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ALMOST SPECIFICATION AND RENEWALITY IN SPACING SHIFTS

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Abstract. Let (Σ_P, σ_P) be the space of a spacing shifts where $P \subset \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\Sigma_P = \{s \in \{0, 1\}^{\mathbb{N}_0} : s_i = s_j = 1 \text{ if } |i - j| \in P \cup \{0\}\}$ and σ_P the shift map. We will show that Σ_P is mixing if and only if it has almost specification property with at least two periodic points. Moreover, we show that if $h(\sigma_P) = 0$, then Σ_P is almost specified and if $h(\sigma_P) > 0$ and Σ_P is almost specified, then it is weak mixing. Also, some sufficient conditions for a coded Σ_P being renewal or uniquely decipherable is given. At last we will show that here are only two conjugacies from a transitive Σ_P to a subshift of $\{0, 1\}^{\mathbb{N}_0}$.
Keywords: Spacing shifts, almost specification, renewal, uniquely decipherable.
MSC(2010): Primary: 54H20; Secondary: 37B10, 37A25.

Introduction

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $P \subseteq \mathbb{N}$ and Σ_P be a subshift on $\{0, 1\}^{\mathbb{N}_0}$ defined as

$$\Sigma_P = \{\{s_i\}_{i\in\mathbb{N}_0} : s_i = s_j = 1 \text{ if } |i - j| \in P \cup \{0\}\}.$$

Then, (Σ_P, σ_P) is called the *spacing shifts* associated to P. A rather detailed study of them can be found in [2]. These shifts generate a good source of examples in topological dynamical systems (TDS); in particular, when considered from the combinatorial point of view. For instance, a (Σ_P, σ_P) is mixing, or weak mixing if and only if P is cofinite or thick, respectively. In fact, Lau and Zame [14] introduced spacing shifts to provide examples of maps that are topologically weak mixing but not mixing. On the other side, the spacing shifts may be considered as the opposite to Markov shifts: any 1 appearing in a word depends on all the 1's coming before it. Therefore, there must be restricting conditions for a spacing shifts being a shift of finite type (SFT) or sofic.

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Banks et al proved in [2] that a spacing shifts is SFT if and only if it is mixing. More equivalent conditions will be given in Theorem 2.3. Ir n this paper, we show that a spacing shifts is mixing if and only if it has almost specification with at least two periodic points. In fact, it turns out that almost specification property is a rather good tool for characterizing some dynamical properties of the spacing shifts. For instance, a spacing shifts with almost specification property and positive entropy is weakly mixing (Theorem 2.6). Also, all zero entropy spacing shifts have almost specification property (Theorem 2.5). On the other hand, there is a weak mixing Σ_P with $d(P) = \lim_{n\to\infty} \frac{P\cap\{1,\ldots,n\}}{n} = 1$, zero entropy and yet having almost specification property.

In Section 3, we consider renewal spacing shifts and will give some sufficient conditions on P to have Σ_P as a renewal system (Theorem 3.2). We will show that not all SFT spacing shifts are renewal (Theorem 3.3).

1. Definitions and preliminaries

A TDS is a pair (X, T) such that X is a compact metric space and T is a homeomorphism. *The return time* set is defined to be $N(U, V) = \{n \in \mathbb{Z} : U \cap T^{-n}V \neq \emptyset\}$ where U and V are opene (nonempty and open) sets. A TDS (X, T) is *transitive* if $N(U, V) \neq \emptyset$; and it is *totally transitive* if (X, T^n) is transitive for any n. A TDS (X, T) is *weak mixing* if $N(U, V)$ is a thick set (i.e. containing arbitrarily long intervals of $\mathbb{Z}$) for any two opene sets U and V; and is *strong mixing* if $N(U, V)$ is cofinite for opene sets U, V.

The *topological entropy* of T with respect to a finite open cover α is $h(T, \alpha) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\vee_{i=1}^{n} T^{-i}\alpha)$ where $\mathcal{N}(\alpha)$ denotes the number of sets in a finite subcover of α with the smallest cardinality and the *entropy* of T is $h(T) = \sup_\alpha h(T, \alpha)$.

Let A be a finite alphabet, i.e. a finite set of symbols. The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$, for $(a_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$. If $A^{\mathbb{Z}}$ is endowed with the product topology of the discrete topology on A, then σ is a homeomorphism and $(A^{\mathbb{Z}}, \sigma)$ is a TDS called *two-sided shift* space. Similarly, *one-sided shift* space can be defined on $A^{\mathbb{N}_0}$, then σ is a finite-to-one continuous map. A *subshift* is the restriction of σ to any closed non-empty subset Σ of $A^{\mathbb{N}_0}$ that is invariant under σ. A *word* (*block*) of length n is $a_0a_1\cdots a_{n-1} \in A^n$ if there is $x \in \Sigma$ such that $x_i = a_i$, $0 \leq i \leq n-1$. The *language* $\mathcal{L}(\Sigma)$ is the collection of all words of Σ and $\mathcal{L}_n(\Sigma)$ is the collection of all words in Σ of length n. Also, a *cylinder* is defined as $[a_0\cdots a_n]_p^q = \{x \in \Sigma : x_p = a_0, \ldots, x_q = a_n\}$. For a subshift, the *topological entropy* of Σ is $h(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log(|\mathcal{L}_n(\Sigma)|)$. Let

$$\mathrm{per}_n(\Sigma) = |\{x \in \Sigma : \sigma^n x = x\}| \quad \text{and} \quad \mathrm{per}(\Sigma) = \bigcup_{n\in\mathbb{N}} \mathrm{per}_n(\Sigma).$$

Shift spaces described by a finite set of forbidden blocks are called *shifts of finite type* (SFT) and their factors are called *sofic*. A shift of finite type is *k-step*, that is, it can be described by a collection of forbidden blocks all of which have length $k+1$ for some $k \in \mathbb{N}$.

2. **Almost specified spacing shifts**

Definition 2.1. A shift space Σ has *specification property* if there exists $N \in \mathbb{N}$ such that for any $\ell \geq N$ and $w^{(1)}, w^{(2)}, \ldots, w^{(m)} \in \mathcal{L}(\Sigma)$ there are $v^{(1)}, v^{(2)}, \ldots, v^{(m)} \in \mathcal{L}_\ell(\Sigma)$ such that

$$(w^{(1)}v^{(1)}w^{(2)}v^{(2)} \cdots w^{(m)}v^{(m)})^\infty \in \Sigma.$$

A weaker concept which is one of our concern in this note is the almost specification property [17]. First, recall that a non-decreasing function $g : \mathbb{N} \to \mathbb{N}$ is called a *mistake function* if $g(n) \leq n$ for all n and $\frac{g(n)}{n} \to 0$.

Definition 2.2. A subshift Σ has *almost specification property* if there exists a mistake function g such that for every $w^{(1)}, \ldots, w^{(n)} \in \mathcal{L}(\Sigma)$, there are words $v^{(1)}, \ldots, v^{(n)} \in \mathcal{L}(\Sigma)$ with $|v^{(i)}| = |w^{(i)}|$ such that $v^{(1)} \ldots v^{(n)} \in \mathcal{L}(\Sigma)$ and each $v^{(i)}$ differs from $w^{(i)}$ in at most $g(|v^{(i)}|)$ places.

A sequence $\{x_i\}_{i\in\mathbb{N}}$ is called δ-*pseudo-orbit* if $d(\sigma(x_i), x_{i+1}) < \delta$ for any $i \geq 1$ and (Σ, σ) has *shadowing property*, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{N}}$, there exists $y \in \Sigma$ such that $d(\sigma^n y, x_n) < \epsilon$ for all $n \geq 1$. In general, in a dynamical system with a shadowing property the properties SFT, weak mixing and specification are equivalent [13, Theorem 1]. Here, without assuming the shadowing property, we have the following.

Theorem 2.3. *Let (Σ_P, σ_P) be a spacing shift. Then, the following are equivalent.*

(1) Σ_P *is topologically mixing;*
(2) Σ_P *is SFT;*
(3) Σ_P *has shadowing property;*
(4) Σ_P *has a non-trivial mixing subsystem;*
(5) Σ_P *has specification property;*
(6) Σ_P *has almost specification property with at least two periodic point.*

Proof. A subshift is SFT if and only if it has shadowing property [19]. Also, in spacing shifts, SFT is equivalent to mixing [2, Theorem 2.4]. So the first three statements are equivalent.

Note that for any Σ_P, $P = N([1], [1])$. Now if Σ_P is mixing, (4) is trivially true. Conversely, if Y is a non-trivially mixing subsystem of Σ_P, then $N([1], [1])$, the return time set in Y, and hence P is cofinite which implies that Σ_P is mixing.

It is known that specification property implies mixing [18, Pr roposition 2] and in mixing subshifts, SFT implies specification property [6].

Since all spacing shifts have the fixed point 0^∞, it remains to show that if a spacing shift has a nonzero periodic point and almost specification, then it is mixing (the converse is clearly true). Let Σ_P be a spacing shift with at least one nonzero periodic point where P is not cofinite. Therefore, we have $k\mathbb{N} \subset P$ for some $k \in \mathbb{N}$ and there is some $m \in \mathbb{N}$, such that $|(k\mathbb{N} + m) \setminus P| = \infty$. This implies that for any N, there is $n \geq N$ such that $kn + m \notin P$. Now let $w^{(1)} = (10^{k-1})^n 10^{m-1} \in \mathcal{L}_{kn+m}(\Sigma_P)$ and $w^{(2)} = (10^{k-1})^n \in \mathcal{L}_{kn}(\Sigma_P)$ and let $v^{(i)}$ be a word with $|v^{(i)}| = |w^{(i)}|$, $i \in \{1, 2\}$ and $v^{(1)}$ differs $w^{(1)}$ in less than $\frac{1}{2k}$ places. Then at least two successive 1's of $w^{(1)}$ will be in the same position of 1's of $v^{(1)}$. Hence, if $v^{(1)}v^{(2)} \in \mathcal{L}(X)$, then all the positions of 1's of $w^{(2)}$ must be different with the corresponding positions of $v^{(2)}$. So,r any mistake function will be larger than $\frac{1}{k}|v^{(2)}|$ and this in turn implies Σ_P does not have almost specification property. □

The proof of the above theorem implies:

Corollary 2.4. *Suppose Σ_P has at least two periodic points. Then Σ_P has almost specification property if and only if P is cofinite.*

The situation is different whenever Σ_P does not have non-zero periodic points. A subclass is when the entropy is zero. First recall that for any $A \subset \mathbb{N}_0$, the *Banach density* of A is defined as

$$d^*(A) = \limsup_{M-N\to\infty} \frac{|A \cap \{N, N+1, \ldots, M\}|}{M - N + 1}.$$

Also for a point $x = \{x_i\}_{i\in\mathbb{N}} \in \Sigma_P$, let

$$A_x = \{i : x_i = 1\}.$$

Theorem 2.5. *If Σ_P has zero entropy, then it has almost specification property.*

Proof. For any $n \in \mathbb{N}$, define $g(n)$ to be the maximum cardinality of entry 1 that can appear in a word of length n. Clearly $g(n)$ is increasing and if $w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ are in $\mathcal{L}(\Sigma_P)$, then $w = w^{(1)}0^{|w^{(2)}|} \cdots 0^{|w^{(k)}|} \in \mathcal{L}(\Sigma_P)$. On the other hand, $h(\Sigma_P) = 0$ if and only if $d^*(A_x) = 0$ for any $x = \{x_i\} \in \Sigma_P$ [1, Theorem 2.4]. Thus

$$\lim_{n\to\infty} \frac{g(n)}{n} = \lim_{n\to\infty} \max_{x_0\cdots x_{n-1}\in\mathcal{L}(\Sigma_P)} \frac{\sum_{i=0}^n x_i}{n} = 0.$$

Therefore, $g(n)$ is a mistake function and so zero entropy implies almost specification property. □

In [11], authors prove that any compact topological system with almost specification and measure of full support is weak mixing. In spacing shifts, we will prove the same conclusion holds for almost specification with positive entropy (Theorem 2.6). This is actually an extension, because if an spacing shifts has a measure with full support then $\mu([1]) > 0$ and by [12, Theorem 13] entropy is positive. However, there are examples in spacing shifts with positive entropy which are not of full support. To see this we recall a lemma from Furstenberg [8, Lemma 3.17]. This lemma states that in a shift space (X, σ) on a set of finite characters Λ, there is an invariant measure μ with $\mu([\alpha]) > 0$ iff the symbol α occurs in some $x \in X$ with upper Banach density. Now a consequence of this result is that if μ is a measure with full support, then for any u, the block u appears in some $x \in X$ with upper Banach density. We use this fact to show that if $P = 2\mathbb{N} \cup \{3\}$, then Σ_P has positive entropy and without any measure with full support. Our space has positive entropy because, $(10)^\infty \in X$. However, if $u = 1001$, then u can appear in any point only finitely many times and so this space is not of full measure.

Recall that a set $A \subset \mathbb{N}$ is thick if for any $M \in \mathbb{N}$, there exists n such that the M consecutive numbers $n, n+1, \dots, n+M \in A$; also, Σ_P is weak mixing if and only if P is thick [2, Theorem 2.1].

Theorem 2.6. *Suppose Σ_P has almost specification property. If $h(\sigma_P) > 0$, then (Σ_P, σ_P) is weak mixing.*

Proof. As stated in the proof of the above theorem $h(\sigma_P) > 0$ if and only if there is $y \in \Sigma_P$ such that $d_y = d^*(A_y) > 0$ [1, Theorem 2.4].

Assume that $g(n)$ is the mistake function and set $k = [\frac{1}{d_y - \delta}]$ where $0 < \delta < d_y$. By this we will have

$$d^*(A_y) > \frac{1}{k}. \tag{2.1}$$

Pick $M \in \mathbb{N}$ and choose N sufficiently large so that $\frac{g(n)}{n} < \frac{1}{2kM}$ whenever $n > N$. Also choose $l > [\frac{N}{2kM}]$ and let $V = v_1 \cdots v_{|V|}$ be a word in y such that $|V| = 2lkM > N$ and $|\{i : v_i = 1\}| \geq \frac{|V|}{k}$. Observe that by (2.1), such V exists and has at least $2lM$ entries equal 1.

Now set $U^{(i)} := V0^{i-1}$ and $V^{(i)} := V$ for $1 \leq i \leq M$. Let $\hat{U}^{(i)}, \hat{V}^{(i)} \in \mathcal{L}(\Sigma_P)$ where $|\hat{U}^{(i)}| = |U^{(i)}|$, $|\hat{V}^{(i)}| = |V^{(i)}|$ and for all $1 \leq i \leq M$, $\hat{U}^{(i)}\hat{V}^{(i)} \in \mathcal{L}(\Sigma_P)$ be the words provided by the definition of almost specification.

We claim that for all $1 \leq i \leq M$, there is $1 \leq s_V \leq |V|$ (resp. $1 \leq t_V \leq |V|$) such that the sth (resp. tth) entry of all $U^{(i)}$ (resp. $V^{(i)} = V$) are 1. Then, there has to be entries 1 in the positions s_V and $|\hat{U}^{(i)}| + t_V$ in $\hat{U}^{(i)}\hat{V}^{(i)}$. This forces to have $|\hat{U}^{(i)}| + t_V - s_V \in P$ or equivalently $|V| + i - 1 + t_V - s_V \in P$, $1 \leq i \leq M$. Since M was arbitrary, P is thick.

Now we set up to prove the claim. Since $g(|V|) < l$, $\hat{V}^{(i)}$ (resp. $\hat{U}^{(i)}$) differs $V = V^{(i)}$ (resp. $U^{(i)}$) in at most l (resp. $l+1$) entries, $1 \le i \le M$. This means that the positions of at least $e = 2lM - (l+1)$ of 1's in $\hat{U}^{(i)}$ and $U^{(i)}$ (resp. $\hat{V}^{(i)}$ and $V^{(i)}$) are identical. Without loss of generality, assume that all other entries except these e entries are 0 and let s_V (resp. t_V) be the first appearance of 1 in $\hat{U}^{(i)}$ (resp. $\hat{V}^{(i)}$). □

The following example shows that the converse of Theorem 2.6 is not true even if one chooses P with full density.

Example 2.7. There is a weak mixing Σ_P with $d(P) = 1$, $h(\sigma_P) = 0$ and having almost specification property.

Proof. Recall that if $y \in \Sigma_P$, then $A_y - A_y \subset P$. Hence, if P is not Δ^*, i.e. there is $B \subset \mathbb{N}, |B| = \infty$ such that $P \cap (B - B) = \emptyset$, then $h(\sigma_P) = 0$. This is because if $d^*(A_y) > 0$, then $A_y - A_y$ and hence P is Δ^* [5]. This enables us to give examples of weak mixing Σ_P with $h(\sigma_P) = 0$ having almost specification property.

For instance, let $B = \{2^n : n \in \mathbb{N}\}$. Then, for any n, $e_n = 2^n = 2^{n+1} - 2^n$ and $e'_n = 2^{n+1} - 2^{n-1}$ are two consecutive elements of $B - B$. But $e'_n - e_n > n$ and so, $P = \mathbb{N} \setminus (B - B)$ is a thick set and as a result Σ_P is weak mixing. By above reasoning we have $h(\sigma_P) = 0$ and using Theorem 2.5, implies that it has almost specification property.

Note that $|(B-B) \cap (2^n, 2^{n+1}]| = n$. That is, $|(B-B) \cap (1, 2^{n+1}]| = \frac{n(n+1)}{2}$ and this means that $d(B - B) = 0$ or equivalently $d(P) = 1$. □

Also, one may use Corollary 2.4 to give examples of spacing shifts which are not almost specified but have positive entropy. For instance, choose P so that it is not cofinite but contains $k\mathbb{N}$ for some $k \in \mathbb{N}$. If P is also chosen to be thick then our Σ_P is weak mixing.

The following example gives a spacing shifts with positive entropy and so that it has only one periodic point, that is, the unique fixed point (0^∞).

Example 2.8. There is a non-weakly mixing spacing shifts with positive entropy and a unique periodic point which does not have almost specification property.

Proof. Kříž provides a subset $A \subset \mathbb{N}$ such that $d^*(A) > 0$ but $A - A$ does not contain any $k\mathbb{N}$ for $k \in \mathbb{N}$ [16]. We have $A = (A \cap (2\mathbb{N} + 1)) \cup (A \cap 2\mathbb{N})$ and with partition regularity, either $A \cap (2\mathbb{N} + 1)$ or $A \cap 2\mathbb{N}$ has positive Banach density; call that B. Then $d^*(B) > 0$ and if $P = B - B$, then $h(\sigma_P) > 0$. Also, Σ_P is not weak mixing since P is not thick and it does not have non-zero periodic points for that requires P containing some $k\mathbb{N}$. So by Theorem 2.6, this spacing shifts does not have almost specification property. □

It is also worth to mention that independently in [1] and [12], using Kříž's example, a Σ_P has been constructed which has positive entropy and it is weakly mixing with a unique fixed point.

2.1. **Spacing shifts as an S-gap shift.** Note that a spacing shifts Σ_P is an S-gap shift with $S = P - 1$ if and only if $P + P \subseteq P$ [2]. The next theorem gives an alternative combinatorial condition on P for Σ_P being an S-gap. We first give the following elementary lemma.

Lemma 2.9. *Suppose $a_1, a_2 \in \mathbb{N}$ and $(a_1, a_2) = \gcd\{a_1, a_2\} = 1$. Then there is $L \in \mathbb{N}$ such that for any $n \geq L$, there are $r_1, r_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $n = r_1 a_1 + r_2 a_2$.*

We recall from [3] that a set P is *ultimately periodic* (with *ultimate period* p and *base* N) if there exist $p \geq 1$ and $N \geq 0$ such that for $n \geq N$

$$n \in P \ \Leftrightarrow n + p \in P.$$

Theorem 2.10. *The spacing shifts Σ_P is an S-gap shift if and only if $P = k\mathbb{N} \cap F$, for some $k \in \mathbb{N}$ and F a cofinite set. In particulare, Σ_P is a transitive sofic.*

Proof. We show that for a set $P \subset \mathbb{N}$, $P + P \subset P$ if and only if $P = k\mathbb{N} \cap F$, for some $k \in \mathbb{N}$ and cofinite F.

This is obvious when $\gcd(P) = 1$, for then P is cofinite. So assume $\gcd(P) = k$. Then without loss of generality we can assume that there are $p_i, p_j \in P$ such that $(p_i, p_j) = k$ (otherwise, $(p_i, p_j) = kl$ for some l and we follow the proof by kl). This means that there are coprime numbers m_i, m_j where $p_i = m_i k$ and $p_j = m_j k$. Since P is closed under addition, for any r, s, $rp_i + sp_j \in P$. So $(rm_i + sm_j)k \in P$ and since (m_i, m_j) are coprime, by Lemma 2.9, there is a cofinite set F generated by m_i and m_j and as a result $P = k\mathbb{N} \cap F$. It is clear that $k\mathbb{N} \cap F$ is ultimately periodic and by [3, Theorem 5.6], Σ_P is sofic. □

Notice that the converse of the above result is not true: Let $P = 2\mathbb{N} - 1$. Then P is ultimately periodic and Σ_P is sofic, but it is not transitive and hence it cannot be an S-gap.

3. Renewal spacing shifts

A shift space X is *coded* if there is a countable set $\mathcal{W} = \{w_1, w_2, \ldots\}$ of words in X called the *generator* of X such that X is the closure of the set of sequences obtained by freely concatenating the words in $\mathcal{W}$ [15]. We denote a coded space Σ by $\Sigma(\mathcal{W})$.

Definition 3.1. A coded system Σ is called *renewal system*, if there is a finite generating set $\mathcal{W}$ such that $\Sigma = \Sigma(\mathcal{W})$. If in addition $\mathcal{W}$ can be chosen so that whenever $u_1 \cdots u_t = v_1 \cdots v_s \in \mathcal{L}(\Sigma)$ with $u_i, v_j \in \mathcal{W}$, we have $t = s$ and $u_i = v_i$, then Σ is called *uniquely decipherable* [10, Definition 1.2].

For instance if $P = k\mathbb{N}$, then $\mathcal{W} = \{0^k, 10^{k-1}\}$ is a generator for Σ_P and it is uniquely decipherable.

A general form for a cofinite set P is

$$P = \{l_1, \dots, l_r, N, N+1, \dots\} \tag{3.1}$$

where $l_i < l_j$ whenever $i < j$ and $l_r < N - 1$.

Set $P_l = \{l_1, \dots, l_r\}$ and $P_N = \{N, N+1, \dots\}$ and note that $P = P_l \cup P_N$.

Theorem 3.2. *Suppose P is cofinite. Then for the following cases Σ_P is uniquely decipherable renewal system.*

(1) *$P + P \subset P$.*

(2) *P as in (3.1) and either $2l_1 \geq N$ or $2l_r < N$.*

Proof. (1) Suppose P is as in (3.1) and $P + P \subset P$. First we show that if

$$\mathcal{W} = \{10^{l_1-1}, \dots, 10^{l_r-1}, \; 10^{N-1}, 10^N, \dots, 10^{2N-2}, 0^N\} \setminus \{10^{N+l_1-1}, \dots, 10^{N+l_r-1}\},$$

then $\Sigma(\mathcal{W}) = \Sigma_P$. The fact that $\Sigma(\mathcal{W}) \subset \Sigma_P$ is a direct consequence of $P + P \subset P$. So let $x = (x_n)_{n\in\mathbb{N}_0} \in \Sigma_P$. We show that x is a sequence made by concatenation of some words in $\mathcal{W}$.

Without loss of generality assume that

$$x_0 = 1 \qquad \text{and} \qquad m_0 = \min\{i > 0 : x_i = 1\}.$$

We have $m_0 \in P$. Let $m_0 = \ell N + q$, $\ell \in \mathbb{N}_0$ and $0 \leq q \leq N - 1$ and set $u_0 = x_0 \cdots x_{m_0-1}$. If $q \in P_l = \{l_1, \dots, l_r\}$, then $u_0 = 10^{q-1}(0^N)^\ell \in \mathcal{L}(\Sigma(\mathcal{W}))$; otherwise, $10^{N+q-1} \in \mathcal{W}$ and hence $u_0 = 10^{N+q}(0^N)^{\ell-1} \in \mathcal{L}(\Sigma(\mathcal{W}))$. Now let $x' = (x'_n)_{n\in\mathbb{N}_0} = \sigma^{m_0}x$. By applying the same routine as x for x' and then using induction, we will have $x \in \Sigma(\mathcal{W})$.

To this end $\Sigma(\mathcal{W})$ is renewal and it remains to show that it is uniquely decipherable. Note that $|\mathcal{W}| < \infty$ and for $0 \leq q < N$, there exists a unique $u \in \mathcal{W} \setminus \{0^N\}$ such that $u = 10^{mN+q}$, $m \in \{0, 1\}$. So if $u_1u_2\cdots u_s = v_1v_2\cdots v_t \in \mathcal{L}(\Sigma(\mathcal{W}))$, $u_i, v_i \in \mathcal{W}$, then $u_1 = v_1$ and similarly $u_i = v_i$ and $s = t$.

(2) When $2l_1 \geq N$, we will have $P + P \subset P$ and the result follows from (1). So suppose $2l_r < N$. Then for all $1 \leq i \leq r$, $l_i\mathbb{N} \cap \{l_r + 1, \dots, N - 1\} = \emptyset$ which implies that $l_i\mathbb{N} \not\subset P$.

To give a suitable generator $\mathcal{W}$ for Σ_P, let B_n be a set consisting of those n-tuples in P_l^n such that all the sums of $s \in \{1, \dots, n\}$ consecutive coordinates are in P. That is, for $1 \leq n \leq r$ let

$$B_n = \{(l_{i_1}, l_{i_2}, \dots, l_{i_n}) \in P_l^n : \cup_{s=1}^n \cup_{\alpha=1}^{n-s+1} \{l_{i_\alpha} + l_{i_{\alpha+1}} + \cdots + l_{i_{\alpha+s-1}}\} \subset P\}.$$

Then $B_1 = P_l$ and B_n may be empty for some $2 \leq n \leq r$. Set $B := \cup_{n=1}^r B_n$ and note that $|B| < \infty$. Now for $b \in B$ define

$$u(b) = 10^{l_{i_1}-1}10^{l_{i_2}-1}1\cdots10^{l_{i_n}-1}10^{N-1}.$$

We show that

$$\mathcal{W} = \{0,\, 10^{N-1}\} \cup \{u(b) : b \in B\}$$

is a generator for Σ_P and $\Sigma_P = \Sigma(\mathcal{W})$ is uniquely decipherable.

Observe that for any $b,\, b' \in B$

$$u(b) = u(b') \quad \Longleftrightarrow \quad b = b', \tag{3.2}$$

and suppose that $u_1 u_2 \cdots u_r = v_1 v_2 \cdots v_s$ where $u_i,\, v_j \in \mathcal{W}$. If $u_1 = 0$ (resp. 10^{N-1}), then $v_1 = 0$ (resp. 10^{N-1}) and if $u_1 = u(b)$ for some b, then $v_1 = u(b')$ for some b' and using (3.2) implies $u_1 = u(b) = u(b') = v_1$. Thus by an induction argument $u_i = v_i$ and $r = s$. This means $\Sigma(\mathcal{W})$ is uniquely decipherable and the necessity is proved if we show that $\Sigma(\mathcal{W}) = \Sigma_P$. But $\Sigma(\mathcal{W})$ is a subsystem of Σ_P and so $\Sigma(\mathcal{W}) \subset \Sigma_P$. Now let $x = (x_n) \in \Sigma_P$. If $x = 0^\infty$ we are done. Otherwise, we may assume $x_0 = 1$ and then by our hypothesis, we must have at least one 10^{N-1} as a word in x starting at a positive position. Let m_0 be the first incident that such a 10^{N-1} occurs and let $u = x_0 \cdots x_{m_0} \cdots x_{m_0+N} = 1x_1 \cdots x_{m_0-1}10^{N-1}$. We show that $u \in \mathcal{W}$ and then substituting x by $x' = \sigma_P^{|u|}x$ and using an induction argument we will have the proof. But the only possibility is that $u = 10^{l_{i_1}-1}10^{l_{i_2}-1} \cdots 10^{l_{i_n}-1}10^{N-1}$ where $l_{i_j} \in P_l$. This in turn means that $b = (l_{i_1}, \ldots, l_{i_n}) \in B_n$ and so $u = u(b) \in \mathcal{W}$ as required. □

Now we exclude some cases that a coded Σ_P cannot be renewal.

Theorem 3.3. *Let P be cofinite as in (3.1). Suppose $P_l \cap [1,\, a+b] \subset a\mathbb{N} \cup b\mathbb{N} \subset P$ but $(a+b) \notin P$ for some positive integers $1 < a < b$. Then, Σ_P is not renewal.*

Proof. Assume that Σ_P is renewal. So there exists a finite set of words $\mathcal{W}$ such that $\Sigma(\mathcal{W}) = \Sigma_P$. Since P contains $a\mathbb{N}$ and $b\mathbb{N}$, we conclude that $(10^{a-1})^\infty,\, (10^{b-1})^\infty \in \Sigma_P$. Hence by the fact that $|\mathcal{W}| < \infty$, there exist $p,\, q,\, s_1,\, s_2,\, t_1,\, t_2 \in \mathbb{N}_0$ such that $w_1 = 0^{s_1}(10^{a-1})^p 10^{t_1}$, $w_2 = 0^{s_2}(10^{b-1})^q 10^{t_2}$ are in $\mathcal{W}$ as subwords of $(10^{a-1})^\infty$ and $(10^{b-1})^\infty$ respectively.

So

$$s_1 + t_1 = a - 1, \;\; s_2 + t_2 = b - 1. \tag{3.3}$$

Also, $w_1 w_2 \in \mathcal{L}(\Sigma_P) \Rightarrow e := t_1 + s_2 + 1 \in P$. On the other hand, $a + b \notin P$ implies $a+b < N$ and since $e \leq t_1+s_1+s_2+t_2+1 \leq a+b-1$, $e \in P_l \cap [1,\, a+b] \subset a\mathbb{N} \cup b\mathbb{N}$. Observe that since $a < b$, we have $e < 2b$. So either $e = b$ or $e = ka$ for some $k \in \mathbb{N}$. If $e = b$, then from

$$w_1 w_2 = 0^{s_1}(10^{a-1})^{p-1}(1\underbrace{0^{a-1})10^{t_1}0^{s_2}}(10^{b-1})^q 10^{t_2} \in \mathcal{L}(\Sigma_P)$$

one must have $a + t_1 + s_2 + 1 = a + b \in P$ which is not allowed by assumption. (If $p = 0$, we can consider $w_1 w_1 w_2$ instead of $w_1 w_2$.) Hence, $e = ka$ for some $k \in \mathbb{N}$. Similarly, we have $w_2 w_1 \in \mathcal{L}(\Sigma_P)$ and so $e' := s_1 + t_2 + 1 = k'a$, $k' \in \mathbb{N}$. This in turn implies $a + b = e + e' = (k + k')a \in a\mathbb{N} \subseteq P$ which is absurd. □

Let P and Σ_P be as in the above theorem. Then, a natural question is if Σ_P is conjugate to a renewal system? In fact, in a more general settings, this question was posed by Adler, that is, he asks: "Is every irreducible shift of finite type conjugate to a renewal system" [9]. Note that this question has not been addressed here.

Definition 3.4. A spacing shifts Σ_P is called *regular spacing shifts* if for any $u \in \mathcal{L}(\Sigma_P)$ there exists $l \in \mathbb{N}$ such that $10^{l-1}u \in \mathcal{L}(\Sigma_P)$.

Example 3.5. (1). Transitive spacing shifts and those with a dense set of periodic points are regular spacing shifts.

(2). Example of a regular but not transitive spacing shifts. Let $P_1 = 2\mathbb{N} \cup 3\mathbb{N}$. First by showing that Σ_{P_1} has a dense set of periodic points, we will conclude that Σ_{P_1} is a regular spacing shifts.

By [2, Theorem 2.7], Σ_{P_1} has a dense set of periodic points if and only if for any $p \in P_1$ there exists $k \in \mathbb{N}$ such that

$$k\mathbb{N} \cup (k\mathbb{N} + p) \cup (k\mathbb{N} - p) \subset P_1.$$

Our example satisfies this condition and so is regular.

Now we show that Σ_{P_1} is not transitive. Let $u = 101$ and $v = 10^2 1$ be two words in Σ_{P_1}. Non-transitivity is proved if we show that $u0^{l-1}v \notin \mathcal{L}(\Sigma_{P_1})$ for all $l \in \mathbb{N}$. Otherwise, l, $l+2$, $l+3$, $l+5 \in P_1$. Since $(l+2)-l = 2$, l, $l+2 \in 2\mathbb{N}$ and so $l+3$, $l+5$ are odd numbers and hence they must be in $3\mathbb{N}$. Trivially this is impossible.

(3). Example for a non-regular spacing shifts. Let $P_2 = 2\mathbb{N} \cup \{3\}$ and let $u = 1001001 \in \mathcal{L}(\Sigma_{P_2})$. Then, $10^{l-1}u \notin \mathcal{L}(\Sigma_{P_2})$ for $l \in \mathbb{N}$. Because, if $l \in 2\mathbb{N}+1$, then l has to be 3 which is impossible here; for this forces to have $9 \in P_2$. If $l \in 2\mathbb{N}$, then we must have another odd number $l+3 \in P_2$ which is again impossible.

Theorem 3.6. *Let Σ_P be a regular spacing shifts and let X be a subshift in $\{0, 1\}^{\mathbb{N}_0}$. Then, Σ_P is conjugate to X if and only if $\Sigma_P = X$. Also, the conjugacy map must be the identity map up to re-indexing the characters in $\{0, 1\}$.*

Proof. Let ϕ be the conjugacy map. Since Σ_P has at most one point of periodic point of period one, the same is true for X and so either $0^\infty \in X$ or $1^\infty \in X$; either is possible and we assume $0^\infty \in X$. (Re-indexing of the characters in $\{0, 1\}$ stated in the conclusion happens here.) Also let Φ be the bijective sliding block coded by anticipation n, and $\phi = \Phi^\infty$.

Claim: If $x_u = u0^\infty = u_0u_1 \cdots u_m0^\infty \in \Sigma_P$ has k entries 1, then k entries of $\phi(x_u)$ are 1.

We prove the claim by induction on k, the number of entries 1 in u. Since 0^∞ is the only periodic point of period 1 in both systems, we must have $\phi(0^\infty) = 0^\infty$

and so $\Phi(0^n) = 0$. So our claim is true when $k = 0$. Hence assume that the claim is true for $k - 1$ and let

$$u = 10^{l_1-1}10^{l_2-1}1 \cdots 10^{l_{k-1}-1}10^{l_k-1},$$

a word of length m in Σ_P with k entries 1. By $\Phi(0^n) = 0$, we have $\phi(u0^\infty) = v0^\infty$ for some $v = v_0v_1 \cdots v_m \in \mathcal{L}(X)$. We show that v has k entries 1. We have $\phi(\sigma x_u) = \sigma(v0^\infty) = v_1v_2 \cdots v_m0^\infty$. Also since σx_u has $k-1$ entries 1, by the induction assumption, $v_1v_2 \cdots v_m$ has the same number of entries 1. Now replace u_0 in u with 0 and call it u', that is, let $u' = 0^{l_1}10^{l_2-1}1 \cdots 10^{l_{k-1}-1}10^{l_k-1}$. Observe that u' and $v'0^\infty = \phi(u'0^\infty)$ have $k-1$ entries 1. But $v'_1v'_2 \cdots v'_m = v_1v_2 \cdots v_m$ (because $u'_1 \cdots u'_m = u_1 \cdots u_m$) and they have $k-1$ entries 1 as well. This implies that $v'_0 = 0$. It remains to determine v_0. If $v_0 = 0$, then $\phi(u'0^\infty) = \phi(u0^\infty)$ and this is absurd by the fact that ϕ is one to one. Therefore, $v_0 = 1$ and as a result v has k entries 1 and the claim is set for the case that u starts with 1. Moreover, in this case x_u and its image start with 1.

Now let $u = 0^{l_0}10^{l_1-1}1 \cdots 10^{l_k-1}$ for some $l_0 \in \mathbb{N}$. By the definition of regular spacing shifts, there exists $l' \in \mathbb{N}$ where $w = 10^{l'-1}0^{l_0}10^{l_1-1}1 \cdots 10^{l_k-1} \in \mathcal{L}(\Sigma_P)$. On the other hand, $x_w = w0^\infty$ and $\sigma^{l'+l_0}(w0^\infty)$ begin with 1 and so $\phi(w0^\infty)$ and $\phi(\sigma^{l'+l_0}(w0^\infty))$ have $k+1$ and k entries 1 respectively and both begin with 1. Thus $\phi(\sigma^{l'}(w0^\infty)) = \phi(u0^\infty)$ has k entries 1 and the proof of the claim is complete.

Now we show that for any $u \in \mathcal{L}(\Sigma_P)$, $\phi(u0^\infty) = u0^\infty$ and clearly this gives the proof. But in the second case of the proof of the claim we have that both x_u and its image start with 0. Combining this fact and the similar fact in the first case, we see that x_u and $\phi(x_u)$ have the same starting entry. This in turn means that $\Phi(u) = u_0$ and so we may actually take the anticipation to be 1. Then, Φ as well as ϕ are identity and we are done. $\square$

Theorem 3.7. *Suppose P_1 is cofinite and $P_1 \subseteq P_2$. Then, Σ_{P_1} is both a subsystem and a factor of Σ_{P_2}.*

Proof. The conclusion follows trivially if $P_1 = P_2$; so assume $P_1 \subsetneq P_2$. Then, Σ_{P_1} is a proper subsystem of Σ_{P_2} and $h(\sigma_{P_1}) < h(\sigma_{P_2})$ ([15, Corollary 4.4.9]). Also, $\mathrm{per}(\Sigma_{P_1}) \subseteq \mathrm{per}(\Sigma_{P_2})$ and in particular, $\mathrm{per}(\Sigma_{P_1}) \searrow \mathrm{per}(\Sigma_{P_2})$, that is, for any periodic point of period p in Σ_{P_1}, there is a periodic point of period q in Σ_{P_2} such that $p|q$. Now, since Σ_{P_1} and Σ_{P_2} are mixing and SFT, by [7, Theorem 5.3], $(\Sigma_{P_1}, \sigma_{P_1})$ is a factor of $(\Sigma_{P_2}, \sigma_{P_2})$. $\square$

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