ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 867–874

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 867–874
Online ISSN: 1735-8515

ON THE TYPE OF CONJUGACY CLASSES AND THE SET OF INDICES OF MAXIMAL SUBGROUPS

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(Communicated by Mohammad Reza Darafsheh)

Abstract. Let G be a finite group. By $MT(G) = (m_1, \cdots, m_k)$ we denote the type of conjugacy classes of maximal subgroups of G, which implies that G has exactly k conjugacy classes of maximal subgroups and $m_1, \ldots, m_k$ are the numbers of conjugates of maximal subgroups of G, where $m_1 \leq \cdots \leq m_k$. In this paper, we give some new characterizations of finite groups by the type of conjugacy classes of maximal subgroups. By $\pi_t(G)$ we denote the set of indices of all maximal subgroups of G. We also investigate the influence of the set of indices of all maximal subgroups on the structure of finite groups.
Keywords: Maximal subgroup, non-abelian simple group, the type of conjugacy classes, the set of indices.
MSC(2010): Primary: 20D05; Secondary: 20D10.

1. Introduction

In this paper all groups are finite. In [14] Wang defined the type of conjugacy classes of maximal subgroups.

Definition 1.1. ([14]). Let G be a group having exactly k conjugacy classes of maximal subgroups and $m_1, \ldots, m_k$ the numbers of conjugates of all maximal subgroups of G, where $m_1 \leq \cdots \leq m_k$. Then the sequence $MT(G) = (m_1, \cdots, m_k)$ is called the type of conjugacy classes of maximal subgroups of G.

In [14], Wang used $MT(G)$ to show that a non-solvable group G has exactly 21 maximal subgroups if and only if $G/\Phi(G)$ is isomorphic to the alternating group A_5, where $\Phi(G)$ is the Frattini subgroup of G.

In [12], we applied $MT(G)$ to characterize some groups having exactly four conjugacy classes of maximal subgroups, some simple groups and the equality

Article electronically published on 30 June, 2017.
Received: 26 January 2016, Accepted: 17 March 2016.
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for $N < G$, respectively. And in [15], we gave a new characterization of all alternating groups and some symmetric groups by $MT(G)$.

Note that the number of conjugates of any normal maximal subgroup equals 1, and the number of conjugates of any non-normal maximal subgroup equals its index.

Let G and N be two groups with $MT(G) = MT(N)$. Let $\pi_t(G)$ be the set of indices of all maximal subgroups of G and $\pi_t(N)$ the set of indices of all maximal subgroups of N. If G and N have no normal maximal subgroups, then $\pi_t(G) = \pi_t(N)$. However, if G and N have at least one normal maximal subgroup, we cannot get $\pi_t(G) = \pi_t(N)$. For example, it is easy to see that $MT(S_5) = MT(A_5 \times \mathbb{Z}_p) = (1, 5, 6, 10)$, where p is an odd prime, but $\pi_t(S_5) = \{2, 5, 6, 10\} \neq \pi_t(A_5 \times \mathbb{Z}_p) = \{p, 5, 6, 10\}$.

Conversely, let G and N be two groups with $\pi_t(G) = \pi_t(N)$, we also cannot get $MT(G) = MT(N)$. For example, it is easy to see that $\pi_t(PSL_2(7)) = \pi_t(\mathbb{Z}_2{}^3 \rtimes \mathbb{Z}_7) = \{7, 8\}$, but $MT(PSL_2(7)) = (7, 7, 8) \neq MT(\mathbb{Z}_2{}^3 \rtimes \mathbb{Z}_7) = (1, 8)$.

For the type of conjugacy classes of maximal subgroups, the following Proposition 1.2 is a direct corollary of [7].

Proposition 1.2. *Let G be a simple group and $N \leq G$. If $MT(N) = MT(G)$, then $N = G$.*

In [13] we proved the following result:

Lemma 1.3. ([13, Lemma 1]). *Let G be a group and $N \leq G$. If $G/\Phi(G)$ is a non-abelian simple group, then $MT(N) = MT(G)$ if and only if $N = G$.*

Lemma 1.3 is not true if $G/\Phi(G)$ is an abelian simple group. For example, let $G = \mathbb{Z}_{p^n}$ and $N = \mathbb{Z}_p \leq G$, where p is a prime and $n \geq 2$. It is easy to see that $G/\Phi(G) \cong \mathbb{Z}_p$ and $MT(N) = MT(G) = (1)$, but $N < G$.

The following Proposition 1.4 is a direct consequence of [8] and Lemma 1.3.

Proposition 1.4. *Let G be a group and N a non-abelian simple subgroup of G. If $MT(N) = MT(G)$, then $N = G$.*

Proposition 1.4 is not true if N is an abelian simple group. For example, let $G = \mathbb{Z}_{p^2}$ and $N = \mathbb{Z}_p \leq G$, where p is a prime. It is obvious that $MT(N) = MT(G)$ but $N < G$.

Motivated by above results, we give a further study of the structure of groups by the type of conjugacy classes of maximal subgroups, some new characterizations of groups are obtained, see Section 3.

Let G be a non-abelian simple group and N a subgroup of G. If $\pi_t(N) \subseteq \pi_t(G)$. By [7], we get:

when $\pi_t(N) = \pi_t(G)$, we have

(1) $N = G$;

when $\pi_t(N) \subset \pi_t(G)$, we have
(2) $N < G$, where $G \cong M_{11}$, $N \cong PSL_2(11)$; or
(3) $N < G$, where $G \cong S_6(2)$, $N \cong U_3(3)$; or
(4) $N < G$, where G is a non-abelian simple group having a maximal subgroup with index a prime q, N is a subgroup of G of order q .

In case (4), by [11, Theorem 12], we know that q is not only the largest prime divisor of $|G|$ but also the smallest number in $\pi_t(G)$.

According to the above result, we propose the following problem:

Problem 1.5. Let G be a non-abelian simple group and N a subgroup of G such that $\pi_t(G) \subseteq \pi_t(N)$. Is it always true that we have $N = G$?

It is obvious that Problem 1.5 holds when G is a minimal simple group. In Section 4 of this paper, we will give a further study of Problem 1.5.

2. Preliminaries

In this paper, we denote $S(G)$ the largest solvable normal subgroup of a group G and $p(G)$ the smallest number in $\pi_t(G)$.

Lemma 2.1. ([2]). *If every maximal subgroup of a group G has prime-power index, then $G/S(G) \cong 1$ or $PSL_2(7)$.*

Lemma 2.2. ([7]). *Let N be a group and G a simple group. If $|N|$ divides $|G|$ and $\pi_t(N) \subseteq \pi_t(G)$, then*
when N is non-solvable, we have
(1) $N \cong G$, or
(2) $G = M_{11}$ and $N \cong PSL_2(11)$ or $SL_2(11)$, or
(3) $G = S_6(2)$ and $N \cong U_3(3)$.
when N is solvable, we have
(4) N is a cyclic group of prime order, or
(5) $N = \langle a, b, c, g \mid a^2 = b^2 = c^2 = g^7 = 1, [a, b] = [a, c] = [b, c] = 1, a^g = c, b^g = a, c^g = bc\rangle$.

Lemma 2.3. ([6, 9]). *Let G be a non-solvable group having exactly n same order classes of maximal subgroups.*
(1) If $n = 2$, then $G/\Phi(G) \cong (\mathbb{Z}_2{}^{3i} \rtimes PSL_2(7)) \times \mathbb{Z}_7{}^j$, where $i, j = 0, 1, \ldots$;
(2) If $n = 3$, then $G/S(G) \cong A_6$; $PSL_2(q)$, $q = 11, 13, 23, 59, 61$; $PSL_3(3)$; $U_3(3)$; $PSL_5(2)$; $PSL_2(2^f)$, f is a prime; $PSL_2(7) \times PSL_2(7) \times \ldots \times PSL_2(7)$.

Lemma 2.4. ([12]). *Let $G = T \times \mathbb{Z}_p$ and $N < G$ such that $MT(N) = MT(G)$, where T is a non-abelian simple group and p is a prime. Then T has a non-solvable proper subgroup that is isomorphic to N.*

Lemma 2.5. ([10]). *Let G be a simple K_4-group, then G is isomorphic to one of the following simple groups:*

(1) A_n, $n = 7, 8, 9, 10$;
(2) M_{11}, M_{12}, J_2;
(3) (*a*) $PSL_2(r)$, *where* $r^2 - 1 = 2^a 3^b u^c$, $a \geq 1$, $b \geq 1$, $c \geq 1$, r *and* u *are primes*, $u > 3$;
(*b*) $PSL_2(2^m)$, *where* $\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t^b, \end{cases}$ $m \geq 1$, u *and* t *are primes*, $t > 3$, $b \geq 1$;
(*c*) $PSL_2(3^m)$, *where* $\begin{cases} 3^m + 1 = 4t \\ 3^m - 1 = 2u^c, \end{cases}$ *or* $\begin{cases} 3^m + 1 = 4t^b \\ 3^m - 1 = 2u, \end{cases}$ $m \geq 1$, u *and* t *are odd primes*, $b \geq 1$, $c \geq 1$;
(*d*) $PSL_2(q)$, $q = 16, 25, 49, 81$; $PSL_3(q)$, $q = 4, 5, 7, 8, 17$; $PSL_4(3)$; $S_4(q)$, $q = 4, 5, 7, 9$; $S_6(2)$; ${O_8}^+(2)$; $G_2(3)$; $U_3(q)$, $q = 4, 5, 7, 8, 9$; $U_4(3)$; $U_5(2)$; $S_Z(8)$; $S_z(32)$; ${}^3D_4(2)$; ${}^2F_4(2)'$.

3. Some results on $MT(G)$

Theorem 3.1. *Let G be a simple group and K a group. Suppose that N is a subgroup of $G \times K$ satisfying $MT(N) = MT(G)$. If $N \cap G \neq 1$, then $N = G$.*

Proof. Since $N \cap G \neq 1$, one has $N \nleq K$ and consequently $1 < N/(N \cap K) \cong NK/K \leq GK/K \cong G$. Note that $MT(N) = MT(G)$. Thus we always have $\pi_t(N) = \pi_t(G)$ whenever G is an abelian simple group or a non-abelian simple group. It follows that $\pi_t(N/(N \cap K)) \subseteq \pi_t(N) = \pi_t(G)$. Note that $MT(N) = MT(G)$. By Lemma 2.2, we have $N/(N \cap K) \cong G$ and consequently $MT(N/(N \cap K)) = MT(G) = MT(N)$. It follows that $N \cap K \leq \Phi(N)$. Moreover, since $N/(N \cap K) \cong G$ is a simple group, we must have $N \cap K = \Phi(N)$. Then $N/\Phi(N)$ is a simple group.

Observe that $(N \cap G)\Phi(N)$ is normal in N. One has $(N \cap G)\Phi(N) = \Phi(N)$ or N. If $(N \cap G)\Phi(N) = \Phi(N)$. Then we have $N \cap G = (N \cap G) \cap \Phi(N) = (N \cap G) \cap (N \cap K) = 1$, which contradicts $N \cap G \neq 1$. Therefore $(N \cap G)\Phi(N) = N$. It follows that $N = N \cap G \leq G$. By Lemma 2.2, we have $N = G$. □

The hypothesis that $N \cap G \neq 1$ in Theorem 3.1 cannot be removed. For example, let $G \cong A_5$ and $K \cong A_5$. Then $G \times K$ has a maximal subgroup N that is also isomorphic to A_5 but $N \neq G$ and $N \neq K$. It is obvious that $N \cap G = 1$ and $MT(N) = MT(G)$.

Corollary 3.2. *Let G be a non-abelian simple group and K a solvable group. Suppose that N is a subgroup of $G \times K$ satisfying $MT(N) = MT(G)$, then $N = G$.*

Proof. We claim $N \cap G \neq 1$. Otherwise, assume $N \cap G = 1$. Since $NG = NG \cap (G \times K) = (NG \cap K) \times G$, one has $N \cong NG/G = (NG \cap K)G/G \cong NG \cap K \leq K$. It follows that N is solvable, which implies that N has at least one normal maximal subgroup. However, since $MT(N) = MT(G)$ and G is a non-abelian simple group, one has that N has no normal maximal subgroups, a contradiction. Hence we get $N \cap G \neq 1$. By Theorem 3.1, we have $N = G$. $\square$

Corollary 3.2 is not true if G is an abelian simple group. For example, let $G = \langle(12)(34)\rangle$, $K = \langle(13)(24)\rangle$ and $N = \langle(14)(23)\rangle$. One has $N \leq G \times K$ and $MT(N) = MT(G)$, but $N \neq G$.

4. Some results on Problem 1.5

In [12], we investigated the following problem:

Problem 4.1. Let $G = T \times \mathbb{Z}_p$ and $N \leq G$, where T is a non-abelian simple group and p is a prime. Is it always true that $MT(N) = MT(G)$ if and only if $N = G$?

Note that if Problem 1.5 holds, then we can get that the Problem 4.1 holds. For the necessity part of the Problem 4.1. Assume $N \neq G$. By Lemma 2.4, there exists a non-solvable proper subgroup H of T such that $N \cong H$. Then $MT(H) = MT(N) = MT(G) = MT(T \times \mathbb{Z}_p)$. It follows that $\pi_t(T) \subseteq \pi_t(H)$. If Problem 1.5 holds, one has $H = T$, a contradiction. Thus we have $N = G$.

Theorem 4.2. *Let $G \cong PSL_2(p^n)$ and N a subgroup of G, where $p^n \geq 4$. If $\pi_t(G) \subseteq \pi_t(N)$, then $N = G$.*

Proof. (1) Suppose $G \cong PSL_2(5)$ or $PSL_2(7)$ or $PSL_2(9)$ or $PSL_2(11)$. It is easy to see that the result holds by [1].

(2) Suppose $p^n \neq 4$, 5, 7, 9, 11. By [5, Theorem 5.2.2], one has $p(G) = p^n+1$. If N is solvable. Since $\pi_t(G) \subseteq \pi_t(N)$ and every maximal subgroup of a solvable group has prime-power index, we have $G \cong PSL_2(7)$ by Lemma 2.1, a contradiction. Thus N is non-solvable. By [4, Theorem 8.27], one has that N might be isomorphic to one of the following groups: A_5, $PSL_2(p^m)$ if $m \mid n$, $PGL_2(p^s)$ if $2s \mid n$.

If $N \cong A_5$. Then $\pi_t(G) \subseteq \{5, 6, 10\}$. It follows that G has at most three same order classes of maximal subgroups. By Lemma 2.3, one has $G \cong A_5 \cong PSL_2(5)$, a contradiction.

Thus $N \cong PSL_2(p^m)$ or $PGL_2(p^s)$. If $N < G$. We have $p^n+1 \nmid |PSL_2(p^m)|$ and $p^n + 1 \nmid |PGL_2(p^s)|$. However, since $\pi_t(G) \subseteq \pi_t(N)$, one has $p^n + 1 \mid |N|$, a contradiction. Hence $N = G$. $\square$

Theorem 4.3. *Let G be a simple K_3-group or a simple K_4-group and N a subgroup of G. If $\pi_t(G) \subseteq \pi_t(N)$, then $N = G$.*

Proof. By [3], Lemma 2.5, [1] and Theorem 4.2, we can easily get that the theorem holds. $\square$

Note that in Problem 1.5 if we assume that G is a general non-solvable group and N is a subgroup of G satisfying $\pi_t(G) \subseteq \pi_t(N)$, we cannot get $N = G$.

For example, let $G = U_3(3) \times N$, where $N = S_6(2)$. It is easy to see that $\pi_t(G) = \pi_t(N)$ but $N < G$. However, we have the following three results, see Theorems 4.4, 4.5 and 4.6.

Theorem 4.4. *Let $G \cong SL_2(p^n)$ and N a subgroup of G, where $p^n \geq 4$. If $\pi_t(G) \subseteq \pi_t(N)$, then $N = G$.*

Proof. (1) Suppose $p = 2$. Then $SL_2(2^n)) \cong PSL_2(2^n)$. By Theorem 4.2, we have $N = G$.

(2) Suppose $p^n = 5$. Then $\pi_t(G) = \{5, 6, 10\}$. Since $\pi_t(G) \subseteq \pi_t(N)$, one has $|N| \geq 30$. Note that $SL_2(5)$ has no proper subgroup H such that $|H| \geq 30$. It follows that $N = G$.

(3) Suppose $p^n = 7, 9, 11$. Arguing as in (2), we can get $N = G$.

(4) Suppose $p^n \neq 5, 7, 9, 11$. By $p(M)$ we denote the smallest index of maximal subgroups of group M. Then $p(SL_2(p^n)) = p(PSL_2(p^n)) = p^n + 1$. If N is solvable. Since $\pi_t(G) \subseteq \pi_t(N)$ and G is non-solvable, we have $G/S(G) \cong PSL_2(7)$. It follows that $G \cong SL_2(7)$, a contradiction. Thus N is non-solvable.

Note that $\Phi(G) \cong \mathbb{Z}_2$. We claim $\Phi(G) \leq N$. Otherwise, assume $\Phi(G) \cap N = 1$. Then $N \cong N\Phi(G)/\Phi(G) \leq G/\Phi(G) \cong PSL_2(p^n)$. Since $\pi_t(G/\Phi(G)) = \pi_t(G)$, one has $\pi_t(G/\Phi(G)) \subseteq \pi_t(N) = \pi_t(N\Phi(G)/\Phi(G))$. By Theorem 4.2, we have $N\Phi(G)/\Phi(G) = G/\Phi(G)$. It follows that $G = N\Phi(G) = N$. Then $\Phi(G) \cap N = \Phi(G) \cap G = \Phi(G) \cong \mathbb{Z}_2 \neq 1$, a contradiction.

Thus $\Phi(G) \leq N$. Since N is non-solvable, one has that $N/\Phi(G)$ is non-solvable and $1 < N/\Phi(G) \leq G/\Phi(G) \cong PSL_2(p^n)$.

If $N < G$. By [4, Theorem 8.27], $N/\Phi(G)$ might be isomorphic to A_5 or $PSL_2(p^m)$ if $m \mid n$ or $PGL_2(p^s)$ if $2s \mid n$.

If $N/\Phi(G) \cong A_5$. Since $\Phi(G) \cong \mathbb{Z}_2$, one has $\pi_t(N) = \{5, 6, 10\}$ or $\{2, 5, 6, 10\}$. Note that G has no normal maximal subgroup and $\pi_t(G) \subseteq \pi_t(N)$. It follows that $\pi_t(G) \subseteq \{5, 6, 10\}$. By Lemma 2.3, one has $G \cong SL_2(5)$, a contradiction.

So $N/\Phi(G) \cong PSL_2(p^m)$ if $m \mid n$ or $PGL_2(p^s)$ if $2s \mid n$. Then $|N| = 2|PSL_2(p^m)|$ if $m \mid n$ or $2|PGL_2(p^s)|$ if $2s \mid n$. Since $N < G$, one has $m < n$ and $s < n$. It follows that $p^n + 1 \nmid 2|PSL_2(p^m)|$ and $p^n + 1 \nmid 2|PGL_2(p^s)|$. However, $\pi_t(G) \subseteq \pi_t(N)$ implies that $p^n + 1 \mid |N|$, a contradiction.

Hence $N = G$. □

Theorem 4.5. *Let $G = T \times \mathbb{Z}_p$ and N a subgroup of G, where $T \cong PSL_2(7)$ and p is a prime. If $\pi_t(G) \subseteq \pi_t(N)$, then one of the following statements holds:*

(1) $N = T$ or $N = G$ if $p = 7$;

(2) $N = G$ if $p \neq 7$.

Proof. (1) Suppose $p = 7$. Then $\pi_t(G) = \pi_t(T \times \mathbb{Z}_p) = \{7, 8\}$.

First assume $\mathbb{Z}_p \leq N$. One has $N = N \cap (T \times \mathbb{Z}_p) = \mathbb{Z}_p \times (N \cap T)$. Since $\pi_t(G) \subseteq \pi_t(N)$ and $\pi_t(G) = \{7, 8\}$, we have $N \cap T = T$ by [1], and it follows that $T \leq N$. Thus $N = T \times \mathbb{Z}_p = G$.

Next assume $\mathbb{Z}_p \not\leq N$. One has $N \times \mathbb{Z}_p = (N \times \mathbb{Z}_p) \cap (T \times \mathbb{Z}_p) = \mathbb{Z}_p \times ((N \times \mathbb{Z}_p) \cap T)$. Then $N \cong (N \times \mathbb{Z}_p) \cap T \leq T$. Since $\pi_t(G) \subseteq \pi_t(N)$, we have $(N \times \mathbb{Z}_p) \cap T = T$ by [1]. It follows that $N \cong T$. We claim $N = T$. Otherwise, if $N \neq T$. One has $N \cap T < N$. Since $N \cap T \trianglelefteq N$, it follows that $N \cap T = 1$. Note that T is a normal maximal subgroup of G. We have $N \cdot T = T \times \mathbb{Z}_p$. Then $|N||T| = |T||\mathbb{Z}_p|$, which implies that $|N| = |\mathbb{Z}_p|$, a contradiction. Hence $N = T$.

(2) Suppose $p \neq 7$. Arguing as above, we can get $N = G$. $\square$

Arguing as in proof of Theorem 4.5, applying Theorem 4.2 and Theorem 4.4, we have:

Theorem 4.6. *Let $G = T \times \mathbb{Z}_q$ and N a subgroup of G, where $T \cong PSL_2(p^n)$ or $SL_2(p^n)$, $p^n \geq 4$ and q is a prime. If $\pi_t(G) \subseteq \pi_t(N)$, then $N = G$ if and only if $q \notin \pi_t(T)$.*

Note that if N is a subgroup of a general non-solvable group G satisfying $\pi_t(G) \subseteq \pi_t(N)$, we cannot get that N is non-solvable. For example, let $G = PSL_2(7) \times N$, where $N = {\mathbb{Z}_2}^3 \rtimes \mathbb{Z}_7$. It is easy to see that $\pi_t(G) = \pi_t(N) = \{7, 8\}$ but N is solvable. However, we have the following two results:

Theorem 4.7. *Let $G = T \times \mathbb{Z}_p$ and N a subgroup of G, where T is a non-abelian simple group and p is a prime. If $\pi_t(G) \subseteq \pi_t(N)$, then N is non-solvable.*

Proof. Assume that N is solvable. Since $\pi_t(G) \subseteq \pi_t(N)$ and G is non-solvable, one has $G/S(G) \cong T \cong PSL_2(7)$ by Lemma 2.1. Therefore, $G \cong PSL_2(7) \times \mathbb{Z}_p$. It follows that $N \cong PSL_2(7)$ or $PSL_2(7) \times \mathbb{Z}_p$ by Theorem 4.5, this contradicts that N is solvable. Hence N is non-solvable. $\square$

Recall that a group A is called a B-free group if any quotient group of every subgroup of A is not isomorphic to B. Arguing as in proof of Theorems 4.5 and 4.7, we have:

Theorem 4.8. *Let $G = T \times K$ and N a subgroup of G, where T is a non-abelian simple group and K is a $({\mathbb{Z}_2}^3 \rtimes \mathbb{Z}_7)$-free solvable group. If $\pi_t(G) \subseteq \pi_t(N)$, then N is non-solvable.*

Acknowledgements

The authors are thankful to the referee who gives valuable comments. J. Shi was supported in part by NSFC (11201401 and 11561021) and Shandong Province Higher Educational Science and Technology Program (J16LI02). Z. Wu was supported by the Basic Ability Improvement Project for Young and Middle Aged Teachers in Guangxi (NO. 2017KY1302). R. Hou was supported by Shandong Province Higher Educational Science and Technology Program (J16LI02).

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