ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 853–865

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 853–865
Online ISSN: 1735-8515

SOME RESULTS ON THE SYMMETRIC DOUBLY STOCHASTIC INVERSE EIGENVALUE PROBLEM

W.-R. XU AND G.-L. CHEN*

(Communicated by Abbas Salemi)

Abstract. The *symmetric doubly stochastic inverse eigenvalue problem* (hereafter SDIEP) is to determine the necessary and sufficient conditions for an n-tuple $\sigma = (1, \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathbb{R}^n$ with $|\lambda_i| \leq 1,\ i = 1, 2, \ldots, n$, to be the spectrum of an $n \times n$ symmetric doubly stochastic matrix A. If there exists an $n \times n$ symmetric doubly stochastic matrix A with σ as its spectrum, then the list σ is *s.d.s. realizable*, or such that A s.d.s. realizes σ. In this paper, we propose a new sufficient condition for the existence of the symmetric doubly stochastic matrices with prescribed spectrum. Finally, some results about how to construct new s.d.s. realizable lists from the known lists are presented.
Keywords: Inverse eigenvalue problem, symmetric doubly stochastic matrix, sufficient condition.
MSC(2010): Primary: 65F18; Secondary: 15A51, 15A18, 15A12.

1. Introduction

A real square matrix with nonnegative entries all of whose row sums or column sums are equal to 1 is referred to as stochastic. Moreover, if all of its row sums and column sums are equal to 1, then it is said to be doubly stochastic. A real matrix $A = (a_{ij})_{n\times n}$ is said to be generalized doubly stochastic if all its row sums and column sums are the same constant, say α, i.e.

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = \alpha,\ i, j = 1, \ldots, n.$$

Therefore, a real matrix of order n is called generalized symmetric doubly stochastic if it is symmetric and generalized doubly stochastic.

Article electronically published on 30 June, 2017.
Received: 25 July 2015, Accepted: 14 March 2016.
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A list $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ of real numbers is *s.d.s. realizable* if there exists an $n \times n$ symmetric doubly stochastic matrix A with σ as its spectrum. In other words, the matrix A s.d.s. realizes σ.

In our paper, we present a wide range of bibliography on the SDIEP [1]-[26]. But so far, the SDIEP has only been solved for the case $n = 3$ by Perfect and Mirsky [18]. The case $n = 4$ for symmetric doubly stochastic matrices of trace zero has also been solved by them in the above reference. Subsequently, Mourad also solved the above case $n = 4$ by a mapping and convexity technique in [14]. In this reference, the case $n = 4$ with trace two is also solved. Because a complete characterization is unknown for all real n-tuples, the problem still remains open for the cases $n = 4$ with other nonzero trace and $n \geq 5$.

From [9], we know that there are six known realizability criteria for the SDIEP, i.e. Perfect and Mirsky's realizability criterion [18, Theorem 8], Soules' realizability criterion [23, Corollary 2.7], Zhu's realizability criterion [26, Theorem 5.1], Mourad's 1st realizability criterion [17] and [9, Theorem 6], Mourad's 2nd realizability criterion [17] and [9, Theorem 7] and Rojo's realizability criterion [20, Theorems 6, 8, 10, 11, 12 and 13].

Now we give two lists

$$\sigma_1 = (1, \frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$$

and

$$\sigma_2 = (1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}).$$

Through a computation, we find that these two lists do not satisfy any of the six known realizability criteria. But they are indeed s.d.s. realizable by the following two symmetric doubly stochastic matrices

$$A_1 = \frac{1}{48}\begin{pmatrix} 3 & 11 & 5 & 5 & 15 & 7 & 1 & 1 \\ 11 & 3 & 5 & 5 & 7 & 15 & 1 & 1 \\ 5 & 5 & 9 & 5 & 1 & 1 & 21 & 1 \\ 5 & 5 & 5 & 9 & 1 & 1 & 1 & 21 \\ 15 & 7 & 1 & 1 & 3 & 11 & 5 & 5 \\ 7 & 15 & 1 & 1 & 11 & 3 & 5 & 5 \\ 1 & 1 & 21 & 1 & 5 & 5 & 9 & 5 \\ 1 & 1 & 1 & 21 & 5 & 5 & 5 & 9 \end{pmatrix},$$

and

$$A_2 = \frac{1}{432}\begin{pmatrix} 25 & 25 & 25 & 25 & 16 & 241 & 25 & 25 & 25 \\ 25 & 25 & 25 & 25 & 16 & 25 & 241 & 25 & 25 \\ 25 & 25 & 25 & 25 & 16 & 25 & 25 & 241 & 25 \\ 25 & 25 & 25 & 25 & 16 & 25 & 25 & 25 & 241 \\ 16 & 16 & 16 & 16 & 304 & 16 & 16 & 16 & 16 \\ 241 & 25 & 25 & 25 & 16 & 25 & 25 & 25 & 25 \\ 25 & 241 & 25 & 25 & 16 & 25 & 25 & 25 & 25 \\ 25 & 25 & 241 & 25 & 16 & 25 & 25 & 25 & 25 \\ 25 & 25 & 25 & 241 & 16 & 25 & 25 & 25 & 25 \end{pmatrix},$$

respectively.

These inspire our interest to find a new realizability criterion to supplement the known realizability criteria.

Throughout our paper, we denote by $\mathbf{e}_i$, $\mathbf{I}_i$ and $\mathbf{0}$ the i-dimensional column vector of all ones, the identity matrix of order i and zero matrix with appropriate size respectively. Let $A(c : d,\ p : q)$ denote the submatrix of an $n \times n$ matrix A formed by taking the entries in the c-th row till to the d-th row and the p-th column till to the q-th column of the matrix A, where $1 \leq c \leq d \leq n$ and $1 \leq p \leq q \leq n$.

Finally, following the notations in [9] the set

$$\begin{aligned} \mathfrak{F}_n = \{\sigma = (1, \lambda_2, \ldots, \lambda_n) \subset \mathbb{R}^n \mid 1 \geq |\lambda_i| \text{ for all } i = 2, \ldots, n \\ \text{and } 1 + \textstyle\sum_{i=2}^{n} \lambda_i \geq 0\} \end{aligned}$$

signifies the polytope of $\mathbb{R}^n$ that strictly contains all the possible spectra of $n \times n$ symmetric doubly stochastic matrices. Meanwhile, the set

$$\Theta_n = \{\sigma \in \mathfrak{F}_n \mid \sigma \text{ is s.d.s. realizable}\}$$

denotes all the s.d.s. realizable lists.

2. A new realizability criterion for the SDIEP

In 1998, Elsner, Nabben and Neumann [4] introduced the following characterization of a Soules matrix.

Lemma 2.1 ([4])**.** *Let $R = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^{n\times n}$ be a matrix with columns $r_1, r_2, \ldots, r_n$, where r_1 is positive. Then the following conditions are equivalent:*

(1) *R is a Soules matrix.*

(2) $\displaystyle\sum_{i=1}^{m} r_i r_i^{\mathrm{T}} \geq 0$ *for $m = 1, 2, \ldots, n$, and* $\displaystyle\sum_{i=1}^{n} r_i r_i^{\mathrm{T}} = \mathbf{I}_n$.

In the above lemma, if there exists a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, such that the matrix $A = R\Lambda R^{\mathrm{T}} \geq \mathbf{0}$, then the ordered set $\{r_1, r_2, \ldots, r_n\}$ was called *Soules basis* in [4].

In 2012, Mourad [15] stated that if A is an $n\times n$ generalized doubly stochastic matrix whose each row and column sum is equal to s, then the sum of the components of the eigenvector corresponding to any eigenvalue except for the eigenvalue s is zero. Furthermore, the unit eigenvector associated with the eigenvalue s of the matrix A is $\frac{1}{\sqrt{n}}\mathbf{e}_n$.

Therefore, in order to construct an $n\times n$ symmetric doubly stochastic matrix for the SDIEP, we let $r_1 = \frac{1}{\sqrt{n}}\mathbf{e}_n$ in Lemma 2.1. From the above results, an immediate consequence is the following.

Corollary 2.2. *Let $P = (\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n) \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, where $x_i \in \mathbb{R}^n$ for any $i = 2, \ldots, n$. Then $\{\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n\}$ is a Soules basis if and only if the following n-tuples*

$$\sigma(s,t) = \{1, \underbrace{1,\ldots,1}_{s\ times}, \underbrace{0,\ldots,0}_{t\ times}\},$$

are s.d.s. realizable by the matrices $P\mathrm{diag}(\sigma(s,t))P^{\mathrm{T}}$, where $s,t \geq 0$ and $s+t = n-1$.

From the above corollary, we explicitly find that:

Observation 2.3. Perfect and Mirsky's realizability criterion and Soules' realizability criterion can be obtained via Soules basis.

In [23], a well-known Soules matrix is presented, that is,

$$U_k = (u_{ij})_{k\times k} = \begin{pmatrix} \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k(k-1)}} & \frac{1}{\sqrt{(k-1)(k-2)}} & \cdots & \frac{1}{\sqrt{4\times 3}} & \frac{1}{\sqrt{3\times 2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k(k-1)}} & \frac{1}{\sqrt{(k-1)(k-2)}} & \cdots & \frac{1}{\sqrt{4\times 3}} & \frac{1}{\sqrt{3\times 2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k(k-1)}} & \frac{1}{\sqrt{(k-1)(k-2)}} & \cdots & \frac{1}{\sqrt{4\times 3}} & \frac{-2}{\sqrt{3\times 2}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k(k-1)}} & \frac{1}{\sqrt{(k-1)(k-2)}} & \cdots & 0 & 0 & 0 \\ \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k(k-1)}} & \frac{-(k-2)}{\sqrt{(k-1)(k-2)}} & \cdots & 0 & 0 & 0 \\ \frac{1}{\sqrt{k}} & \frac{-(k-1)}{\sqrt{k(k-1)}} & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

For simplicity, $u_{ij} = 0$ for $i+j > k+2$, $u_{ij} = -\frac{i-1}{\sqrt{i(i-1)}}$ for $i+j = k+2$ and $u_{i1} = \frac{1}{\sqrt{k}}$, $u_{ij} = \frac{1}{\sqrt{(k-j+2)(k-j+1)}}$ $(j \neq 1)$ for $i+j < k+2$. All columns of the matrix U_k consist of not only a Soules basis, but also an orthonormal basis of unit eigenvectors of a $k\times k$ generalized doubly stochastic matrix. From [9], we know that Perfect and Mirsky's realizability criterion can also be obtained by using the Soules matrix U_n. Therefore, this criterion can be realized by Soules basis. Furthermore, in [17], an $n\times n$ orthogonal matrix V_β^σ is presented to realize Soules' realizability criterion. It is obvious to see that all columns of the matrix V_β^σ consists of a Soules basis. From Corollary 2.2, it can be verified by the fact that the n-tuples $\sigma(s,t)$ are all s.d.s. realizable. Therefore, Soules' realizability criterion can also be obtained via Soules basis.

Observation 2.4. Mourad's 1st realizability criterion and Mourad's 2nd realizability criterion can not be obtained via Soules basis.

Mourad [17] generalized Soules matrix to obtain the above two realizability criteria. But it can be seen that the n-tuple $\sigma(n-2,1)$ does not satisfy them. Therefore, Mourad's 1st and 2nd realizability criteria can not be achieved via Soules basis from Corollary 2.2.

Observation 2.5. Zhu's realizability criterion also can not be obtained via Soules basis.

In [26], the authors constructed a Householder matrix to obtain the above realizability criterion. But the Householder matrix is not a Soules matrix, because all the column vectors of the Householder matrix does not form a Soules basis. The reason is that the n-tuple $\sigma(1,n-2)$, $(n \geq 4)$ does not satisfy Zhu's realizability criterion.

In this section, we use the well-known Soules matrix U_k to construct an $n\times n$ orthogonal matrix P and explore the conditions under which $A = P\Lambda P^{\mathrm{T}} \geq 0$ and can be symmetric doubly stochastic with $\Lambda = \operatorname{diag}(1, \lambda_2, \ldots, \lambda_n)$. But the column vectors of matrix P does not form a Soules basis. For a set $\sigma \in \mathfrak{F}_n$, we present our new realizability criterion for n even and odd, respectively.

When $n = 2m$ $(m \geq 2)$ for $m \in \mathbb{Z}$, the authors in [24] constructed an orthogonal matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} U_m & U_m \\ U_m & -U_m \end{pmatrix}$$

to get the following realizability criterion.

Theorem 2.6 ([24])**.** *Let $n = 2m$ $(m \geq 2)$ for $m \in \mathbb{Z}$ and let $\sigma \in \mathfrak{F}_n$ with $1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$. If*

$$\frac{1+\lambda_{m+1}}{m} + \frac{\lambda_2+\lambda_{m+2}}{m(m-1)} + \frac{\lambda_3+\lambda_{m+3}}{(m-1)(m-2)} + \cdots + \frac{\lambda_m+\lambda_{2m}}{2\cdot 1} \geq 0 \tag{2.1}$$

and

$$\begin{cases} 1-\lambda_2-\lambda_{m+1}+\lambda_{m+2} \geq 0, \\ \lambda_i-\lambda_{i+1}-\lambda_{m+i}+\lambda_{m+i+1} \geq 0, \quad i = 2,\ldots,m-1, \end{cases} \tag{2.2}$$

hold, then the list σ belongs to Θ_n.

For example, the $2m$-tuple $\sigma(m,m-1)$ does not satisfy the above realizability criterion. Therefore, the above realizability criterion can not be obtained via Soules basis and the orthogonal matrix R is not a Soules matrix. Through calculation, we easily find that the list $\sigma_1 \in \mathfrak{F}_8$ satisfies the conditions (2.1) and (2.2), though it doesn't satisfy any of the known realizability criteria for the SDIEP.

When $n = 2m-1$ $(m \geq 3)$ for $m \in \mathbb{Z}$, we use the Soules matrix U_k to construct an orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c|c} \mathbf{e}_{m-1}\left(\frac{\sqrt{2}}{\sqrt{2m-1}}, \quad \frac{1}{\sqrt{(2m-1)(m-1)}} \right) & U_m(1:m-1,\ 3:m) & U_{m-1} \\ \hline \left(\frac{\sqrt{2}}{\sqrt{2m-1}}, \quad \frac{-2(m-1)}{\sqrt{(2m-1)(m-1)}} \right) & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{e}_{m-1}\left(\frac{\sqrt{2}}{\sqrt{2m-1}}, \quad \frac{1}{\sqrt{(2m-1)(m-1)}} \right) & U_m(1:m-1,\ 3:m) & -U_{m-1} \end{array} \right).$$

From Mourad's statement in [15], all columns in the matrix Q constitutes an orthonormal basis of unit eigenvectors of a generalized doubly stochastic matrix of order $2m-1$. Then we present the following result.

Theorem 2.7. *Let $n = 2m-1$ $(m \geq 3)$ for $m \in \mathbb{Z}$ and let $\sigma \in \mathfrak{F}_n$ with $1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$. If*

(2.3)
$$\frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} + \sum_{k=3}^{m} \frac{\lambda_k + \lambda_{m+k-1}}{2(m-k+2)(m-k+1)} + \frac{\lambda_{m+1}}{2m-2} \geq 0,$$

$$\frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} - \frac{\lambda_3}{2m-2} - \frac{\lambda_{m+1}}{2m-2} + \frac{\lambda_{m+2}}{2m-2} \geq 0 \tag{2.4}$$

and

$$\begin{aligned} &\frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} + \sum_{k=3}^{m-j+1} \frac{\lambda_k - \lambda_{m+k-1}}{2(m-k+2)(m-k+1)} \\ &\qquad - \frac{\lambda_{m-j+2} - \lambda_{2m-j+1}}{2j} - \frac{\lambda_{m+1}}{2m-2} \geq 0, \ j = 2, 3, \ldots, m-2, \end{aligned} \tag{2.5}$$

hold, then the list σ belongs to Θ_n.

Proof. When $n = 2m-1$ $(m \geq 3)$ for $m \in \mathbb{Z}$, we set $\Lambda = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_n)$, where $1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$.

Then we can get an $n \times n$ generalized symmetric doubly stochastic matrix $A = (a_{ij})_{n\times n} = Q\Lambda Q^{\mathrm{T}}$ all of whose row sums and column sums are equal to 1.

The diagonal entries of the matrix A satisfy $a_{jj} = a_{m+j,m+j}$ for any $j = 1, 2, \ldots, m-1$, and $a_{mm} \geq a_{m-1,m-1} \geq \cdots \geq a_{33} \geq a_{22} = a_{11}$. Moreover,

$$a_{11} = \frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} + \sum_{k=3}^{m} \frac{\lambda_k + \lambda_{m+k-1}}{2(m-k+2)(m-k+1)} + \frac{\lambda_{m+1}}{2m-2}.$$

If the condition (2.3) holds, then $a_{11} \geq 0$. Consequently, all the diagonal entries of the matrix A are nonnegative.

Because the matrix A is symmetric, we consider the entries in its strictly upper triangular position. Then from the matrix A, we have the following results.

(1) When $j = m+1, m+2, \ldots, 2m-2$ and $k = 1, 2, \ldots, m-2$, the entries

$$a_{m-1,j} = a_{k,2m-1} = \frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} - \frac{\lambda_3}{2m-2} - \frac{\lambda_{m+1}}{2m-2} + \frac{\lambda_{m+2}}{2m-2}.$$

If the condition (2.4) holds, then these entries are nonnegative.

(2) When $i = 2, 3, \ldots, m-2$ and $j = 1, \ldots, i-1$, the entries

$$\begin{aligned} a_{i,m+j} = a_{1,m+i} = \quad & \frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} \\ & + \sum_{k=3}^{m-i+1} \frac{\lambda_k - \lambda_{m+k-1}}{2(m-k+2)(m-k+1)} \\ & - \frac{\lambda_{m-i+2} - \lambda_{2m-i+1}}{2i} - \frac{\lambda_{m+1}}{2m-2}. \end{aligned}$$

When $i = 2, 3, \ldots, m-3$ and $j = i+1, i+2, \ldots, m-2$, the entries

$$\begin{aligned} a_{i,m+j} = a_{1,m+j} = \quad & \frac{1}{2m-1} + \frac{\lambda_2}{(2m-1)(2m-2)} \\ & + \sum_{k=3}^{m-j+1} \frac{\lambda_k - \lambda_{m+k-1}}{2(m-k+2)(m-k+1)} \\ & - \frac{\lambda_{m-j+2} - \lambda_{2m-j+1}}{2j} - \frac{\lambda_{m+1}}{2m-2}. \end{aligned}$$

If the condition (2.5) holds, then the above entries are nonnegative.

(3) Since $1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$, all the other entries in the strictly upper triangular position of the matrix A involve $\frac{1}{n}(1-\lambda_2)$ and all the other terms involve sums of nonnegative terms. Thus these other entries are nonnegative.

Finally, we conclude that if the conditions (2.3),(2.4) and (2.5) are satisfied, then the matrix A is a symmetric doubly stochastic matrix. Therefore, the list σ belongs to Θ_n. $\square$

The list $\sigma_2 \in \mathfrak{F}_9$ satisfies the conditions (2.3),(2.4) and (2.5). Thus it demonstrates that our realizability criterion is effective. In addition, the $(2m-1)$-tuple $\sigma(m, m-2)$ does not satisfy the above realizability criterion, then this realizability criterion can not be obtained via Soules basis and the orthogonal matrix Q is not a Soules matrix.

Remark 2.8. Let $\sigma \in \mathfrak{F}_n$ with $1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$. Mourad's 1st realizability criterion is equivalent to our new realizability criterion for the cases $n = 4$ and $n = 5$.

From Theorem 2.7, we can also get the following result concerning the inverse eigenvalue problem for symmetric positive doubly stochastic matrices.

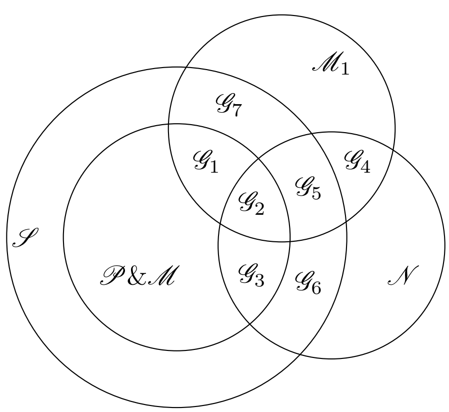
Corollary 2.9. *Let $n = 2m-1$ $(m \geq 3)$ for $m \in \mathbb{Z}$ and the elements in the list $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ satisfy $1 + \lambda_2 + \cdots + \lambda_n > 0$ and $1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n > -1$. If the conditions (2.3), (2.4) and (2.5) are all strict, then the list σ can be the spectrum of an $n \times n$ symmetric positive doubly stochastic matrix.*

Proof. In the proof of Theorem 2.7, if the conditions (2.3), (2.4) and (2.5) are all strict and $1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n > -1$, then the entries in the upper triangular position of the matrix A are all positive. Thus the list σ is the spectrum of an n-by-n symmetric positive doubly stochastic matrix A. □

Mourad's 1st realizability criterion, Mourad's 2nd realizability criterion and this new realizability criterion are all obtained by generalizing Soules matrix, not by constructing Soules basis. In addition, Zhu's realizability criterion is obtained by using Householder matrices, rather than Soules matrices. Rojo's realizability criterion is obtained by fast Fourier transformation, also not involving Soules matrix.

Because Mourad's 2nd realizability criterion does not refine Mourad's 1st realizability criterion effectively, and Rojo's realizability criterion requires irreducible realizable matrices with special structure. Therefore, we will establish the following relationship between the general realizability criteria involving Soules matrix, i.e. Perfect and Mirsky's realizability criterion, Soules' realizability criterion, Mourad's 1st realizability criterion and our new realizability criterion.

Observation 2.10. Let $\mathscr{P}\&\mathscr{M}, \mathscr{S}, \mathscr{M}_1$ and $\mathscr{N}$ be the point sets of $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ which can be s.d.s. realized by Perfect and Mirsky's realizability criterion, Soules' realizability criterion, Mourad's 1st realizability criterion and our new realizability criterion respectively, where $\sigma \in \mathfrak{F}_n$ with non-increasing order. Then the Venn diagram of the point sets $\mathscr{P}\&\mathscr{M}, \mathscr{S}, \mathscr{M}_1$ and $\mathscr{N}$ can be presented as follows:



$$
\begin{aligned}
\mathscr{G}_1 &= \mathscr{P}\&\mathscr{M} \bigcap \mathscr{M}_1 - \mathscr{G}_2 \\
\mathscr{G}_2 &= \mathscr{P}\&\mathscr{M} \bigcap \mathscr{M}_1 \bigcap \mathscr{N} \\
\mathscr{G}_3 &= \mathscr{P}\&\mathscr{M} \bigcap \mathscr{N} - \mathscr{G}_2 \\
\mathscr{G}_4 &= \mathscr{M}_1 \bigcap \mathscr{N} - \mathscr{G}_2 - \mathscr{G}_5 \\
\mathscr{G}_5 &= (\mathscr{S} - \mathscr{P}\&\mathscr{M}) \bigcap \mathscr{M}_1 \bigcap \mathscr{N} \\
\mathscr{G}_6 &= (\mathscr{S} - \mathscr{P}\&\mathscr{M}) \bigcap \mathscr{N} - \mathscr{G}_5 \\
\mathscr{G}_7 &= (\mathscr{S} - \mathscr{P}\&\mathscr{M}) \bigcap \mathscr{M}_1 - \mathscr{G}_5
\end{aligned}
$$

The diagram can be verified by the following lists.

(1) $(1, 0, 0, 0, 0, -\frac{1}{3}) \in \mathscr{G}_1$
(2) $(1, 1, 0, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}) \in \mathscr{G}_2$
(3) $(1, \frac{1}{2}, \frac{1}{2}, \frac{7}{24}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}) \in \mathscr{G}_3$

(4) $(1, \frac{1}{6}, -\frac{1}{2}, -\frac{2}{3}) \in \mathscr{G}_4$
(5) $(1, \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}) \in \mathscr{G}_5$
(6) $(1, \frac{1}{3}, \frac{1}{3}, 0, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}) \in \mathscr{G}_6$
(7) $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}) \in \mathscr{G}_7$
(8) $(1, \frac{1}{3}, \frac{1}{3}, 0, 0, -\frac{2}{3}) \in \mathscr{M}_1 - \mathscr{N} - \mathscr{S}$
(9) $(1, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in \mathscr{N} - \mathscr{M}_1 - \mathscr{S}$
(10) $(1, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}) \in \mathscr{P}\&\mathscr{M} - \mathscr{N} - \mathscr{M}_1$
(11) $(1, 1, 1, 0, -\frac{1}{2}, -\frac{2}{3}) \in \mathscr{S} - \mathscr{P}\&\mathscr{M} - \mathscr{N} - \mathscr{M}_1$.

Because Zhu's realizability criterion can be seen as a general realizability criterion, we might as well give three examples to compare it with our new realizability criterion. For convenience, $\mathscr{Z}$ signifies the point set of $\sigma = (1, \lambda_2, \ldots, \lambda_n)$ which can be s.d.s. realized by Zhu's realizability criterion, where $\sigma \in \mathfrak{F}_n$ with non-increasing order.

Observation 2.11. $(1, \frac{1}{2}, \frac{1}{2}, 0, -1) \in \mathscr{N} - \mathscr{Z}$, $(1, 0, 0, 0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) \in \mathscr{Z} - \mathscr{N}$, $(1, \frac{14}{40}, \frac{14}{40}, \frac{1}{5}, 0, -\frac{7}{40}, -\frac{7}{40}, -\frac{7}{40}, -\frac{7}{40}) \in \mathscr{Z} \bigcap \mathscr{N}$.

In conclusion, for the SDIEP, although there exist some points not satisfying any of the known realizability criteria, we have used our new realizability criterion to supplement the existing realizability criteria effectively.

3. Constructing new s.d.s realizable lists from the known lists

In this section, we will present some results about how to construct new s.d.s. realizable lists from the known lists.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be an n-by-n symmetric doubly stochastic matrix constructed by Soules basis $\{\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n\}$, where $x_i \in \mathbb{R}^n$ for any $i = 2, \ldots, n$. Let the spectrum of A be $\sigma(A) = (1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n)$. Then the matrix*

$$B = (1-t)A + tECE^{\mathrm{T}}$$

is also an n-by-n symmetric doubly stochastic matrix with spectrum

$$\begin{aligned}\sigma(B) \quad = \quad &(\ 1, (1-t)\lambda_2 + t, \ldots, (1-t)\lambda_k + t, (1-t)\lambda_{k+1}, \ldots, (1-t)\lambda_r, \\ &(1-t)\lambda_{r+1}, \ldots, (1-t)\lambda_n\ ),\end{aligned}$$

where $0 \le t \le 1$, $X = (\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n)$, $E = X(1:n, 1:r)$ and

$$C = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k\ times}, 0, \ldots, 0) \in \mathbb{R}^{r\times r}$$

for any $k = 1, 2, \ldots, r$.

Proof. Because $\{\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n\}$ is a Soules basis, then X is a Soules matrix. From Corollary 2.2, $\sigma(k-1, n-k)$ is s.d.s. realizable by the $n \times n$ matrix

$ECE^{\mathrm{T}} = X\mathrm{diag}(\sigma(k-1,n-k))X^{\mathrm{T}}$. Then the matrix $B = (1-t)A + tECE^{\mathrm{T}}$ $(0 \le t \le 1)$ is an $n \times n$ symmetric doubly stochastic matrix.

Moreover, the matrix A is constructed by Soules basis $\{\frac{1}{\sqrt{n}}\mathbf{e}_n, x_2, \ldots, x_n\}$, so $X^{\mathrm{T}}AX = \mathrm{diag}(\sigma(A))$. Then

$$\begin{aligned} X^{\mathrm{T}}BX &= (1-t)X^{\mathrm{T}}AX + t\begin{pmatrix} \mathbf{I}_r \\ \mathbf{0} \end{pmatrix} C \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \end{pmatrix} \\ &= (1-t)\mathrm{diag}(\sigma(A)) + t\begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \end{aligned}$$

Therefore, the spectrum of B is $\sigma(B) = (1, (1-t)\lambda_2 + t, \ldots, (1-t)\lambda_k + t, (1-t)\lambda_{k+1}, \ldots, (1-t)\lambda_r, (1-t)\lambda_{r+1}, \ldots, (1-t)\lambda_n)$. $\square$

In particular, if we take $k = 1$ in the above theorem, then $\sigma(B) = (1, (1-t)\lambda_2, (1-t)\lambda_3, \ldots, \ldots, (1-t)\lambda_n)$ is s.d.s. realizable. Next, the following result is presented.

Theorem 3.2. *Let $A \in \mathbb{R}^{m\times m}$ be a symmetric doubly stochastic matrix and the spectrum of $\frac{m}{n}A$ be $(\frac{m}{n}, \lambda_2, \ldots, \lambda_m)$. Let $B \in \mathbb{R}^{(n-m)\times(n-m)}$ $(n > m)$ be a symmetric doubly stochastic matrix and the spectrum of $\frac{n-m}{n}B$ be $(\frac{n-m}{n}, \mu_2, \ldots, \mu_{n-m})$. Then $(1, 0, \lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_{n-m})$ is s.d.s. realizable.*

Proof. Firstly, we construct the following $n \times n$ symmetric doubly stochastic matrix

$$C = \begin{pmatrix} \frac{m}{n}A & \frac{1}{n}\mathbf{e}_m\mathbf{e}_{n-m}^{\mathrm{T}} \\ \frac{1}{n}\mathbf{e}_{n-m}\mathbf{e}_m^{\mathrm{T}} & \frac{n-m}{n}B \end{pmatrix}.$$

Then there exists a nonsingular matrix $T = \begin{pmatrix} \mathbf{I}_m & \mathbf{0} \\ -\frac{1}{m}\mathbf{e}_{n-m}\mathbf{e}_m^{\mathrm{T}} & \mathbf{I}_{n-m} \end{pmatrix}$ such that

$$TCT^{-1} = \begin{pmatrix} \frac{m}{n}A + \frac{n-m}{mn}\mathbf{e}_m\mathbf{e}_m^{\mathrm{T}} & \frac{1}{n}\mathbf{e}_m\mathbf{e}_{n-m}^{\mathrm{T}} \\ \mathbf{0} & \frac{n-m}{n}B - \frac{1}{n}\mathbf{e}_{n-m}\mathbf{e}_{n-m}^{\mathrm{T}} \end{pmatrix}.$$

Because the spectra of $\frac{m}{n}A + \frac{n-m}{mn}\mathbf{e}_m\mathbf{e}_m^{\mathrm{T}}$ and $\frac{n-m}{n}B - \frac{1}{n}\mathbf{e}_{n-m}\mathbf{e}_{n-m}^{\mathrm{T}}$ are, respectively, $(1, \lambda_2, \ldots, \lambda_m)$ and $(0, \mu_2, \ldots, \mu_{n-m})$ by Theorem 3.1, then the list $(1, 0, \lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_{n-m})$ is the spectrum of C. $\square$

If we take B to be $\frac{1}{n-m}\mathbf{e}_{n-m}\mathbf{e}_{n-m}^{\mathrm{T}}$, then an immediate consequence is the following.

Corollary 3.3. *Let $A \in \mathbb{R}^{m\times m}$ be a symmetric doubly stochastic matrix and the spectrum of $\frac{m}{n}A$ be $(\frac{m}{n}, \lambda_2, \ldots, \lambda_m)$.*

(1) Then $(1, \lambda_2, \ldots, \lambda_m, \underbrace{0, \ldots, 0}_{n-m \text{ times}})$ $(n > m)$ is s.d.s. realizable.

(2) If A is also positive, then $(1, \lambda_2, \ldots, \lambda_m, \underbrace{0, \ldots, 0}_{n-m \text{ times}})$ $(n > m)$ is positively s.d.s. realizable.

Proof. From the proof of Theorem 3.2, we can construct a symmetric doubly stochastic matrix

$$D = \begin{pmatrix} \frac{m}{n}A & \frac{1}{n}\mathbf{e}_m\mathbf{e}_{n-m}^{\mathrm{T}} \\ \frac{1}{n}\mathbf{e}_{n-m}\mathbf{e}_m^{\mathrm{T}} & \frac{1}{n}\mathbf{e}_{n-m}\mathbf{e}_{n-m}^{\mathrm{T}} \end{pmatrix}$$

by taking $B = \frac{1}{n-m}\mathbf{e}_{n-m}\mathbf{e}_{n-m}^{\mathrm{T}}$. Then the matrix D can s.d.s. realize the new list $(1, \lambda_2, \ldots, \lambda_m, \underbrace{0, \ldots, 0}_{n-m\ times})$ $(n > m)$. If A is also positive, then D is a symmetric positive doubly stochastic matrix. At this moment, $(1, \lambda_2, \ldots, \lambda_m, \underbrace{0, \ldots, 0}_{n-m\ times})$ $(n > m)$ can be positively s.d.s. realized by it. Therefore, the proof is completed. $\square$

Finally, we consider the following function set

$$\begin{aligned} \mathfrak{M} \quad := \quad & \{f \text{ be holomorphic over the interval } [-1,1] \; : \; f(1) = 1, \\ & f^{(k)}(0) \geq 0, k = 1, 2, \ldots \} \end{aligned}$$

and result in [18].

Theorem 3.4 ([18])**.** *Let λ be an eigenvalue of an $n \times n$ doubly stochastic matrix and $f \in \mathfrak{M}$, then $f(\lambda)$ is also an eigenvalue of a doubly stochastic matrix of order n.*

Then we have the following result.

Theorem 3.5. *Let $\sigma \in \mathfrak{F}_n$ and $\sigma \in \Theta_n$. If $f \in \mathfrak{M}$, then $f(\sigma) \in \mathfrak{F}_n$ and $f(\sigma) \in \Theta_n$, where $f(\sigma) = (1, f(\lambda_2), \ldots, f(\lambda_n))$.*

Proof. Because $f \in \mathfrak{M}$, the Taylor expansion of f around the origin is convergent, i.e. $f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} z^k, |z| \leq 1$. Then $f(1) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!}$. Assume the list σ can be s.d.s. realized by a symmetric doubly stochastic matrix A of order n. Then from the spectral mapping theorem in [25], we know that the spectrum of the matrix $f(A)$ is $f(\sigma)$. Because the power of a symmetric doubly stochastic matrix is also symmetric doubly stochastic, $f(A) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} A^k$ is symmetric doubly stochastic. Therefore, we get $f(\sigma) \in \Theta_n$ and $f(\sigma) \in \mathfrak{F}_n$. $\square$

Conjecture 3.6. Let $\sigma \in \mathfrak{F}_n$ and σ is not in Θ_n. There may exist a function $f \in \mathfrak{M}$ such that $f(\sigma) \in \mathfrak{F}_n$ and $f(\sigma) \in \Theta_n$.

For example, the following list

$$\sigma_3 = (1, 1, 1, \frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, -1)$$

in [13] is not in Θ_8. If we take a function $f(x) = x$ in the set $\mathfrak{M}$, then $f(\sigma_3)$ is not in Θ_8. But if we take a function $g(x) = x^3$ in the set $\mathfrak{M}$, then $g(\sigma_3) \in \Theta_8$. Because we can partition the list

$$g(\sigma_3) = (1, 1, 1, \frac{1}{64}, -\frac{27}{64}, -\frac{27}{64}, , -\frac{27}{64}, -1)$$

into three sublists

$$g(\sigma_{31}) = (1, -1),\ g(\sigma_{32}) = (1, -\frac{27}{64}),\ g(\sigma_{33}) = (1, \frac{1}{64}, -\frac{27}{64}, -\frac{27}{64}).$$

By Perfect and Mirsky's realizability criterion, we apparently know that the above three sublists are all s.d.s. realizable. Therefore, $g(\sigma_3) \in \Theta_8$.

Acknowledgments

The authors are grateful to Prof. Abbas Salemi and the anonymous referees for their valuable comments and suggestions which improved the original manuscript. This work is supported by the National Natural Science Foundation of China (Grant No. 11471122).

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