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THE NORM OF PRE-SCHWARZIAN DERIVATIVES ON BI-UNIVALENT FUNCTIONS OF ORDER α

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Abstract. In the present investigation, we give the best estimates for the norm of the pre-Schwarzian derivative $T_f(z) = \frac{f''(z)}{f'(z)}$ for bi-starlike functions and a new subclass of bi-univalent functions of order α, where $\|T_f\| = \sup_{|z|<1}(1-|z|^2)|\frac{f''(z)}{f'(z)}|$.
Keywords: Bi-univalent functions, bi-starlike functions, subordination, pre-Schwarzian derivatives.
MSC(2010): Primary: 30C45; Secondary: 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of A, consisting of all univalent functions $f(z)$ in Δ, is denoted by S. Obviously, every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z, z \in \Delta$, and $f(f^{-1}(w)) = w$, $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \; ; w \in \Delta,$$

which implies that f^{-1} is analytic. The derivative of f^{-1}, can be calculated by using elementary calculus. It is easy to see that if $w = f(z)$ and $w+\epsilon = f(z+\delta)$, then

$$\delta = f^{-1}(w + \epsilon) - f^{-1}(w) \longrightarrow 0$$

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as $\epsilon \longrightarrow 0$. Since f^{-1} is continuous, so in this case we have

$$f(z+\delta) - f(z) \neq 0 \ , \ f(z) \neq 0.$$

Since f is a univalent function as a result, when $\epsilon \longrightarrow 0$, one gets

$$\frac{f^{-1}(w+\epsilon) - f^{-1}(w)}{\epsilon} = \frac{\delta}{f(z+\delta) - f(z)} \longrightarrow \frac{1}{f'(z)}.$$

Therefore, we have

$$\frac{d}{dw}(f^{-1}(w)) = \frac{1}{f'(z)}. \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. We denote by σ the class of bi-univalent functions in Δ of the form (1.1). For examples of bi-univalent functions see the recent work of Srivastava et.al., [7].

Let f and g be analytic in Δ. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ on Δ, see [3].

The pre-Schwarzian derivative of f is denoted by

$$T_f(z) = \frac{f''(z)}{f'(z)}.$$

We define the norm of T_f by

$$\|T_f\| = \sup_{|z|<1} (1 - |z|^2)|\frac{f''(z)}{f'(z)}|.$$

This norm have a significant meaning in the theory of Teichmuller spaces. For a univalent function f, it is well known that $\|T_f\| < 6$. This is the best possible estimation. On the other hand, the following result is important to note.

Theorem 1.1. *Let f be analytic and locally univalent in Δ. Then*

(i) *if $\|T_f\| \leq 1$, then f is univalent, and*
(ii) *if $f \in S^*(\alpha)$, then $\|T_f\| \leq 6 - 4\alpha$.*

Part (i) is due to Becker [1] and sharpness of the constants is due to Becker and Pommerenke [2]. Part (ii) is due Yamashita [4, 8]. The norm estimations for typical subclasses of univalent functions are investigated by many authors, see [5–8].

In this paper we shall give the best estimation for subclasses of bi-univalent functions .

2. Main results

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $S^*_\sigma(\alpha)$, if the following conditions are satisfied:

$$f \in \sigma \qquad \text{and} \qquad \mathrm{Re}\{\frac{zf^{'}(z)}{f(z)}\} > \alpha,$$

$$f \in \sigma \qquad \text{and} \qquad \mathrm{Re}\{\frac{wg^{'}(w)}{g(w)}\} > \alpha,$$

where $w = f(z), \quad g = f^{-1}, \quad 0 \le \alpha < 1.$

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $S^*_\sigma(\alpha)$, $0 \le \alpha < 1$. Then*

$$\|T_f\| \le \min\{6 - 4\alpha, 4\alpha + 2\}. \tag{2.1}$$

Proof. It follows from Theorem 1.1 that $\|T_f\| \le 6 - 4\alpha$, where $0 \le \alpha < 1$. Also by Definition 2.1 we have $\mathrm{Re}\{\frac{wg^{'}(w)}{g(w)}\} > \alpha$. We set $h(z) = \frac{wg^{'}(w)}{g(w)}$. Then by assumption h is a holomorphic function on Δ satisfying $h(0) = 1$, and $h(\Delta) \subset \{w \in C; Re\, w > \alpha\}$. The univalent map $p(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$ on Δ satisfies $p(0) = 1$ and

$$h(w) = \frac{wg^{'}(w)}{g(w)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \tag{2.2}$$

so there exists a holomorphic function $\varphi : \Delta \longrightarrow \Delta$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ on Δ such that

$$h = p \circ \varphi = \frac{1 + (1 - 2\alpha)\varphi(z)}{1 - \varphi(z)}. \tag{2.3}$$

Using (1.2) we have,

$$h(z) = \frac{f(z)}{zf'(z)} = \frac{1 + (1 - 2\alpha)\varphi(z)}{1 - \varphi(z)}. \tag{2.4}$$

By logarithmic differentiation on (2.4), we have

$$\log f(z) - \log z - \log f^{'}(z) = \log(1 + (1 - 2\alpha)\varphi(z)) - \log(1 - \varphi(z))$$
$$\frac{f^{'}(z)}{f(z)} - \frac{1}{z} - \frac{f^{''}(z)}{f'(z)} = \frac{(1 - 2\alpha)\varphi^{'}(z)}{1 + (1 - 2\alpha)\varphi(z)} + \frac{\varphi^{'}}{1 - \varphi(z)}.$$

Since φ belongs to class of Schwarz functions, we have $\varphi(z) \prec z$ on Δ. So we can set $\varphi = id_\Delta$; we also have

$$T_{f_\alpha}(z) = \frac{f^{''}(z)}{f^{'}(z)} = \frac{(2-6\alpha+4\alpha^2)z}{(1-z)(1+(1-2\alpha)z)}. \tag{2.5}$$

We conclude

$$(1-|z|^2)|T_f(z)| \leq (1-|z|^2)|T_{f_\alpha}(z)|. \tag{2.6}$$

So we can estimate as,

$$\begin{aligned}(1-|z|^2)|T_f(z)| &\leq \sup_{|z|<1}(1-|z|^2)|\frac{f^{''}(z)}{f^{'}(z)}| \\ &= \sup_{|z|<1} \frac{(2-6\alpha+4\alpha^2)|z|(1+|z|)}{|1+(1-2\alpha)z|}. \end{aligned} \tag{2.7}$$

We can see that for upper bound of $\|T_{f_\alpha}(z)\|$, that z have to lead to 1. So we have,

$$\lim_{z \longrightarrow 1} \frac{(2-6\alpha+4\alpha^2)|z|(1+|z|)}{|1+(1-2\alpha)z|} = \frac{2(2-6\alpha+4\alpha^2)}{1+(1-2\alpha)} = 4\alpha+2. \tag{2.8}$$

Hence by (2.6) and (2.8) we conclude,

$$\|T_f\| \leq 4\alpha+2.$$

This complete the proof. □

Definition 2.3. A function $f(z)$ given by (1.1) is said to be in the class $\nu_\sigma^*(\alpha)$, if the following conditions are satisfied:

$$f \in \sigma \qquad \text{and} \qquad \operatorname{Re}\left\{(\frac{z}{f(z)})^2 f^{'}(z)\right\} > \alpha,$$
$$f \in \sigma \qquad \text{and} \qquad \operatorname{Re}\left\{(\frac{w}{g(w)})^2 g^{'}(w)\right\} > \alpha,$$

where $w = f(z), \quad g = f^{-1}, \quad 0 \leq \alpha < 1, w \in \Delta$.

Theorem 2.4. *Let the function $f(z)$ given by (1.1) be in the class $\nu_\sigma^*(\alpha)$, $0 \leq \alpha < 1$. Then*

$$\|T_f\| \leq \min\{10-8\alpha, 6-8\alpha\}. \tag{2.9}$$

Proof. Since $\operatorname{Re}\{(\frac{z}{f(z)})^2 f^{'}(z)\} > \alpha$, we set $h(z) = (\frac{z}{f(z)})^2 f^{'}(z)$, by a similar argument we conclude that

$$(\frac{z}{f(z)})^2 f^{'}(z) \prec \frac{1+(1-2\alpha)z}{1-z}.$$

So we have,

$$(\frac{z}{f(z)})^2 f^{'}(z) = \frac{1+(1-2\alpha)\varphi(z)}{1-\varphi(z)}. \tag{2.10}$$

By logarithmic differentiation on (2.10) we have

$$2(\log z - \log f(z)) + \log f^{'}(z) = \log(1+(1-2\alpha)\varphi(z)) - \log(1-\varphi(z)),$$
$$\frac{2}{z} - \frac{2f^{'}(z)}{f(z)} + \frac{f^{''}(z)}{f^{'}(z)} = \frac{(1-2\alpha)\varphi^{'}(z)}{1+(1-2\alpha)\varphi(z)} + \frac{\varphi^{'}}{1-\varphi(z)},$$
$$\frac{f^{''}(z)}{f^{'}(z)} = \frac{2f^{'}(z)}{f(z)} - \frac{2}{z} + \frac{(1-2\alpha)\varphi^{'}(z)}{1+(1-2\alpha)\varphi(z)} + \frac{\varphi^{'}}{1-\varphi(z)}.$$

So, we get

$$T_f(z) = \frac{f^{''}(z)}{f^{'}(z)} = \frac{2+2(1-2\alpha)\varphi(z)}{z(1-\varphi(z))} - \frac{2}{z} + \frac{(1-2\alpha)\varphi^{'}(z)}{1+(1-2\alpha)\varphi(z)} + \frac{\varphi^{'}}{1-\varphi(z)}.$$

Setting $\varphi = id_\Delta$, we also have

$$T_{f_\alpha}(z) = \frac{f^{''}(z)}{f^{'}(z)} = \frac{2(1+(1-2\alpha)z)}{z(1-z)} - \frac{2}{z} + \frac{1-2\alpha}{1+(1-2\alpha)z} + \frac{1}{1-z},$$

or equivalently,

$$T_{f_\alpha}(z) = \frac{-6\alpha+6+(4-12\alpha+8\alpha^2)z}{(1-z)(1+(1-2\alpha)z)}.$$

We conclude that

$$(1-|z|^2)|T_f(z)| \le (1-|z|^2)|T_{f_\alpha}(z)|. \tag{2.11}$$

So we can estimate as

$$\sup_{|z|<1}(1-|z|^2)|T_f(z)| \le \sup_{|z|<1}(1-|z|^2)|T_{f_{(\alpha)}}(z)|$$
$$\le \sup_{|z|<1} \frac{(1+|z|)(-6\alpha+6+(4-12\alpha+8\alpha^2)|z|)}{(1+(1-2\alpha)|z|)}. \tag{2.12}$$

We can see that for upper bound of $\|T_{f_\alpha}(z)\|$, that z have to lead to 1,

$$\lim_{z\longrightarrow 1} \frac{(1+|z|)(-6\alpha+6+(4-12\alpha+8\alpha^2)|z|)}{1+(1-2\alpha)|z|}$$
$$= 10-8\alpha. \tag{2.13}$$

Hence by (2.12) and (2.13) we conclude that

$$\|T_f\| \le 10-8\alpha. \tag{2.14}$$

For the second part of the proof, we have
$f \in \sigma$ and $Re\{(\frac{w}{g(w)})^2 g^{'}(w)\} > \alpha$, where $g = f^{-1}, w \in \Delta$ and $0 \le \alpha < 1$.

By $g^{'} = \frac{d}{dw}(f^{-1}(w)) = \frac{1}{f'(z)}$, we have $Re\{(\frac{f(z)}{z})^2 \frac{1}{f'(z)}\} > \alpha$. We set $h(z) = (\frac{f(z)}{z})^2 \frac{1}{f'(z)}$. By a similar argument, we conclude that

$$(\frac{f(z)}{z})^2 \frac{1}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z}.$$

So we have,

$$(\frac{f(z)}{z})^2 \frac{1}{f'(z)} = \frac{1+(1-2\alpha)\varphi(z)}{1-\varphi(z)}. \tag{2.15}$$

By logarithmic differentiation of (2.15) we have,

$$T_f(z) = \frac{f^{''}(z)}{f'(z)} = \frac{2+2(1-2\alpha)\varphi(z)}{z(1-\varphi(z))} - \frac{2}{z} - \frac{(1-2\alpha)\varphi^{'}(z)}{1+(1-2\alpha)\varphi(z)} - \frac{\varphi^{'}}{1-\varphi(z)}.$$

Setting $\varphi = id_\Delta$, we also have

$$T_{f_\alpha}(z) = \frac{f^{''}(z)}{f'(z)} = \frac{2(1-\alpha)+(8\alpha^2-12\alpha+4)z}{(1-z)(1+(1-2\alpha)z)}.$$

So, we obtain

$$\begin{aligned}\|T_{f_\alpha}(z)\| &\leq \sup_{|z|<1}(1-|z|^2)\frac{2(1-\alpha)+(8\alpha^2-12\alpha+4)|z|}{(1-|z|)(1+(1-2\alpha)|z|)}\\ &= \sup_{|z|<1}\frac{(1+|z|)[2(1-\alpha)+(8\alpha^2-12\alpha+4)|z|}{(1+(1-2\alpha)|z|)},\end{aligned}$$

and finally,

$$(1-|z|^2)|T_f(z)| \leq (1-|z|^2)|T_{f_\alpha}(z)|.$$

Therefore, using what we get

$$\|T_f(z)\| \leq \sup_{|z|<1}(1-|z|^2)|T_{f_\alpha}(z)| = \sup_{|z|<1}\frac{(1+|z|)[2(1-\alpha)+(8\alpha^2-12\alpha+4)|z|]}{(1+(1-2\alpha)|z|)}. \tag{2.16}$$

We can see that for upper bound of $\|T_{f_\alpha}(z)\|$, that z have to lead to 1,

$$\begin{aligned}&\lim_{z\longrightarrow 1}\frac{(1+|z|)[2(1-\alpha)+(8\alpha^2-12\alpha+4)|z|]}{1+(1-2\alpha)|z|}\\ &= 6-8\alpha.\end{aligned} \tag{2.17}$$

Hence using (2.16) and (2.17) we conclude that

$$\|T_f\| \leq 6-8\alpha. \tag{2.18}$$

Combining (2.14) and (2.18) completes the proof. □

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