . Then the sequence generated by the proximal point algorithm (4.2) converges strongly to a point  $p \in A^{-1}(\mathbf{0})$ .*

*Proof.* Let  $x \in A^{-1}(\mathbf{0})$ . By (4.3), we have

$$d(x_n, x) \leq d(x_{n-1}, x), \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} d^2(x_n, x_{n-1}) < \infty,$$

which implies  $(d(x_n, A^{-1}(\mathbf{0})))$  is nonincreasing and  $\lim_n d(x_n, x_{n-1}) = 0$ . On the other hand, by condition  $I$  and (4.2), we get

$$f(d(x_n, A^{-1}(\mathbf{0}))) \leq |\langle [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \overrightarrow{x_n \hat{y}} \rangle| \leq \frac{1}{\lambda_n} d(x_n, x_{n-1}) d(x_n, y)$$

for some  $y \in X$ . This implies

$$\lim_n f(d(x_n, A^{-1}(\mathbf{0}))) = 0,$$

and the properties of  $f$  implies

$$\lim_n d(x_n, A^{-1}(\mathbf{0})) = 0.$$

Therefore, for any  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  and  $z \in A^{-1}(\mathbf{0})$  such that  $d(x_N, z) < \epsilon$ , which implies  $d(x_n, z) < \epsilon$ , for all  $n > N$ , because the sequence  $(d(x_n, z))$  is nonincreasing for all  $z \in A^{-1}(\mathbf{0})$ . Thus, for any  $k \in \mathbb{Z}^+$ , there exists  $N_k \in \mathbb{N}$  and  $z_k \in A^{-1}(\mathbf{0})$  such that  $d(x_n, z_k) < \frac{1}{2^{k+2}}$  for all  $n \geq N_k$ . Set  $S_z(\epsilon) = \{x \in X : d(x, z) \leq \epsilon\}$ . We show that  $(S_{z_k}(\frac{1}{2^k}))$  is a nested sequence of nonempty closed sets such that  $\text{diam } S_{z_k}(\frac{1}{2^k}) \rightarrow 0$ . Clearly,  $\text{diam } S_{z_k}(\frac{1}{2^k}) \rightarrow 0$  and for each  $k \in \mathbb{Z}^+$ ,  $S_{z_k}(\frac{1}{2^k})$  is closed. Suppose for each  $k \in \mathbb{Z}^+$ ,  $N_{k+1} \geq N_k$ . For every  $k \in \mathbb{Z}^+$ , we have

$$\begin{aligned} d(z_k, z_{k+1}) &\leq d(z_k, x_{N_{k+1}}) + d(x_{N_{k+1}}, z_{k+1}) \\ &< \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} = \frac{3}{2^{k+3}} \\ &< \frac{1}{2^{k+1}}. \end{aligned}$$

Hence, for all  $k \in \mathbb{Z}^+$ ,  $S_{z_k}(\frac{1}{2^k}) \neq \emptyset$ . On the other hand, if  $x \in S_{z_{k+1}}(\frac{1}{2^{k+1}})$ , then

$$\begin{aligned} d(x, z_k) &\leq d(x, z_{k+1}) + d(z_{k+1}, z_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \\ &= \frac{1}{2^k}, \end{aligned}$$

which implies  $x \in S_{z_k}(\frac{1}{2^k})$ . Thus,  $S_{z_{k+1}}(\frac{1}{2^{k+1}}) \subset S_{z_k}(\frac{1}{2^k})$ . Hence, by the Cantor intersection theorem,  $\bigcap_{k \in \mathbb{Z}^+} S_{z_k}(\frac{1}{2^k})$  is a singleton. Let

$$\bigcap_{k \in \mathbb{Z}^+} S_{z_k}(\frac{1}{2^k}) := \{p\}.$$

We have

$$d(z_k, p) \leq \frac{1}{2^k}, \quad \text{for all } k \in \mathbb{Z}^+,$$

which implies  $z_k \rightarrow p$ . Since  $A^{-1}(\mathbf{0})$  is  $w$ -sequentially closed,  $p \in A^{-1}(\mathbf{0})$ . Consequently, for all  $n \geq N_k$ ,

$$\begin{aligned} d(x_n, p) &\leq d(x_n, z_k) + d(z_k, p) \\ &< \frac{1}{2^{k+2}} + d(z_k, p), \end{aligned}$$

which implies  $x_n \rightarrow p$ . This completes the proof.  $\square$

In the sequel, we approximate a minimizer of a proper, lower semicontinuous and convex function in a Hadamard space. Let  $X$  be a Hadamard space with dual  $X^*$ . Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper, lower semicontinuous and convex function and  $(\lambda_n)$  be a sequence of positive real numbers. Using part (iii) of Theorem 3.9, the operator  $\partial f$  satisfies the range condition. Thus, we can introduce the proximal point algorithm for  $\partial f$  in the Hadamard space  $X$  as follows

$$(4.6) \quad \begin{cases} [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}] \in \partial f(x_n), \\ x_0 \in X. \end{cases}$$

In the following theorem, it will be shown that the sequence generated by (4.6) is  $w$ -convergent to a point of  $(\partial f)^{-1}(\mathbf{0})$ .

**Theorem 4.9.** *Let  $X$  be a Hadamard space with dual  $X^*$ , and let  $X$  satisfy the condition  $Q$ . Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper, lower semicontinuous and convex function such that  $(\partial f)^{-1}(\mathbf{0}) \neq \emptyset$ . Suppose  $(\lambda_n)$  is a sequence of positive real numbers such that  $\sum_{n=1}^{\infty} \lambda_n = \infty$ . Then the sequence generated by the proximal point algorithm (4.6)  $w$ -converges to a point  $p \in (\partial f)^{-1}(\mathbf{0})$ .*

*Proof.* By subdifferential inequality in Definition 3.8 and (4.6), for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} f(x_n) - f(x_{n-1}) &\leq \langle [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \overrightarrow{x_{n-1} x_n} \rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x_{n-1}}, \overrightarrow{x_{n-1} x_n} \rangle \\ &= -\frac{1}{\lambda_n} d^2(x_n, x_{n-1}), \end{aligned}$$

which implies  $f(x_n) \leq f(x_{n-1})$  for all  $n \in \mathbb{N}$ . Hence, we get

$$(4.7) \quad f(x_k) \leq f(x_n) \quad \forall k \geq n.$$

Let  $p \in (\partial f)^{-1}(\mathbf{0})$ . Then by subdifferential inequality and part (i) of Theorem 3.9, for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} 0 &\leq f(x_n) - f(p) \\ &\leq \langle [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}], \overrightarrow{p x_n} \rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x_{n-1}}, \overrightarrow{p x_n} \rangle \\ &= \frac{1}{2\lambda_n} (d^2(x_{n-1}, p) - d^2(x_n, p) - d^2(x_n, x_{n-1})), \end{aligned}$$

which implies

$$(4.8) \quad 0 \leq \lambda_n(f(x_n) - f(p)) \leq \frac{1}{2}(d^2(x_{n-1}, p) - d^2(x_n, p)), \quad \forall n \in \mathbb{N}.$$

It follows that  $d(x_{n-1}, p) \leq d(x_n, p)$  for all  $n \in \mathbb{N}$ . Thus,  $\lim_n d(x_n, p)$  exists for all  $p \in (\partial f)^{-1}(\mathbf{0})$ . On the other hand, for all  $k \geq n$ , (4.8) and (4.7) imply

$$0 \leq \lambda_n(f(x_k) - f(p)) \leq \frac{1}{2}(d^2(x_{n-1}, p) - d^2(x_n, p)).$$

Summing up from  $n = 1$  to  $k$ , we get

$$\begin{aligned} 0 &\leq \sum_{n=1}^k \lambda_n(f(x_k) - f(p)) \\ &\leq \sum_{n=1}^k \frac{1}{2}(d^2(x_{n-1}, p) - d^2(x_n, p)) \\ &\leq \frac{1}{2}d^2(x_0, p). \end{aligned}$$

Dividing the above inequality by  $\sum_{n=1}^k \lambda_n$ , we obtain

$$0 \leq f(x_k) - f(p) \leq \frac{d^2(x_0, p)}{2 \sum_{n=1}^k \lambda_n}.$$

Letting  $k \rightarrow \infty$ , by assumptions, we get  $f(x_k) \rightarrow f(p)$ . Now, by the condition  $Q$ , let  $(x_{n_j})$  be a subsequence of the sequence  $(x_n)$  such that  $x_{n_j} \rightarrow q$ . By assumptions and Corollary 3.7,  $f$  is  $w$ -sequentially lower semicontinuous, thus we have

$$f(q) \leq \liminf_j f(x_{n_j}) = \lim_k f(x_k) = f(p),$$

which by (i) of Theorem 3.9, implies  $q \in (\partial f)^{-1}(\mathbf{0})$ . Now, Lemma 4.2 gives the desired result.  $\square$

**Theorem 4.10.** *Let  $X$  be a Hadamard space with dual  $X^*$ . Let  $X$  satisfies the condition  $Q$  and let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper, lower semicontinuous and convex function such that  $(\partial f)^{-1}(\mathbf{0}) \neq \emptyset$ . Suppose  $(\lambda_n)$  is a sequence of positive real numbers such that  $\sum_{n=1}^{\infty} \lambda_n = \infty$  and the sequence  $(x_n)$  is generated by the proximal point algorithm (4.6). If for every  $n \in \mathbb{N}$  and every  $k \leq n$ , there exists  $\alpha_{n,k} > 0$  such that the inclusion*

$$[\alpha_{n,k} \overrightarrow{(\frac{1}{2}x_n \oplus \frac{1}{2}x_k)}(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1})] \in \partial f(\frac{1}{2}x_n \oplus \frac{1}{2}x_k)$$

holds, then  $(x_n)$  converges strongly to a point  $q \in (\partial f)^{-1}(\mathbf{0})$ .



*Proof.* Let  $p \in (\partial f)^{-1}(\mathbf{0})$ . Using the monotonicity of  $\partial f$  and (4.6) we get that  $\lim_n d(x_n, p)$  exists. On the other hand, by subdifferential inequality and assumptions, for every  $n \in \mathbb{N}$  and every  $k \leq n$  there exists  $\alpha_{n,k} > 0$  such that

$$\begin{aligned} f(p) &\geq f\left(\frac{1}{2}x_n \oplus \frac{1}{2}x_k\right) + \overrightarrow{[\alpha_{n,k}(\frac{1}{2}x_n \oplus \frac{1}{2}x_k)(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1})], (\frac{1}{2}x_n \oplus \frac{1}{2}x_k)p} \\ &\geq f(p) + \overrightarrow{(\frac{1}{2}x_n \oplus \frac{1}{2}x_k)(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1}), (\frac{1}{2}x_n \oplus \frac{1}{2}x_k)p} \\ &\geq f(p) + \frac{\alpha_{n,k}}{2}(d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_k, p) - d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1}, p)). \end{aligned}$$

Thus, for every  $n \in \mathbb{N}$  and every  $k \leq n$ , we get

$$(4.9) \quad d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_k, p) \leq d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1}, p).$$

Substituting  $k$  by  $n, n-1, n-2, \dots, 0$  in (4.9), and using  $\frac{1}{2}x_n \oplus \frac{1}{2}x_n = x_n$ , we have

$$d^2(x_n, p) \leq d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_k, p) \quad \forall k \leq n,$$

which by the  $CN$ -inequality, for all  $k \leq n$  implies

$$\begin{aligned} d^2(x_n, x_k) &\leq 2d^2(x_n, p) + 2d^2(x_k, p) - 4d^2(\frac{1}{2}x_n \oplus \frac{1}{2}x_k, p) \\ &\leq 2(d^2(x_k, p) - d^2(x_n, p)). \end{aligned}$$

Thus  $(x_n)$  is a Cauchy sequence. Let  $q = \lim_n x_n$ . By Theorem 4.9,  $x_n \rightarrow p \in (\partial f)^{-1}(\mathbf{0})$ . Since  $x_n$  converges to  $q$  and  $w$ -converges to  $p$ , then  $p = q$ . This completes the proof.  $\square$

The following example is related to Theorem 4.10. In fact, we give a proper, lower semicontinuous and convex function  $f : X \rightarrow ]-\infty, +\infty]$  such that  $(\partial f)^{-1}(\mathbf{0}) \neq \emptyset$  and for every  $n \in \mathbb{N}$  and every  $k \leq n$ , there exists  $\alpha_{n,k} > 0$  such that

$$\overrightarrow{[\alpha_{n,k}(\frac{1}{2}x_n \oplus \frac{1}{2}x_k)(\frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1})] \in \partial f(\frac{1}{2}x_n \oplus \frac{1}{2}x_k),$$

where  $(x_n)$  is generated by the proximal point algorithm.

**Example 4.11.** Define  $f : ]0, \infty[ \rightarrow ]-\infty, +\infty]$  with  $f(x) = \frac{1}{2}x^2$ . Clearly,  $f$  is a proper, lower semicontinuous and convex function with  $\partial f(x) = \{x\}$  and  $(\partial f)^{-1}(\mathbf{0}) = \{0\}$ . If  $x_0 > 0$  in the proximal point algorithm (4.6), then for every  $n \in \mathbb{N}$  and every  $k \leq n$ , we have

$$\begin{aligned}
\frac{x_n + x_k}{2} &= \frac{x_n + x_k}{x_{k-1} - x_k} \left( \frac{x_{k-1} - x_k}{2} \right) \\
&= \overrightarrow{\left[ \frac{x_n + x_k}{x_{k-1} - x_k} \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_k \right) \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1} \right) \right]} \\
&\in \left\{ \frac{x_n + x_k}{2} \right\} \\
&= \partial f \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_k \right).
\end{aligned}$$

Hence, for every  $n \in \mathbb{N}$  and every  $k \leq n$ , there exists  $\alpha_{n,k} = \frac{x_n + x_k}{x_{k-1} - x_k} > 0$  that satisfies

$$\overrightarrow{\left[ \alpha_{n,k} \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_k \right) \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_{k-1} \right) \right]} \in \partial f \left( \frac{1}{2}x_n \oplus \frac{1}{2}x_k \right).$$

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### REFERENCES

- [1] B. Ahmadi Kakavandi, Weak topologies in complete CAT(0) metric spaces, *Proc. Amer. Math. Soc.* **141** (2013), no. 3, 1029–1039.
- [2] B. Ahmadi Kakavandi and M. Amini, Duality and subdifferential for convex functions on complete CAT(0) metric spaces, *Nonlinear Anal.* **73** (2010), no. 10, 3450–3455.
- [3] M. Bačák, The proximal point algorithm in metric spaces, *Israel J. Math.* **194** (2013), no. 2, 689–701.
- [4] I.D. Berg and I.G. Nikolaev, Quasilinearization and curvature of Aleksandrov spaces, *Geom. Dedicata* **133** (2008) 195–218.
- [5] O.A. Boikanyo and G. Morosanu, Modified Rockafellar's algorithms, *Math. Sci. Res. J.* **13** (2009), no. 5, 101–122.
- [6] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Fundamental Principles of Mathematical Sciences, Springer, Berlin, 1999.
- [7] H. Brézis and P.L. Lions, Produits infinis dérésolvantes, *Israel J. Math.* **29** (1978), no. 4, 329–345.
- [8] K.S. Brown, *Buildings*, Springer, New York, 1989.
- [9] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Grad. Stud. Math. 33, Amer. Math. Soc. Providence, RI, 2001.
- [10] S. Dhompongsa and B. Panyanak, On  $\Delta$ -convergence theorems in CAT(0) spaces, *Comput. Math. Appl.* **56** (2008) 2572–2579.
- [11] B. Djafari Rouhani and H. Khatibzadeh, On the proximal point algorithm, *J. Optim. Theory Appl.* **137** (2008), no. 2, 411–417.
- [12] R. Espínola and A. Fernández-León, CAT( $\kappa$ )-spaces, weak convergence and fixed points, *J. Math. Anal. Appl.* **353** (2009), no. 1, 410–427.
- [13] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 1984.

- [14] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, with appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by S.M. Bates, Progr. Math. 152, Birkhäuser, Boston, MA, 1999.
- [15] O. Güler, On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.* **29** (1991), no. 2, 403–419.
- [16] M.T. Heydari and S. Ranjbar, Halpern-type proximal point algorithm in complete CAT(0) metric spaces, *An. Ştiinţ. Univ. Ovidius Constanta Ser. Mat.* **24** (2016), no. 3, 141–159.
- [17] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Lectures Math. ETH Zürich, Birkhäuser, Basel 1997.
- [18] H. Khatibzadeh, Some remarks on the proximal point algorithm, *J. Optim. Theory Appl.* **153** (2012), no. 3, 769–778.
- [19] H. Khatibzadeh and S. Ranjbar, A variational inequality in complete CAT(0) spaces, *J. Fixed Point Theory Appl.* **17** (2015), no. 3, 557–574.
- [20] H. Khatibzadeh and S. Ranjbar, Monotone operators and the proximal point algorithm in complete CAT(0) metric spaces, *J. Aust. Math. Soc.* **2016** (2016) DOI:10.1017/s1446788716000446.
- [21] W.A. Kirk, Fixed point theorems in CAT(0) spaces and  $\mathbb{R}$ -trees, *Fixed Point Theory Appl.* **2004** (2004), no. 4, 309–316.
- [22] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Anal.* **68** (2008), no. 12, 3689–3696.
- [23] C. Li, G. López and V. Martín-Márquez, Monotone vector fields and the proximal point algorithm, *J. Lond. Math. Soc.* **679** (2009), no. 3, 663–683.
- [24] T.C. Lim, Remarks on some fixed point theorems, *Proc. Amer. Math. Soc.* **60** (1976) 179–182.
- [25] B. Martinet, Régularisation d'inéquations variationnelles par approximations successives, *Revue Française d'Informatique et de Recherche Opérationnelle*, **4** (1970) 154–158.
- [26] G. Morosanu, *Nonlinear Evolution Equations and Applications*, Editura Academiei Romane (and D. Reidel publishing Company), Bucharest, 1988.
- [27] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), no. 5, 877–898.

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