ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 7, pp. 2111–2124

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 7, pp. 2111–2124
Online ISSN: 1735-8515

GROUND STATE SOLUTIONS FOR A CLASS OF NONLINEAR ELLIPTIC EQUATIONS WITH FAST INCREASING WEIGHT

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(Communicated by Asadollah Aghajani)

Abstract. This paper is devoted to get a ground state solution for a class of nonlinear elliptic equations with fast increasing weight. We apply the variational methods to prove the existence of ground state solution.
Keywords: Self-similar solution, variational methods, ground state solution.
MSC(2010): Primary: 35J61; Secondary: 35C06, 35J20.

1. Introduction

In this paper we consider the equation

$$(\mathcal{P}) \qquad -\operatorname{div}(K(x)\nabla u) = K(x)f(x,u), \qquad x \in \mathbb{R}^N,$$

where $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $N \geq 3$ and $K(x) := \exp(|x|^2/4)$. Note that equation $(\mathcal{P})$ is equivalent to

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(x,u). \tag{1.1}$$

This problem is closely related to the study of self-similar solutions for the heat equation as quoted in the works of Escobedo and Kavian [17] (see also [7,18–21,34,35]). In this direction, equations like (1.1) arise naturally when one seeks for solutions of the form $w(t,x) := t^{-1/(p-2)}u(t^{-1/2}x)$ for the evolution equation

$$w_t - \Delta w = |w|^{p-2}w, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{1.2}$$

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1.2).

Article electronically published on December 30, 2017.
Received: 8 July 2015, Accepted: 27 March 2016.
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Using variational methods, Escobedo and Kavian [17] considered the following equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{q-2}u, \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where $2 < q \leq 2^*$. For a long time, there were no results about this type of equation using variational methods. Until very recently, Furtado et al., [21] studied the critical exponent case by using the concentration-compactness principle at infinity. Later, Furtado et al., [19] established a Trudinger-Moser type inequality in a weighted Sobolev space. The inequality is applied in the study of a perturbed elliptic equation and the nonlinearity term has exponential critical growth and the perturbed term belongs to the dual of an appropriate function space. They proved that the problem has at least two weak solutions. Furthermore, Catrina et al. [7], Furtado et al. [18] and Furtado et al. [20] studied more general weight and Ohya [34, 35] studied the p-Laplacian case. Motivated by the previous results, in this paper we want to study the ground state solutions of ($\mathcal{P}$).

For the superlinear case, we give the following assumptions.

- (S1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, for some $2 < p < 2^* = 2N/(N-2)$, $C > 0$, $|f(x,t)| \leq C(|t| + |t|^{p-1})$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\lim_{t\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly in $x \in \mathbb{R}^N$.
- (S2) $f(x,t) = o(t)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^N$.
- (S3) There exists $\theta > 1$ such that $\theta\mathcal{F}(x,t) \geq \mathcal{F}(x,st)$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $F(x,t) = \int_0^t f(x,s)ds$ and $\mathcal{F}(x,t) = f(x,t)t - 2F(x,t)$.

Now, we have the following theorem.

Theorem 1.1. *Suppose conditions* (S1)–(S3) *hold, then problem* ($\mathcal{P}$) *has a ground state solution.*

Remark 1.2. In order to get ground state solution for superlinear equations, the following superlinear condition of Ambrosetti and Rabinowitz is assumed:

(AR) there is $\mu > 2$ such that $0 < \mu F(x,t) \leq tf(x,t)$ for $x \in \mathbb{R}^N$ and $t \neq 0$.

The role of (AR) is to ensure the boundedness of the Palais–Smale (PS) sequences of the functional corresponding to problem ($\mathcal{P}$). However, many functions which are superlinear at infinity do not satisfy condition (AR) for any $\mu > 2$. In fact, (AR) implies that $F(x,t) \geq C|t|^\mu$ for some $C > 0$. For example, the superlinear function

$$f(x,t) = t\log(1 + |t|)$$

does not satisfy (AR). However, it satisfies our condition (S1)–(S3).

The mountain pass solution has the least energy for the corresponding functional (minimizing the functional on the Nehari manifold $\mathcal{N}$). Instead of minimizing the functional on the Nehari manifold $\mathcal{N}$, we will prove our result by a mountain pass type argument. A crucial step is to prove that some kind of almost critical sequence is bounded. We adopt a technique developed in [23] to show that any Cerami sequence is bounded. Finally, we show the existence of ground states by using a technique of Jeanjean and Tanaka [24].

For the asymptotically linear case, we assume

(H1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,0)=0$, $f(x,t) \equiv 0$ for all $t \leq 0$ and all $x \in \mathbb{R}^N$.
(H2) $\limsup_{t\to 0^+} \frac{f(x,t)}{t} < \lambda_1 < \liminf_{t\to+\infty} \frac{f(x,t)}{t} \leq \limsup_{t\to+\infty} \frac{f(x,t)}{t} < +\infty$, uniformly in $x \in \mathbb{R}^N$, where $\lambda_1 = \inf\big\{\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} K(x)u^2 dx = 1\big\}$ and $K(x) := \exp(|x|^2/4)$.

Then we can obtain the following theorem.

Theorem 1.3. *Assume that* (H1) *and* (H2) *hold. Then problem* ($\mathcal{P}$) *admits a positive ground state solution.*

The paper is organized as follows. In Section 2, we introduce a variational setting of the problem and present some preliminary results. In Section 3, we apply a variant version of the Mountain Pass Theorem to prove the existence of ground state solutions of ($\mathcal{P}$).

2. Preliminaries

We shall denote by X the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx\right)^{1/2},$$

which is induced by the inner product

$$(u,v) := \int_{\mathbb{R}^N} K(x)(\nabla u \cdot \nabla v)dx.$$

For each $q \in [2, 2^*]$ we denote by $L_K^q(\mathbb{R}^N)$ the following space

$$L_K^q(\mathbb{R}^N) := \left\{u \text{ is measurable in } \mathbb{R}^N : \|u\|_q := \left(\int_{\mathbb{R}^N} K(x)|u|^q dx\right)^{1/q} < \infty\right\}.$$

Due to the rapid decay at infinity of the functions belonging to X, we have the following embedding result proved in [17].

Proposition 2.1. *The embedding $X \hookrightarrow L_K^q$ is continuous for all $q \in [2, 2^*]$ and it is compact for all $q \in [2, 2^*)$.*

By using the above result, we can prove that the functional $I(u) : X \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \int_{\mathbb{R}^N} K(x)F(x,u)dx, \tag{2.1}$$

is well defined. Standard calculations and Proposition 2.1 show that $I \in C^1(X, \mathbb{R})$ and the derivative of I at the point u is given by

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} K(x)\nabla u \cdot \nabla v dx - \int_{\mathbb{R}^N} K(x) f(x,u) v dx,$$

for any $v \in X$. Hence, the critical points of I are precisely the weak solutions of problem ($\mathcal{P}$). We remark that the compactness of the embedding $X \hookrightarrow L^2_K$ and standard spectral theory for self-adjoint compact operators shows that the linear problem

$$(\mathcal{LP}) \qquad - \operatorname{div}(K(x)\nabla u) = \lambda K(x) u, \qquad x \in \mathbb{R}^N,$$

has a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$. More precisely, we have the following lemma.

Lemma 2.2. *Let*

$$\lambda_1 = \inf\left\{\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} K(x) u^2 dx = 1\right\}$$

be an eigenvalue of the operator $-\Delta(K(x)u)$ and there exists a corresponding eigenfunction $\phi_1(x)$ with $\phi_1(x) > 0$ for all $x \in \mathbb{R}^N$.

We recall that a sequence $\{u_n\} \subset X$ is said to be a $(C)_c$-sequence if $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$. The functional I is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence of I has a convergent subsequence. To obtain a nonzero critical point of the functional I, we need a variant version of the Mountain Pass Theorem as follows.

Theorem 2.3 ([38])**.** *Let X be a real Banach space with its dual space X^*. Suppose that $I \in C^1(X, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \eta_1 < \eta_2 \le \inf_{\|u\|=\rho} I(u),$$

for some η_1, η_2, $\rho > 0$, and $e \in X$ with $\|e\| > \rho$. Define

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$. If I satisfies the $(C)_c$-condition, then c is a critical value of I.

3. **Proof of theorems**

3.1. **Proof of Theorem 1.1.**

As an obvious consequence of (S1) and (S2), we have the following lemma.

Lemma 3.1. *There exists $r > 0$ and $\varphi \in X$ such that $\|\varphi\| > r$ and*

$$b := \inf_{\|u\|=r} I(u) > I(0) = 0 \ge I(\varphi). \tag{3.1}$$

Remark 3.2. In fact, (S2) implies that as $u \to 0$ we have

$$\langle I'(u), u\rangle = \|u\|^2 + o(\|u\|^2), \quad I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2).$$

Therefore, it is also easy to see that:

(i) There exists $\rho_0 > 0$ such that $\|u\| \geq \rho_0$, where u is any nontrivial critical point of I.

(ii) For any $c > 0$, there exists $\rho_c > 0$ such that if $I(u_n) \to c$, then $\|u_n\| \geq \rho_c$.

By Lemma 3.1 we see that I has a mountain pass geometry: that is, setting

$$\Gamma = \{\gamma \in (C[0,1], X) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\},$$

we have $\Gamma \neq \emptyset$. By a special version of the Mountain Pass theorem (see [16]), for the mountain pass level

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{3.2}$$

there exists a $(C)_c$ sequence $\{u_n\}$ for I. Moreover, by (3.1) we see that $c > 0$.

Next, we show that this $(C)_c$ sequence is bounded. Before that, we deduce from (S3) that

$$\mathcal{F}(x,t) := f(x,t)t - 2F(x,t) \geq 0, \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Actually, let $s = 0$ in (S3), we can get this conclusion easily.

Now we are ready to prove the following lemma.

Lemma 3.3. *Suppose that* (S1), (S2) *and* (S3) *hold, and let* $c \in \mathbb{R}$. *Then any* $(C)_c$ *sequence of* I *is bounded.*

Proof. Let $\{u_n\}$ be a $(C)_c$ sequence of I. If $\{u_n\}$ is unbounded, up to a subsequence we may assume that

$$I(u_n) \to c, \quad \|u_n\| \to \infty, \quad \|I'(u_n)\|\|u_n\| \to 0.$$

In particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x) \left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) dx = \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right) = c. \tag{3.3}$$

Let $v_n = \|u_n\|^{-1}u_n$, then $\{v_n\}$ is bounded in X, and there exists $v \in X$ such that, passing to a subsequence if necessary,

$$\begin{aligned} v_n &\rightharpoonup v \text{ in } X, \\ v_n &\to v \text{ in } L^q_K(\mathbb{R}^N) \quad \forall q \in [2, 2^*), \\ v_n(x) &\to v(x) \text{ for a.e. } x \in \mathbb{R}^N. \end{aligned} \tag{3.4}$$

If $v(x) \not\equiv 0$ we have $|u_n(x)| \to +\infty$, and using (S1) we obtain

$$\frac{F(x, u_n(x))}{|u_n|^2}|v_n|^2 \to +\infty. \tag{3.5}$$

The set $\Theta = \{x \in \mathbb{R}^N : v(x) \neq 0\}$ has positive Lebesgue measure, using (3.5) we have

$$\begin{aligned}\frac{1}{2} - \frac{c + o(1)}{\|u_n\|^2} &= \int_{\mathbb{R}^N} \frac{K(x)F(x, u_n)}{\|u_n\|^2}dx \\ &\geq \int_{\Theta} \frac{K(x)F(x, u_n)}{|u_n|^2}|v_n|^2 dx \to +\infty,\end{aligned}$$

which is impossible.

If $v(x) \equiv 0$, we shall derive a contradiction as follow. Given a real number $R > 0$, by (S1) and (S2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x, Rt)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^p, \tag{3.6}$$

we can get that,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} K(x)|F(x, Rv_n)|dx \leq \limsup_{n\to\infty} \left(\varepsilon\|v_n\|_2^2 + C_\varepsilon\|v_n\|_p^p\right).$$

By (3.4), we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)F(x, Rv_n)dx = 0. \tag{3.7}$$

As in [23], we choose a sequence $\{t_n\} \subset [0, 1]$ such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n). \tag{3.8}$$

Given $l > 0$, since for n large enough we have $\sqrt{4l}\|u_n\|^{-1} \in (0, 1)$, using (3.7) with $R = \sqrt{4l}$, we obtain

$$I(t_n u_n) \geq I((4l)^{1/2}v_n) = 2l - \int_{\mathbb{R}^N} K(x)F(x, (4l)^{1/2}v_n)dx \geq l. \tag{3.9}$$

That is, $I(t_n u_n) \to +\infty$. But $I(0) = 0$, $I(u_n) \to c$, using (3.8) we see that $t_n \in (0, 1)$, and

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|\nabla(t_n u_n)|^2 dx - \int_{\mathbb{R}^N} K(x)f(x, t_n u_n)t_n u_n dx &= \langle I'(t_n u_n), t_n u_n\rangle \\ &= t_n \frac{d}{dt}\Big|_{t=t_n} I(tu_n) = 0.\end{aligned}$$

Now, by assumption (S3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\qquad \geq \int_{\mathbb{R}^N} K(x)\left(\frac{1}{2}f(x,t_nu_n)t_nu_n - F(x,t_nu_n)\right)dx \\
&\qquad = \frac{1}{\theta}\left(\frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla(t_nu_n)|^2dx - \int_{\mathbb{R}^N} K(x)F(x,t_nu_n)dx\right) \\
&\qquad = \frac{1}{\theta}I(t_nu_n) \to +\infty.
\end{aligned}$$

This contradicts (3.3). Therefore we have proved that $\{u_n\}$ is bounded. □

Proof of Theorem 1.1. We use the standard argument (see e.g. [1] for the case that (AR) holds) to show that $(\mathcal{P})$ has a nontrivial solution u. Using the technique in the proof [24, Theorem 4.5], we show that problem $(\mathcal{P})$ has a ground state. Letting

$$m = \inf\{I(u) : u \in X \setminus \{0\} \text{ and } I'(u) = 0\}. \tag{3.10}$$

Assume that u is an arbitrary critical point of I. Since (S3) implies

$$\mathcal{F}(x,t) \geq 0, \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.11}$$

we deduce

$$I(u) = I(u) - \frac{1}{2}\langle I'(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^N} K(x)\mathcal{F}(x,u)dx \geq 0, \tag{3.12}$$

and $m \geq 0$. Therefore $0 \leq m \leq I(u) < +\infty$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I such that $I(u_n) \to m$. According to Remark 3.2(i) we see that

$$\|u_n\| \geq \rho_0 \tag{3.13}$$

for some $\rho_0 > 0$. Since u_n is critical point, we also have

$$(1 + \|u_n\|)\|I'(u_n)\| \to 0.$$

Thus $\{u_n\}$ is a Cerami sequence at the level m. By Lemma 3.3, $\{u_n\}$ is bounded in X. Up to a subsequence $\{u_n\}$ (still denoted by $\{u_n\}$) converges weakly to some u, a critical point of I. Moreover, using (3.11) and applying Fatou's lemma, we deduce

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{2}\langle I'(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^N} K(x)\mathcal{F}(x,u)dx \\
&\leq \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} K(x)\mathcal{F}(x,u_n)dx \\
&= \liminf_{n\to\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right) = m.
\end{aligned}$$

Therefore, u is a nontrivial critical point of I with $I(u) = m$. Now, Theorem 1.1 is proved. □

3.2. **Proof of Theorem 1.3.** First, we show that the functional I has the mountain pass geometry.

Lemma 3.4. *Under the conditions of Theorem 1.3, we have*

(i) *There exist ρ, $\delta > 0$ such that $I(u) \geq \delta$, $\forall u \in X$ with $\|u\| = \rho$.*
(ii) *There exists $e \in X$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof. (i) By (H1), (H2), there exist δ_0, $C_0 > 0$ such that

$$f(x,t) \leq (\lambda_1 - \delta_0)t + C_0 t^{2^*-1}, \quad \forall x \in \mathbb{R}^N, \forall t \geq 0, \tag{3.14}$$

which implies that

$$2F(x,t) \leq (\lambda_1 - \delta_0)t^2 + C_0 t^{2^*}, \quad \forall x \in \mathbb{R}^N, \forall t \geq 0.$$

Then we get

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \int_{\mathbb{R}^N} K(x)F(x,u)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{\lambda_1 - \delta_0}{2}\int_{\mathbb{R}^N} K(x)u^2 dx - C_0\int_{\mathbb{R}^N} K(x)u^{2^*} dx \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda_1 - \delta_0}{\lambda_1}\right)\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - C_0\int_{\mathbb{R}^N} K(x)u^{2^*} dx \\
&\geq \frac{\delta_0}{2\lambda_1}\|u\|^2 - \frac{C_0}{S^{\frac{N}{N-2}}}\|u\|^{\frac{2N}{N-2}} \\
&= \frac{\delta_0}{2\lambda_1}\|u\|^2\left[1 - \frac{2\lambda_1 C_0}{\delta_0 S^{\frac{N}{N-2}}}\|u\|^{\frac{2N}{N-2}}\right].
\end{aligned}$$

It is not difficult to see that there exists sufficiently small $\rho > 0$ such that

$$I(u) \geq \delta, \quad \forall u \in X \text{ with } \|u\| = \rho.$$

(ii) Using (H2), there exist δ_1, $M_1 > 0$ such that

$$2F(x,t) \geq (\lambda_1 + \delta_1)t^2, \quad \forall x \in \mathbb{R}^N, \forall t \geq M_1. \tag{3.15}$$

Take $R_1 > 0$ large enough such that

$$\|\phi_1\|^2_{L^2(B_{R_1})} \geq \frac{\lambda_1 + \frac{1}{2}\delta_1}{\lambda_1 + \delta_1}\|\phi_1\|_2^2. \tag{3.16}$$

Since $\phi_1(x) > 0$ in $\mathbb{R}^N$, there exists $t_1 > 0$ such that

$$t_1\phi_1(x) > M_1, \quad \forall |x| \leq R_1.$$

Then, by (3.15), (3.16) and Lemma 2.2, we can see that, for $t \geq t_1$,

$$\begin{aligned}
I(t\phi_1) &= \frac{t^2}{2}\int_{\mathbb{R}^N} K(x)\nabla\phi_1^2 dx - \int_{\mathbb{R}^N} K(x)F(x,t\phi_1)dx \\
&= \frac{t^2}{2}\lambda_1\|\phi_1\|_2^2 - \int_{B_{R_1}} K(x)F(x,t\phi_1)dx - \int_{B_{R_1}^c} K(x)F(x,t\phi_1)dx \\
&\leq \frac{t^2}{2}\lambda_1\|\phi_1\|_2^2 - \frac{t^2}{2}(\lambda_1+\delta_1)\int_{B_{R_1}} \phi_1^2 dx \\
&\leq -\frac{t^2}{4}\delta_1\|\phi_1\|_2^2.
\end{aligned}$$

Take $e = \tilde{t}\phi_1$ with $\tilde{t} > t_1$. It is easily seen that $I(e) < 0$. □

To obtain critical points of the functional I, we need to show that I satisfies the $(C)_c$-condition.

Lemma 3.5. *Assume that* (H1) *and* (H2) *hold. Then the functional I satisfies the $(C)_c$-condition for any $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset X$ such that

$$I(u_n) \to c, \quad (1+\|u_n\|)\|I'(u_n)\| \to 0. \tag{3.17}$$

We assume by contradiction that $\|u_n\| \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then $\|w_n\| = 1$, and there exists $w_0 \in X$ such that, passing to a subsequence if necessary,

$$\begin{aligned}
w_n &\rightharpoonup w_0 \text{ in } X, \\
w_n &\to w_0 \text{ in } L_K^2(\mathbb{R}^N), \\
w_n(x) &\to w_0(x) \text{ for a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

We claim that $w_0(x) \not\equiv 0$. In fact, if not, we assume that $w_0(x) \equiv 0$, that is, $w_n \to 0$ in $L_K^2(\mathbb{R}^N)$. From (3.17) and (H1) it follows that

$$\begin{aligned}
o(1) &= \frac{\langle I'(u_n), u_n\rangle}{\|u_n\|} \\
&= \int_{\mathbb{R}^N} K(x)\nabla w_n^2 dx - \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{\|u_n\|} w_n dx \\
&= 1 - \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{\|u_n\|} w_n^2 dx \\
&\to 1,
\end{aligned}$$

which is a contradiction. Taking $w_n^-(x) = \max\{-w_n(x), 0\}$ as test function, using (3.17) again we get

$$\begin{aligned} o(1) &= \frac{\langle I'(u_n), w_n^-\rangle}{\|u_n\|} \\ &= \int_{\mathbb{R}^N} K(x)\nabla(w_n^-)^2 dx + \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{\|u_n\|} w_n^- dx \\ &= \|w_n^-\|^2 + \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{\|u_n\|}(w_n^-)^2 dx. \end{aligned}$$

From (H1) it follows that $w_n^-(x) = o(1)$. Then we have $w_0^-(x) = 0$ for a.e. $x \in \mathbb{R}^N$ and thus $w_0(x) \geq 0$ and $w_0(x) \not\equiv 0$.

Define

$$\Omega_1 = \{x \in \mathbb{R}^N : w_0(x) = 0\}, \quad \Omega_2 = \{x \in \mathbb{R}^N : w_0(x) > 0\}.$$

If $x \in \Omega_1$, using (H1), (H2), there exists $C_1 > 0$ such that

$$0 \leq \frac{f(x,t)}{t} \leq C_1, \quad \forall x \in \mathbb{R}^N, \forall t \in \mathbb{R}, \tag{3.18}$$

which implies that

$$\frac{|f(x,u_n(x))|}{\|u_n\|} = \left|\frac{f(x,u_n(x))}{u_n(x)} w_n(x)\right| \leq C_1|w_n(x)| \to 0 \text{ for a.e. } x \in \Omega_1.$$

Then we have

$$\frac{f(x,u_n(x))}{\|u_n\|} \to 0 = (\lambda_1 + \delta_1)w_0(x) \text{ for a.e. } x \in \Omega_1, \tag{3.19}$$

where δ_1 is taken as in (3.15). If $x \in \Omega_2$, then $u_n(x) = w_n(x)\|u_n\| \to +\infty$. Using (H2) we get

$$\liminf_{n\to+\infty} \frac{f(x,u_n(x))}{\|u_n\|} = \liminf_{n\to\infty} \frac{f(x,u_n(x))}{u_n(x)} w_n(x) \geq (\lambda_1 + \delta_1)w_0(x) \text{ for a.e. } x \in \Omega_2. \tag{3.20}$$

In view of $w_n \rightharpoonup w_0$ in X, it follows that $\int_{\mathbb{R}^N} w_n\phi_1 dx \to \int_{\mathbb{R}^N} w_0\phi_1 dx$. By (3.17) and Lemma 2.2, we have

$$\begin{aligned} o(1) &= \frac{\langle I'(u_n), \phi_1\rangle}{\|u_n\|} \\ &= \int_{\mathbb{R}^N} K(x)\nabla w_n \cdot \nabla\phi_1 dx - \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{\|u_n\|}\phi_1 dx \\ &= \lambda_1 \int_{\mathbb{R}^N} w_n\phi_1 dx - \int_{\mathbb{R}^N} K(x)\frac{f(x,u_n)}{u_n} w_n\phi_1 dx. \end{aligned}$$

Then, from (3.19), (3.20) and Fatou's lemma, it follows that

$$\lambda_1 \int_{\mathbb{R}^N} w_0\phi_1 dx \geq (\lambda_1 + \delta_1) \int_{\mathbb{R}^N} w_0\phi_1 dx.$$

Since, $\phi_1(x) > 0$ and $w_0 \geq 0$, $w_0 \not\equiv 0$, it follows that $\int_{\mathbb{R}^N} w_0\phi_1 dx > 0$. Thus $\lambda_1 \geq \lambda_1 + \delta_1$, a contradiction. This shows that $\{u_n\}$ is bounded in X. Thus there exists $u_0 \in X$ such that, passing to a subsequence if necessary,

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ in } X, \\ u_n &\to u_0 \text{ in } L^2_K(\mathbb{R}^N), \\ u_n(x) &\to u_0(x) \text{ for a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Together with (H1), (H2), it is not difficult to obtain that

$$\int_{\mathbb{R}^N} K(x)f(x,u_n)(u_n - u_0)dx = o(1).$$

Noting that $\langle I'(u_n), u_n - u_0\rangle \to 0$, it follows that

$$\|u_n - u_0\|^2 = \int_{\mathbb{R}^N} K(x)f(x,u_n)(u_n - u_0)dx + o(1) = o(1),$$

which implies that $u_n \to u_0$ in X. This completes the proof. $\square$

Proof of Theorem 1.3. Set $\mathcal{M} = \{u \in X \setminus \{0\} : I'(u) = 0\}$. By Lemmas 3.5, 3.4 and Theorem 2.3, it is easy to obtain a nontrivial critical point u_0 of I, which implies that $\mathcal{M} \neq \emptyset$. We claim that

$$\eta = \inf\{u \in X \setminus \{0\} : u \in \mathcal{M}\} > 0.$$

Assume by contradiction that there is a sequence $u_n \subset X \setminus \{0\}$ with $\|u_n\| \to 0$ such that $I'(u_n) = 0$, which implies that $\|u_n\|^2 = \int_{\mathbb{R}^N} K(x)f(x,u_n)u_n dx$. As shown in Lemma 3.4, there exists $C_1 > 0$ such that

$$\|u_n\|^2 \leq \frac{\lambda_1 - \delta_0}{\lambda_1}\|u_n\|^2 + C_1\|u_n\|^{2^*}.$$

Then we can obtain a contradiction and thus the claim is right.

Now we shall show that I is bounded from below on $\mathcal{M}$. Indeed, if not, there exists a sequence $\{u_n\} \subset \mathcal{M}$ such that $I(u_n) < -n$, $\forall n \in \mathbb{N}$. Using (3.14), we can get that $I(u) \geq \frac{\delta_0}{2\lambda_1}\|u\|^2 - C_2\|u\|^{2^*}$ for some $C_2 > 0$, which implies that $\|u_n\| \to +\infty$. Note that $(1 + \|u_n\|)\|I'(u_n)\| = 0$ for all $u_n \in \mathcal{M}$. As in the proof of Lemma 3.5, we can obtain that $\{u_n\}$ is bounded in X, which is a contradiction. Thus I is bounded from below on $\mathcal{M}$. Define

$$c_{min} = \inf\{I(u) : u \in \mathcal{M}\}.$$

Clearly, $c_{min} \leq I(u_0)$. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for c_{min}, i.e., $I(u_n) \to c_{min}$ and $I'(u_n) = 0$. Then $\{u_n\}$ is a $(C)_{c_{min}}$ sequence. Together with Lemmas 3.5, $\{u_n\}$ is bounded in X and it has a convergence subsequence, still denoted by $\{u_n\}$, such that $u_n \to \hat{u}$ in X. Thus, $\hat{u}$ in X is a nontrivial critical point of I with $I(\hat{u}) = c_{min} > 0$ and hence $\hat{u}$ is a ground state solution of $(\mathcal{P})$. $\square$

Acknowledgements

This work is supported by National Natural Science Foundation of China (No. 11471267) and Research Fund of Chongqing Technology and Business University (No. 2015-56-09).

References

[1] A. Ambrosetti and A. Malchiodi, Nonlinear Analysis and Semilinear Elliptic Problems, Cambridge Univ. Press, Cambridge, 2007.

[2] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973) 349–381.

[3] M. Badiale and E. Serra, Semilinear Eliptic Equations for Beginners, Existence Results via the Variational Approach, Universitext, Springer, London, 2011.

[4] T. Bartsch and Z.Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on R^N, *Comm. Partial Differential Equations* **20** (1995), no. 9-10, 1725–1741.

[5] K.J. Brown, The Nehari manifold for a semilinear elliptic equation involving a sublinear term, *Calc. Var. Partial Differential Equations* **22** (2005), no. 4, 483–494.

[6] K.J. Brown and Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, *J. Differential Equations* **193** (2003), no. 2, 481–499.

[7] F. Catrina, M. Furtado and M. Montenegro, Positive solutions for nonlinear elliptic equations with fast increasing weights, *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), no. 6, 1157–1178.

[8] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* **3** (1995), no. 4, 493–512.

[9] J. Chabrowski, Weak Convergence Methods for Semilinear Elliptic Equations, World Scientific Publ. River Edge, 1999.

[10] J. Chabrowski and D.G. Costa, On existence of positive solutions for a class of Caffarelli-Kohn-Nirenberg type equations, *Colloq. Math.* **120** (2010), no. 1, 43–62.

[11] J. Chabrowski and D.G. Costa, On a class of Schrödinger-type equations with indefinite weight functions, *Comm. Partial Differential Equations*, **33** (2008), no. 7-9, 1368–1394.

[12] C.Y. Chen, Y.C. Kuo and T.F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, *J. Differential Equations* **250** (2011), no. 4, 1876–1908.

[13] V. Coti Zelati and P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on R^n, *Comm. Pure Appl. Math.* **45** (1992), no. 10, 1217–1269.

[14] P. Drábek and I.S. Pohozaev, Positive solutions for the p-Laplacian: application of the fibering method, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 4, 703–726.

[15] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974) 324–353.

[16] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer-Verlag, Berlin, 1990.

[17] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation, *Nonlinear Anal.* **11** (1987), no. 10, 1103–1133.

[18] M.F. Furtado, O. Myiagaki and J.P.P. da Silva, On a class of nonlinear elliptic equations with fast increasing weight and critical growth, *J. Differential Equations* **249** (2010), no. 5, 1035–1055.

[19] M.F. Furtado, O. Myiagaki, J.P.P. da Silva and B. Uberlandio, A Trudinger-Moser inequality in a weighted Sobolev space and applications, *Math. Nachr.* **287** (2014), no. 11-12, 1255–1273.

[20] M.F. Furtado, O. Myiagaki and J.P.P. da Silva, Two solutions for an elliptic equation with fast increasing weight and concave–convex nonlinearities, *J. Math. Anal. Appl.* **416** (2014), no. 2, 698–709.

[21] M.F. Furtado, J.P.P. da Silva and M.S. Xavier, Multiplicity of self-similar solutions for a critical equation, *J. Differential Equations* **254** (2013), no. 7, 2732–2743.

[22] A. Haraux and F.B. Weissler, Nonuniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.* **31** (1982), no. 2, 167–189.

[23] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on R^N, *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), no. 4, 787–809.

[24] L. Jeanjean and K. Tanaka, A positive solution for an asymptotically linear elliptic problem on R^N autonomous at infinity, *ESAIM Control Optim. Calc. Var.* **7** (2002) 597–614.

[25] L. Jeanjean and K. Tanaka, Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, *Calc. Var. Partial Differential Equations* **21** (2004), no. 3, 287–318.

[26] W. Kryszewski and A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equation, *Adv. Differential Equations* **3** (1998), no. 3, 441–472.

[27] P.L. Lions, The concentration-compactness principle in the calculus of variations, The limit case. I, *Rev. Mat. Iberoamericana* **1** (1985), no. 1, 145–201.

[28] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, *Rev. Mat. Iberoamericana* **1** (1985), no. 2, 45–121.

[29] T.F. Ma, M. Ramos and L. Sanchez, Multiple solutions for a class of nonlinear boundary value problems near resonance: a variational approach, in: Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996), *Nonlinear Anal.* **30** (1997), no. 6, 3301–3311.

[30] J. Mawhin, K. Schmitt, Nonlinear eigenvalue problems with the parameter near resonance, *Ann. Polon. Math.* **51** (1990) 241–248.

[31] Y. Naito, A variational approach to self-similar solutions for semilinear heat equations, in: EQUADIFF 2003, pp. 555–557, World Scientific Publ, Hackensack, NJ, 2005.

[32] Y. Naito, Self-similar solutions for a semilinear heat equation with critical Sobolev exponent, *Indiana Univ. Math. J.* **57** (2008), no. 3, 1283–1315.

[33] Z. Nehari, On a class of nonlinear second-order differential equations, *Trans. Amer. Math. Soc.* **95** (1960) 101–123.

[34] H. Ohya, Exponentially decaying solutions of quasilinear elliptic equations in R^N, *Adv. Math. Sci. Appl.* **13** (2003), no. 1, 287–299.

[35] H. Ohya, Existence results for some quasilinear elliptic equations involving critical Sobolev exponents, *Adv. Differential Equations* **9** (2004), no. 11-12, 1339–1368.

[36] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992), no. 2, 270–291.

[37] M. Ramos and L. Sanchez, A variational approach to multiplicity in elliptic problems near resonance, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 2, 385–394.

[38] M. Schechter, A variation of the mountain pass lemma and applications, *J. Lond. Math. Soc. (2)* **44** (1991), no. 3, 491–502.

[39] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9** (1992), no. 3, 281–304.

[40] T.F. Wu, Multiple positive solutions for a class of concave-convex elliptic problems in R^N involving sign-changing weight, *J. Funct. Anal.* **258** (2010), no. 1, 99–131.

[41] T.F. Wu, Three positive solutions for Dirichlet problems involving critical Sobolev exponent and sign-changing weight, *J. Differential Equations* **249** (2010), no. 7, 1549–1578.1

[42] T.F. Wu, Existence and multiplicity of positive solutions for a class of nonlinear boundary value problems, *J. Differential Equations* **252** (2012), no. 5, 3403–3435.
[43] Y. Xiaohui, The Nehari manifold for elliptic equation involving the square root of the Laplacian, *J. Differential Equations* **252** (2012), no. 2, 1283–1308.

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