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DIAGONAL ARGUMENTS AND FIXED POINTS

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Abstract. A universal schema for diagonalization was popularized by N.S. Yanofsky (2003), based on a pioneering work of F.W. Lawvere (1969), in which the existence of a (diagonolized-out and contradictory) object implies the existence of a fixed-point for a certain function. It was shown that many self-referential paradoxes and diagonally proved theorems can fit in that schema. Here, we fit more theorems in the universal schema of diagonalization, such as Euclid's proof for the infinitude of the primes and new proofs of G. Boolos (1997) for Cantor's theorem on the non-equinumerosity of a set with its powerset. Then, in Linear Temporal Logic, we show the non-existence of a fixed-point in this logic whose proof resembles the argument of Yablo's paradox (1985, 1993). Thus, Yablo's paradox turns for the first time into a genuine mathematico-logical theorem in the framework of Linear Temporal Logic. Again the diagonal schema of the paper is used in this proof; and it is also shown that G. Priest's inclosure schema (1997) can fit in our universal diagonal/fixed-point schema. We also show the existence of dominating (Ackermann-like) functions (which dominate a given countable set of functions, such as primitive recursive functions) in the schema.
Keywords: Diagonal argument, self-reference, fixed-points, Yablo's paradox, (linear) temporal logic.
MSC(2010): Primary: 18A10; Secondary: 18A15, 03B44, 03A05.

1. **Introduction**

Cantor's Diagonal Argument was introduced in his third proof of the well-known theorem on the non–denumerability of the reals; the argument shows that there can be no surjective function from a set A to its powerset $\mathscr{P}(A)$: for any function $F : A \to \mathscr{P}(A)$ the set $D_F = \{x \in A \mid x \notin F(x)\}$ is not in the range of F because for any $a \in A$ we have $a \in D_F \longleftrightarrow a \notin F(a)$ which is equivalent with $a \in (D_F \setminus F(a)) \cup (F(a) \setminus D_F)$, so $D_F \neq F(a)$. This argument

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is also shown in Russell's Paradox: the collection $R = \{x \mid x \notin x\}$ of sets is not a set, since for any set A, $A \in R \longleftrightarrow A \notin A$, so $A \neq R$.

Many other theorems in mathematics (logic and set theory, computability theory, complexity theory, etc.) use diagonal arguments; Tarski's theorem on the undefinability of truth, and Gödel's theorem on the incompleteness of sufficiently strong and (ω–)consistent theories are two prominent examples. In 2003, Noson S. Yanofsky in [18] mentioned some earlier descriptions for "many of the classical paradoxes and incompleteness theorems in a categorial fashion", by using "the language of category theory (and of cartesian closed categories in particular)" one can demonstrate some paradoxical phenomena and show the above mentioned theorems of Cantor, Tarski and Gödel; the goal of [18] was "to make these amazing results available to a larger audience". In that paper, a universal schema has been considered in the language of sets and functions (not categories) and paradoxes such as the Liar, the strong liar, Russell, Grelling, Richard, Time Travel, and Löb, and also the theorems of Cantor ($A \lneqq \mathscr{P}(A)$), Turing (undecidability of the Halting problem, and existence of a non–re set), Baker-Gill-Solovay (the existence of an oracle $\mathcal{O}$ such that $\mathbf{P}^{\mathcal{O}} \neq \mathbf{NP}^{\mathcal{O}}$), Carnap (the diagonalization lemma), Gödel (first incompleteness theorem), Rosser (incompleteness of sufficiently strong and consistent theories), Tarski (undefinability of truth in sufficiently strong languages), Parikh (existence of sentences with very long proofs), Kleene (Recursion Theorem), Rice (undecidability of non–trivial properties of recursive functions), and von Neumann (existence of self–reproducing machines) are shown as instances (see also Gaifman's paper [7] for a unification of Gödel-Carnap's diagonal lemma and Kleene's recursion theorem). Indeed, [18] was based on [11] in which Lawvere used the framework of cartesian closed categories to unify many diagonal arguments in set theory and logic; Yanofsky [18] simplified Lawvere's framework to a set-theoretical diagrammatical template.

In this paper, we fit some other theorems and proofs into the above mentioned universal schema of Yanofsky; these include Euclid's Theorem on the infinitude of the primes, Boolos' proof of the existence of some explicitly definable counterexamples to the non–injectivity of functions $F : \mathscr{P}(A) \to A$ for any set A, Yablo's paradox in a form of a mathematical theorem in the framework of linear temporal logic as a non–existence of some certain fixed–points, and the existence of dominating functions for a given countable set of functions such as Ackermann's function which dominates all the primitive recursive functions. In the rest of the introduction we fix our notation and introduce the common framework.

1.1. **Cantor's Theorem by Fixed–Points.** Let B, C and D be arbitrary sets. Any function $f : B \times C \to D$ corresponds to a function $\widehat{f} : C \to D^B$ where $\widehat{f}(c)(b) = f(b, c)$ for any $b \in B$ and $c \in C$ (the set D^B consists of all the functions from B to D). Conversely, for any function $F : C \to D^B$ there exists some $f : B \times C \to D$ such that $\widehat{f} = F$; it is enough to take $f(b,c) = F(c)(b)$ for any $b \in B$ and $c \in C$. In the other words $\widehat{-} : D^{B\times C} \cong (D^B)^C$. Let $f : B\times C \to D$ be a fixed function. A function $g : B \to D$ is called *representable* by f at a fixed $c_0 \in C$, when for any $x \in B$, $g(x) = f(x, c_0)$ holds. In the other words, $g = \widehat{f}(c_0)$. So, the function $\widehat{f} : C \to D^B$ is onto if and only if every function $B \to D$ is representable by f at some $c_0 \in C$.

Theorem 1.1 (Cantor's Diagonal Theorem)**.** *Assume the function $\alpha : D \longrightarrow D$, for a set D, does not have any fixed point (i.e., $\alpha(d) \neq d$ for all $d \in D$). Then for any set B and any function $f : B \times B \to D$ there exists a function $g : B \to D$ that is not representable by f; i.e., for all $b \in B$, $g(-) \neq f(-, b)$.*

Proof. The desired function $g : x \mapsto \alpha(f(x,x))$ can be constructed as follows:

$$\begin{tikzcd}
B\times B \arrow[r, "f"] & D \arrow[d, "\alpha"] \\
B \arrow[u, "\triangle_B"] \arrow[r, "g"'] & D
\end{tikzcd}$$

where $\triangle_B$ is the diagonal function of B ($\triangle_B(x) = \langle x, x\rangle$). If g is representable by f at $b \in B$, then $g(x) = f(x, b)$ for any $x \in B$, and in particular $g(b) = f(b, b)$. On the other hand by the definition of g we have $g(x) = \alpha(f(x, x))$ and in particular (for $x = b$) $g(b) = \alpha(f(b, b))$. It follows that $f(b, b)$ is a fixed–point of α; a contradiction. Therefore, the function g is not representable by f (at any $b \in B$). $\square$

For any set A we have $\mathscr{P}(A) \cong \mathbf{2}^A$ where $\mathbf{2} = \{0, 1\}$ and $\mathbf{2}^A$ is the set of all functions from A to $\mathbf{2}$. So, Cantor's diagonal theorem is equivalent to the non–existence of a surjection $A \to \mathbf{2}^A$. Putting it another way, Cantor's diagonal theorem says that for any function $f : A \times A \to \mathbf{2}$ there exists a function $g : A \to \mathbf{2}$ which is not representable by f (at any member of A). In this new setting, Cantor's (diagonal) proof goes as follows: let $\triangle_A : A \to A \times A$ be the diagonal function of A ($\triangle_A(x) = \langle x, x\rangle$) and let $\alpha : \mathbf{2} \to \mathbf{2}$ be a fixed function. Define $g : A \to \mathbf{2}$ by $g(x) = \alpha(f(\triangle_A(x)))$. If g is representable by f and fixed $a \in A$, then $f(a, a) = g(a) = \alpha(f(a, a))$, which shows that α has a fixed–point (namely, $f(a, a)$). So, for reaching a contradiction, we need to take a function $\alpha : \mathbf{2} \to \mathbf{2}$ which does not have any fixed–point; and the only such function (without any fixed–point) is the negation function $\texttt{neg} : \mathbf{2} \to \mathbf{2}$, $\texttt{neg}(i) = 1 - i$ for $i = 0, 1$. For a function $F : A \to \mathscr{P}(A)$ let $f : A \times A \to \mathbf{2}$ be

$$f(a, a') = \begin{cases} 1 & \text{if } \; a \in F(a') \\ 0 & \text{if } \; a \notin F(a'). \end{cases}$$

The function g constructed by the diagram

$$\begin{tikzcd} A \times A \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\ A \arrow[u, "\triangle_A"] \arrow[r, "g"'] & \mathbf{2} \end{tikzcd}$$

is the characteristic function of the set $D_F = \{x \in A \mid x \notin F(x)\}$. Saying that "$g$ is not representable by f (at any $a \in A$)" is equivalent to saying that "the set D_F is not in the range of F (i.e., $D_F \neq F(a)$ for any $a \in A$)".

In the rest of the paper we will fit many other theorems in the diagram which was applied in the proof of Theorem 1.1 by varying the sets B, D and the functions f, α. In the most cases of the current paper, we assume $D = 2$ and $\alpha = neg$ as in the above diagram.

2. Euclid's theorem on the infinitude of the primes

Our first application of Cantor's Diagonal Theorem is a surprising one: the ancient theorem of Euclid stating that there are infinitely many prime numbers. We use (almost) the classical proof of Euclid which seems far from being a diagonal argument. Indeed there are many different proofs of this theorem in the literature, and ours is not a new one; we just fit the classical proof of Euclid's Theorem in a diagonal diagram as above.

Theorem 2.1 (Euclid)**.** *There are infinitely many prime numbers in* $\mathbb{N}$.

Proof. Define the function $f : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$ as follows:

$$f(n,m) = \begin{cases} 1 \text{ if all the prime factors of } (n!+1) \text{ are less than } m, \\ 0 \text{ if some prime factor of } (n!+1) \text{ is greater than or equal to } m. \end{cases}$$

For example, $f(4,9) = 1$ because $4!+1 = 25$ and it has no other prime factor but 5 and $5 < 9$; it can be seen that $f(4,m) = 0$ for all $m \leqslant 5$ and $f(4,m) = 1$ for all $m > 5$. Indeed, for any $n \in \mathbb{N}$ we have $f(n,n) = 0$ because no prime factor of $n!+1$ can be less than n: for any $d < n$ if $d \mid (n!+1)$ then from $d \mid n!$ it follows that $d \mid 1$ so d cannot be a prime. Now, consider the function $g : \mathbb{N} \to \mathbf{2}$ constructed as

$$\begin{tikzcd} \mathbb{N} \times \mathbb{N} \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\ \mathbb{N} \arrow[u, "\triangle_{\mathbb{N}}"] \arrow[r, "g"'] & \mathbf{2}. \end{tikzcd}$$

If all the prime numbers are less than $\mathfrak{p} \in \mathbb{N}$ then the function g is representable by f at $\mathfrak{p}$: for any $n \in \mathbb{N}$, $f(n, \mathfrak{p}) = 1$ and $g(n) = \mathtt{neg}(f(n, n)) = 1$; whence $g(n) = f(n, \mathfrak{p})$ for all $n \in \mathbb{N}$. A contradiction follows as before: if such a number $\mathfrak{p}$ exists, then $f(\mathfrak{p}, \mathfrak{p})$ becomes a fixed–point of $\mathtt{neg}$. So, there exists no $\mathfrak{p} \in \mathbb{N}$ such that all the primes are non–greater than $\mathfrak{p}$; whence there must be infinitely many primes. $\square$

This surprising argument, we believe, deserves another closer look: define the function $F : \mathbb{N} \to \mathscr{P}(\mathbb{N})$ by

$F(n) = \{x \in \mathbb{N} \mid n \text{ is greater than or equal to all the prime factors of } (x!{+}1)\}$.

Cantor's Theorem says that F cannot be surjective, or more explicitly, the (anti–diagonal) set $D_F = \{n \mid n \notin F(n)\}$ is not equal to any $F(m)$. A number–theoretic argument shows that $D_F = \mathbb{N}$ because for any n all the prime factors of $(n! + 1)$ are greater than n (see the proof of the above Theorem 2.1). On the other hand if $\mathfrak{p} \in \mathbb{N}$ is the greatest prime, then $F(\mathfrak{p}) = \mathbb{N} = D_F$, a contradiction!

3. Some other proofs for Cantor's theorem

In 1997, the late George Boolos published another proof [3] for Cantor's Theorem, by showing that there cannot be any injection from the powerset of a set to the set. This proof has been (implicitly or explicitly) mentioned also in [9, 15] (but without referring to the earlier [3]). The first proof is essentially Cantor's Diagonal Argument; in fact the proof of the following theorem gives some more information than mere non–injectivity of any function $h : \mathscr{P}(A) \to A$, i.e., the existence of some $C, D \subseteq A$ such that $h(C) = h(D)$ and $C \neq D$. Let us emphasize that an elementary reasoning shows that if there is an injective function from a set A to a set B, then there is a surjective function from B to A (the converse is also true, but needs the axiom of choice). Therefore, "the nonexistence of an injective function from $\mathscr{P}(A)$ to A" is an immediate consequence of "the nonexistence of a surjective function from A to $\mathscr{P}(A)$". Boolos noticed that, proving nonexistence of a surjection from A to $\mathscr{P}(A)$ proceeds by exhibiting a set (namely D_f of the introduction) which is missing from the range of f; but the easy reduction from "there is no injective h from $\mathscr{P}(A)$ to A" to "there is no surjective f from A to $\mathscr{P}(A)$" does not definably yield counterexamples to the injectivity of h.

Theorem 3.1. *No function $h : \mathscr{P}(A) \to A$ can be injective.*

Proof. Let $h : \mathscr{P}(A) \to A$ be a function. Define $f : \mathscr{P}(A) \times \mathscr{P}(A) \to \mathbf{2}$ by

$$f(X, Y) = \begin{cases} 1 & \text{if } h(X) \notin Y, \\ 0 & \text{if } h(X) \in Y, \end{cases}$$

and let $g : \mathscr{P}(A) \to \mathbf{2}$ be the following function

$$\begin{tikzcd} \mathscr{P}(A) \times \mathscr{P}(A) \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\ \mathscr{P}(A) \arrow[u, "\triangle_{\mathscr{P}(A)}"] \arrow[r, "g"'] & \mathbf{2}. \end{tikzcd}$$

Let $\mathcal{D}_h = \{a \in A \mid \exists Y \subseteq A\ (h(Y) = a\ \&\ a \notin Y)\}$. Note that for any $X \subseteq A$ we have $h(X) \notin X \longrightarrow h(X) \in \mathcal{D}_h$. We show that if h is injective then g is representable by f at $\mathcal{D}_h$. For, if h is injective then for any $X \subseteq A$ we have

$$\begin{array}{lll} h(X) \in \mathcal{D}_h & \longrightarrow & \exists Y \subseteq A\ (h(Y) = h(X)\ \&\ h(X) \notin Y) \\ & \longrightarrow & \exists Y\ (Y = X\ \&\ h(X) \notin Y) \\ & \longrightarrow & h(X) \notin X. \end{array}$$

Therefore, we get $h(X) \notin X \longleftrightarrow h(X) \in \mathcal{D}_h$ for all $X \subseteq A$. So, for any $X \subseteq A$, one gets

$$\begin{array}{lll} f(X, \mathcal{D}_h) = 0 & \longleftrightarrow & h(X) \in \mathcal{D}_h \\ & \longleftrightarrow & h(X) \notin X \\ & \longleftrightarrow & f(X, X) = 1 \\ & \longleftrightarrow & g(X) = \mathtt{neg}(f(X, X)) = 0. \end{array}$$

Thus, we conclude that $g(X) = f(X, \mathcal{D}_h)$. The contradiction (that $\mathtt{neg}$ possesses a fixed–point) follows as before, implying that the function h cannot be injective. $\square$

Corollary 3.2. *For any function $h : \mathscr{P}(A) \to A$ there are some $C, D \subseteq A$ such that $h(C) = h(D) \in D \setminus C$ (and so $C \neq D$).*

Proof. For any $X \subseteq A$ we had $h(X) \notin X \longrightarrow h(X) \in \mathcal{D}_h$, whence from the implication $h(\mathcal{D}_h) \notin \mathcal{D}_h \longrightarrow h(\mathcal{D}_h) \in \mathcal{D}_h$, we can conclude that $h(\mathcal{D}_h) \in \mathcal{D}_h$. Thus, there exists some $\mathcal{C}_h$ such that $h(\mathcal{C}_h) = h(\mathcal{D}_h)$ and $h(\mathcal{D}_h) \notin \mathcal{C}_h$. So, for these subsets $\mathcal{C}_h, \mathcal{D}_h$ of A we have $h(\mathcal{C}_h) = h(\mathcal{D}_h) \in \mathcal{D}_h \setminus \mathcal{C}_h$. $\square$

Boolos [3] found out that the set $\mathcal{D}_h$ in the above proof has an explicit definition:

$$\mathcal{D}_h = \{a \in A \mid \exists Y \subseteq A\ (h(Y) = a\ \&\ a \notin Y)\}.$$

However, the set $\mathcal{C}_h$ was not defined explicitly, and its mere existence was shown. So, this proof of non–injectivity was not constructive (did not explicitly construct two sets C and D such that $h(C) = h(D)$ and $C \neq D$). For a constructive proof, Boolos [3] proceeds as follows (cf. [9, 15]).

Fix a function $h : \mathscr{P}(A) \to A$. Call a subset $B \subseteq A$ an h*–woset* (h well ordered set) when there exists a well ordering $\prec$ on B such that $b = h(\{x \in B \mid x \prec b\})$ for any $b \in B$. For example, $\{h(\emptyset)\}$ is an h–woset, and indeed any non–empty h–woset must contain $h(\emptyset)$. Some other examples of h–wosets are

$$\{h(\emptyset), h\big(\{h(\emptyset)\}\big)\} \text{ and } \{h(\emptyset), h\big(\{h(\emptyset)\}\big), h\Big(\Big\{h(\emptyset), h\big(\{h(\emptyset)\}\big)\Big\}\Big)\}, \text{ etc.}$$

We need the following two facts about the h–wosets:

(1) If B and C are two h–wosets with the well ordering relations $\prec_B$ and $\prec_C$ then exactly one (and only one) of the following holds:
  (i) $(B, \prec_B)$ is an initial segment of $(C, \prec_C)$, or
  (ii) $(C, \prec_C)$ is an initial segment of $(B, \prec_B)$, or
  (iii) $(B, \prec_B) = (C, \prec_C)$.

(2) For any h–woset B, if $h(B) \notin B$ then the set $\Phi(B) = B \cup \{h(B)\}$ is an h–woset, and B is an initial segment of $\Phi(B)$.

The statement (1) corresponds to Zermelo's theorem that any two well ordered sets are comparable to each other: either they are isomorphic or one of them is isomorphic to an initial segment of the other one. It follows from (1) that the union of all h–wosets is an h–woset, denoted by $\mathcal{W}_h$; thus $\mathcal{W}_h$ is the greatest h–woset. For (2) let B be an h–woset with the well ordering $\prec_B$ such that $h(B) \notin B$. Then $\Phi(B)$ is an h–woset with the well ordering $\prec_{\Phi(B)} = \prec_B \cup (B \times \{h(B)\})$.

The proof of Boolos [3] continues as follows (see also [9]): since $\Phi(\mathcal{W}_h) = \mathcal{W}_h$ then $h(\mathcal{W}_h) \in \mathcal{W}_h$. Also for $\mathcal{V}_h = \{x \in \mathcal{W}_h \mid x \prec_{\mathcal{W}_h} h(\mathcal{W}_h)\}$ we have $h(\mathcal{W}_h) = h(\mathcal{V}_h)$ and $\mathcal{W}_h \neq \mathcal{V}_h$ because $h(\mathcal{W}_h) \notin \mathcal{V}_h$. Indeed, the result is stronger than this (and Corollary 3.2) since the sets $\mathcal{W}_h$ and $\mathcal{V}_h$ were explicitly defined in such way that $\mathcal{V}_h \subsetneqq \mathcal{W}_h$ holds and $h(\mathcal{V}_h) = h(\mathcal{W}_h) \in \mathcal{W}_h \setminus \mathcal{V}_h$. As another partial surprise we show that this proof is also diagonal and fits in our universal framework.

Theorem 3.3 (Boolos)**.** *For any set A and function $h : \mathscr{P}(A) \to A$ there exist explicitly definable subsets $V, W \subseteq A$ such that $V \subsetneqq W$ and $h(V) = h(W) \in W \setminus V$.*

Proof. Let $\mathbf{W}_h$ be the class of all h–wosets; i.e., all subsets $B \subseteq A$ on which there exists a (unique) well ordering $\prec_B$ such that $b = h(\{x \in B \mid x \prec_B b\})$ for all $b \in B$. Define $\Phi : \mathbf{W}_h \to \mathbf{W}_h$ by

$$\Phi(X) = \begin{cases} X \cup \{h(X)\} & \text{if } h(X) \notin X \\ X & \text{if } h(X) \in X \end{cases}$$

$$\text{with} \qquad \prec_{\Phi(X)} = \begin{cases} \prec_X \cup (X \times \{h(X)\}) & \text{if } h(X) \notin X \\ \prec_X & \text{if } h(X) \in X. \end{cases}$$

Define the function $f : \mathbf{W}_h \times \mathbf{W}_h \to \mathbf{2}$ by

$$f(X, Y) = \begin{cases} 1 & \text{if } \Phi(X) \text{ is isomorphic to } Y \text{ or an initial segment of it} \\ 0 & \text{if } Y \text{ is isomorphic to an initial segment of } \Phi(X). \end{cases}$$

Let $\mathcal{W}_h$ be the greatest element of $\mathbf{W}_h$ (as above). Then $f(X, \mathcal{W}_h) = 1$ for all $X \in \mathbf{W}_h$. We claim that

$(*)$ there exists some $Z \in \mathbf{W}_h$ such that $h(Z) \in Z$ or equivalently $\Phi(Z) = Z$.

Assume (for a moment) that the claim is false. Then for all $X \in \mathbf{W}_h$, X

is (isomorphic to) an initial segment of $\Phi(X)$; whence $f(X,X)=0$. Let $g:\mathbf{W}_h\to\mathbf{2}$ be defined by the following diagram

$$\begin{tikzcd}
\mathbf{W}_h\times\mathbf{W}_h \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\
\mathbf{W}_h \arrow[u, "\triangle_{\mathbf{W}_h}"] \arrow[r, "g"'] & \mathbf{2}.
\end{tikzcd}$$

It follows from assuming the falsity of the claim $(*)$ that

$$g(X)=\mathtt{neg}(f(X,X))=1=f(X,\mathcal{W}_h).$$

Thus g is representable by f (at $\mathcal{W}_h$) and the usual contradiction (the existence of a fixed–point for $\mathtt{neg}$) follows. So, the claim $(*)$ is true, which implies that there exists some $Z\in\mathbf{W}_h$ such that $h(Z)\in Z$ or equivalently $\Phi(Z)=Z$. It can be seen that then $\mathcal{W}_h=Z$, so $\Phi(\mathcal{W}_h)=\mathcal{W}_h$ and $h(\mathcal{W}_h)\in\mathcal{W}_h$. Whence, for the (explicitly definable) set $\mathcal{V}_h=\{x\in\mathcal{W}_h\mid x\prec_{\mathcal{W}_h}h(\mathcal{W}_h)\}$ $(\subseteq A)$ we will have $\mathcal{V}_h\subsetneqq\mathcal{W}_h$ and $h(\mathcal{V}_h)=h(\mathcal{W}_h)\in\mathcal{W}_h\setminus\mathcal{V}_h$. Note that $\mathcal{W}_h$ was also defined explicitly. $\square$

Let us reiterate what was proved:

(Corollary 3.2) For any function $h:\mathscr{P}(A)\to A$ a subset $\mathcal{D}_h\subseteq A$ was explicitly defined in such a way that there exists some $\mathcal{C}_h\subseteq A$ (without an explicit definition) such that $\mathcal{C}_h\neq\mathcal{D}_h$ and $h(\mathcal{C}_h)=h(\mathcal{D}_h)\in\mathcal{D}_h\setminus\mathcal{C}_h$.

(Theorem 3.3) For any function $h:\mathscr{P}(A)\to A$ two subset $\mathcal{V}_h\subseteq A$ and $\mathcal{W}_h\subseteq A$ were explicitly defined in such a way that $\mathcal{V}_h\subsetneqq\mathcal{W}_h$ and $h(\mathcal{V}_h)=h(\mathcal{W}_h)\in\mathcal{W}_h\setminus\mathcal{V}_h$.

4. Yablo's paradox

There was a general belief that all the paradoxes stem from a kind of circularity (or involve some self–reference, or use a diagonal argument). In contrary to this belief, Stephen Yablo in 1985 designed a paradox that seemingly avoided self-reference; see [16, 17]. Let us have a brief review of Yablo's Paradox. Consider the sequence of sentences $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$:

$$\mathcal{Y}_n \text{ is true } \iff \forall k>n\ (\mathcal{Y}_k \text{ is untrue}).$$

The paradox follows from the following deductions. For each $n\in\mathbb{N}$,

$$\begin{aligned}
\mathcal{Y}_n \text{ is true } &\Longrightarrow \forall k>n\ (\mathcal{Y}_k \text{ is untrue})\\
&\Longrightarrow (\mathcal{Y}_{n+1} \text{ is untrue}) \text{ and } \forall k>n+1\ (\mathcal{Y}_k \text{ is untrue})\\
&\Longrightarrow (\mathcal{Y}_{n+1} \text{ is untrue}) \text{ and } (\mathcal{Y}_{n+1} \text{ is true}),\\
&\Longrightarrow \text{Contradiction!}
\end{aligned}$$

Thus $\mathcal{Y}_n$ is not true. So,

$$\forall k\ (\mathcal{Y}_k \text{ is untrue}),$$

and in particular

$$\forall k > 0\ (\mathcal{Y}_k \text{ is untrue}),$$

and so $\mathcal{Y}_0$ must be true (and untrue at the same time); a contradiction!

4.1. **Propositional Linear Temporal Logic.** The propositional linear temporal logic (LTL) is a logical formalism that can refer to time; in LTL one can encode formulae about the future, e.g., a condition will eventually be true, a condition will be true until another fact becomes true, etc. LTL was first proposed for the formal verification of computer programs in 1977 by Amir Pnueli [13]. For a modern introduction to LTL and its syntax and semantics see e.g. [10]. Two modality operators in LTL that we will use are the "next" modality $\bigcirc$ and the "always" modality $\Box$. The formula $\bigcirc\varphi$ holds (in the current moment) when φ is true in the "next step", and the formula $\Box\varphi$ is true (in the current moment) when φ is true "now and forever" ("always in the future"). In the other words, $\Box$ is the reflexive and transitive closure of $\bigcirc$. It can be seen that the formula $\bigcirc\neg\varphi \longleftrightarrow \neg\bigcirc\varphi$ is always true (is a law of LTL, see T1 on page 27 of [10]), since φ is untrue in the next step if and only if it is not the case that "φ is true in the next step". Also the formula $\bigcirc\Box\psi$ is true when ψ is true from the next step onward, that is ψ holds in the next step, and the step after that, and the step after that, etc. The same holds for $\Box\bigcirc\psi$; indeed the (equivalence) formula $\bigcirc\Box\psi \longleftrightarrow \Box\bigcirc\psi$ is a law of LTL (T12 on page 28 of [10]). Whence, we have the equivalences $\bigcirc\Box\neg\varphi \leftrightarrow \Box\bigcirc\neg\varphi \leftrightarrow \Box\neg\bigcirc\varphi$ in LTL.

The intended models (semantics) of LTL are systems $\langle \mathbb{N}, \Vdash \rangle$ where $\Vdash \subseteq \mathbb{N}\times\mathtt{Atoms}$ is an arbitrary relation which can be extended to all formulas as follows:

$$\begin{array}{lll} n \Vdash \varphi \wedge \psi & \text{if and only if} & n \Vdash \varphi \text{ and } n \Vdash \psi, \\ n \Vdash \neg\varphi & \text{if and only if} & n \nVdash \varphi, \\ n \Vdash \bigcirc\varphi & \text{if and only if} & (n+1) \Vdash \varphi, \\ n \Vdash \Box\varphi & \text{if and only if} & m \Vdash \varphi \text{ for every } m \geq n. \end{array}$$

A formula τ is called valid (an LTL tautology) when for any model $\langle \mathbb{N}, \Vdash \rangle$ and any $n \in \mathbb{N}$ we have $n \Vdash \tau$. We can readily check the validity of $\bigcirc\neg\varphi \longleftrightarrow \neg\bigcirc\varphi$ as follows:

$$n \Vdash \bigcirc\neg\varphi \iff (n+1) \Vdash \neg\varphi \iff (n+1) \nVdash \varphi \iff n \nVdash \bigcirc\varphi \iff n \Vdash \neg\bigcirc\varphi.$$

Also the validity of $\bigcirc\Box\psi \longleftrightarrow \Box\bigcirc\psi$ can be readily checked:

$$\begin{array}{lll} n \Vdash \bigcirc\Box\psi & \iff & (n+1) \Vdash \Box\psi \\ & \iff & \forall k \geq n+1 \big(k \Vdash \psi\big) \\ & \iff & \forall k \geq n \big[(k+1) \Vdash \psi\big] \\ & \iff & \forall k \geq n \big(k \Vdash \bigcirc\psi\big) \\ & \iff & n \Vdash \Box\bigcirc\psi. \end{array}$$

Now we show the non–existence of a formula $\mathscr{Y}$ that satisfies the equivalence

$$\mathscr{Y} \longleftrightarrow \bigcirc\Box\neg\mathscr{Y} \quad \big(\longleftrightarrow \Box\bigcirc\neg\mathscr{Y} \longleftrightarrow \Box\neg\bigcirc\mathscr{Y}\big);$$

in other words $\mathscr{Y}$ is a fixed–point of the operator

$$x \mapsto \bigcirc\Box\neg x \ \big(\equiv \Box\bigcirc\neg x \equiv \Box\neg\bigcirc x\big).$$

Following [18] we can demonstrate this by the following diagram

$$\begin{tikzcd} \mathsf{LTL} \times \mathsf{LTL} \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\ \mathsf{LTL} \arrow[u, "\triangle_{\mathsf{LTL}}"] \arrow[r, "g"'] & \mathbf{2} \end{tikzcd}$$

where LTL is the set of sentences in the language of LTL and f is defined by

$$f(X, Y) = \begin{cases} 1 & \text{if } \ X \not\equiv \bigcirc\Box\neg Y, \\ 0 & \text{if } \ X \equiv \bigcirc\Box\neg Y. \end{cases}$$

Here, g is the characteristic function of all the Yablo–like sentences, the sentences which claim that all they say in the future (from the next step onward) is untrue.

Theorem 4.1. *For any φ, the formula $\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ is not provable in LTL.*

Proof. If LTL proves $\psi \leftrightarrow \bigcirc\Box\neg\psi$ for some (propositional) formula ψ, then for a model $\langle\mathbb{N}, \Vdash\rangle$:

(i) If $m \Vdash \psi$ for some m, then we should have $m \Vdash \bigcirc\Box\neg\psi$ so $(m+1) \Vdash \Box\neg\psi$, hence $(m+i) \Vdash \neg\psi$ for all $i \geq 1$. In particular, we have $(m+1) \Vdash \neg\psi$ and $(m+j) \Vdash \neg\psi$ for all $j \geq 2$ which implies $(m+2) \Vdash \Box\neg\psi$ or equivalently $(m+1) \Vdash \bigcirc\Box\neg\psi$ so $(m+1) \Vdash \psi$, a contradiction!

(ii) So for all k we have $k \Vdash \neg\psi$ or equivalently $k \Vdash \neg\bigcirc\Box\neg\psi$ or $k \Vdash \bigcirc\neg\Box\neg\psi$, thus $(k+1) \Vdash \neg\Box\neg\psi$; hence $(k+n) \Vdash \psi$ for some $n \geq 1$, contradicting (i)!

So, LTL$\not\vdash$ $\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ for all formulas φ. $\Box$

The above proof is very similar to Yablo's argument (in his paradox) presented at the beginning of this section, and it explains that Yablo's paradox has turned into a genuine mathematico–logical theorem (in LTL) for the first time in Theorem 4.1[1], and in the following stronger theorem which can be proved along almost the same line of reasoning.

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[1]Note that Yablo's paradox has already been used to give new proofs of some old theorems e.g. in [5] (Gödel's theorem) or in [12] (Rosser's Theorem); but no new theorem had come out of it.

Theorem 4.2. *For any φ, the formula $\neg\Box\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ is provable in LTL.*

Proof. By [10, Theorem 2.4.10], it suffices to prove that $\neg\Box\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ is valid in any model of LTL, or, equivalently, the formula $\Box\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ is not satisfiable in any node of any model of LTL. For a moment assume that there is a model $\langle\mathbb{N},\Vdash\rangle$ and a node $n \in \mathbb{N}$ for which $n \Vdash \Box\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$. Then we have $\forall i \geq n: \ i \Vdash (\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ which implies that

$$\forall i \geq n : i \Vdash \varphi \iff i \Vdash \bigcirc\Box\neg\varphi \iff i+1 \Vdash \Box\neg\varphi.$$

(i) If for some $j \geq n$ we have $j \Vdash \varphi$, then $j+1 \Vdash \Box\neg\varphi$ and so $j+\ell \nVdash \varphi$ for all $\ell \geq 1$. In particular, $j+1 \nVdash \varphi$ whence $j+2 \nVdash \Box\neg\varphi$ which is in contradiction with $j+1 \Vdash \Box\neg\varphi$.

(ii) If for all $j \geq n$ we have $j \nVdash \varphi$, then $n \nVdash \varphi$ so $n+1 \nVdash \Box\neg\varphi$; hence there must exist some $i > n$ with $i \Vdash \varphi$ which contradicts (i) above.

So, LTL$\vdash \neg\Box\big(\varphi \leftrightarrow \bigcirc\Box\neg\varphi\big)$ for all formulas φ. $\Box$

4.2. **Priest's Inclosure Schema.** In 1997 Priest [14] showed the existence of a formula $Y(x)$ which satisfies the equivalence $Y(n) \leftrightarrow \forall k\!>\!n\ \neg\mathcal{T}(\ulcorner Y(k)\urcorner)$ for every $n \in \mathbb{N}$, where $\mathcal{T}(x)$ is a (supposedly truth) predicate; here $\ulcorner\psi\urcorner$ is the (Gödel) code of the formula ψ. Here we construct a formula $Y(x)$ which, for every $n \in \mathbb{N}$, satisfies $Y(n) \leftrightarrow \forall k > n\ \Psi(\ulcorner Y(k)\urcorner)$ for some Π_1 formula Ψ, by using the Recursion Theorem (of Kleene); for recursion–theoretic definitions and theorems see e.g. [6][2]. Let $\mathbf{T}$ denote Kleene's T Predicate, and for a fixed Π_1 formula $\Psi(x)$ let r be the recursive function defined by $r(x,y) = \mu\, z\big(z > x\ \&\ \neg\Psi(\ulcorner\neg\exists u\mathbf{T}(y,z,u)\urcorner)\big)$; note that $\neg\Psi$ is a Σ_1 formula. By the S^m_n theorem there exists a primitive recursive function s such that $\varphi_{s(y)}(x) = r(x,y)$; here φ_n denotes the unary recursive function with (Gödel) code n, so $\varphi_0, \varphi_1, \varphi_2, \cdots$ lists all the unary recursive functions. By Kleene's Recursion Theorem, there exists some (Gödel code) e such that $\varphi_e = \varphi_{s(e)}$. Therefore,

$$\varphi_e(x) = \varphi_{s(e)}(x) = r(x,e) = \mu\, z\big(z\!>\!x\ \&\ \neg\Psi(\ulcorner\neg\exists u\mathbf{T}(e,z,u)\urcorner)\big).$$

So, for any $x \in \mathbb{N}$, $\exists u\mathbf{T}(e,x,u) \Leftrightarrow \varphi_e(x)\!\downarrow\ \Leftrightarrow \exists z\big(z\!>\!x\ \&\ \neg\Psi(\ulcorner\neg\exists u\mathbf{T}(e,z,u)\urcorner)\big)$ holds, equivalently, we have

$$\neg\exists u\mathbf{T}(e,x,u) \iff \forall z\!>\!x\ \Psi(\ulcorner\neg\exists u\mathbf{T}(e,z,u)\urcorner).$$

Therefore, if $\mathcal{Y}(v) = \neg\exists z\mathbf{T}(e,\underline{v},z)$, then for any $n \in \mathbb{N}$, we have

$$\mathcal{Y}(n) \leftrightarrow \forall k\!>\!n\ \Psi(\ulcorner\mathcal{Y}(k)\urcorner).$$

Let us note that Yablo's paradox occurs when Ψ is taken to be an untruth (or non-satisfaction) predicate; in fact one might be tempted to take $\neg Sat_{\Pi,1}(x,\emptyset)$ (see [8, Theorem 1.75]) as $\Psi(x)$; but by construction $Sat_{\Pi,1}(x,\emptyset)$ is Π_1 and so $\neg Sat_{\Pi,1}(x,\emptyset)$ is Σ_1, and our proof works for $\Psi \in \Pi_1$ only (otherwise the

[2] Of course the mere existence of such a formula $Y(x)$ can be inferred directly from Gödel's Diagonal Lemma.

function r could not be recursive). Actually, the above construction shows that the predicate $Sat_{\Pi,1}(x, \emptyset)$ (in [8]) cannot be Σ_1, which is equivalent to saying that the set of (arithmetical) true Π_1 sentences cannot be recursively enumerable, and this is a consequence of Gödel's first incompleteness theorem[3].

In [14] Priest also introduced his Inclosure Schema and showed that Yablo's paradox is amenable in it (see also [4]). In the following, we show that Priest's Inclosure Schema can fit in Yanofsky's framework [18]. With some inessential modification for better reading, Priest's inclosure schema is defined to be a triple $\langle \Omega, \Theta, \delta \rangle$ where

- Ω is a set of objects;
- $\Theta \subseteq \mathscr{P}(\Omega)$ is a property of subsets of Ω such that $\Omega \in \Theta$;
- $\delta : \Theta \to \Omega$ is a function such that for each $X \in \Theta$, $\delta(X) \notin X$.

The fact any inclosure schema is contradictory can be derived from the fact that by the second item above, $\delta(\Omega)$ must be defined and belong to Ω, but at the same time by the third item $\delta(\Omega) \notin \Omega$. We show how this can be proved by the non–existence of a fixed–point for the negation function.

Theorem 4.3. *If an inclosure schema exists, then* `negation` *has a fixed–point.*

Proof. Assume $\langle \Omega, \Theta, \delta \rangle$ is a (hypothetical) inclosure schema. Put $f : \Theta \times \Theta \to \mathbf{2}$ as follows

$$f(X, Y) = \begin{cases} 1 & \text{if } \delta(X) \in Y, \\ 0 & \text{if } \delta(X) \notin Y, \end{cases}$$

and let $g : \Theta \to \mathbf{2}$ be defined as

$$\begin{tikzcd} \Theta \times \Theta \arrow[r, "f"] & \mathbf{2} \arrow[d, "\mathtt{neg}"] \\ \Theta \arrow[u, "\triangle_\Theta"] \arrow[r, "g"'] & \mathbf{2}. \end{tikzcd}$$

We show that g is representable by f at Ω. For every $X \in \Theta$ we have $f(X, \Omega) = 1$. On the other hand by the property of δ, for any $X \in \Theta$ we have $\delta(X) \notin X$, and so $f(X, X) = 0$, thus $g(X) = \mathtt{neg}(f(X, X)) = 1$. Whence $g(X) = f(X, \Omega)$ holds for all $X \in \Theta$. $\square$

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[3]This line of reasoning also shows the non–existence of a formula $\theta(x)$ (in arithmetical languages) which can satisfy $\theta(x) \leftrightarrow \forall y > x\, \neg\theta(y)$ in $\mathbb{N}$ or in a theory containing Peano's Arithmetic.

5. **Dominating functions**

Ackermann's function is a recursive (computable) function which is not primitive recursive (see e.g. [6]). The class of primitive recursive functions is the smallest class which contains the initial functions, i.e.,

- the constant zero function $\mathtt{z}(x) = 0$,
- the successor function $\mathtt{s}(x) = x + 1$ and
- the projection functions $\mathtt{p}_i^n(x_1, \ldots, x_n) = x_i$ for any $1 \leqslant i \leqslant n \in \mathbb{N}$,

and is closed under

- composition and
- primitive recursion,

i.e., for primitive recursive functions $f, f_1, \ldots, f_n$ the function $\mathtt{comp}(f; f_1, \ldots, f_n)$ defined by $(x_1, \ldots, x_m) \mapsto f(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$ is also primitive recursive, and also for primitive recursive functions g and h the (primitively recursively defined) function $\mathtt{prim.rec}(g, h)$ defined by $(x_1, \ldots, x_n, 0) \mapsto g(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n, x+1) \mapsto h\big(\mathtt{prim.rec}(g, h)(x_1, \ldots, x_n, x), x_1, \ldots, x_n, x\big)$ is also primitive recursive. The class of recursive functions contains the same initial functions and is closed under composition, primitive recursion, and also

- minimization,

i.e., for recursive function f the function $\mathtt{min}(f)$ defined by $(x_1, \ldots, x_n) \mapsto y$ where y is the least natural number that satisfies $f(x_1, \ldots, x_n, y) = 0$ is also recursive; note that then for all $z < y$ we have $f(x_1, \ldots, x_n, z) \neq 0$, and if there is no such y then $\mathtt{min}(f)$ is undefined on $x_1, \ldots, x_n$.

In fact, Ackermann's function is not only a non–primitive recursive (and a recursive) function, but it also dominates all the primitive recursive functions (see e.g. [6]). A function g is said to dominate a function f (or f is dominated by g) when for all but finitely many x's the inequality $g(x) > f(x)$ holds. Here we show a way of dominating a given enumerable list of functions by diagonalization. Before that let us note that the set of all primitive recursive functions can be (recursively) enumerated: let $\#(f)$ denote the (Gödel) code of (a defining program of) the function f and define the Gödel code of a primitive recursive defining program inductively:

- $\#(\mathtt{z}) = 1$,
- $\#(\mathtt{s}) = 2$,
- $\#(\mathtt{p}_i^n) = 2^i \cdot 3^n$,
- $\#(\mathtt{comp}(f; f_1, \ldots, f_n)) = 5^{\#(f)} \cdot 7^{\#(f_1)} \cdots \wp_{n+2}^{\#(f_n)}$,
- $\#(\mathtt{prim.rec}(g, h)) = 3^{\#(g)} \cdot 5^{\#(h)}$.

where $\wp_i$ is the $i-$th prime number (thus, $\wp_0 = 2, \wp_1 = 3, \wp_2 = 5, \wp_3 = 7, \cdots$). Let us note that while $\mathtt{comp}(\mathtt{z}; \mathtt{s}) = \mathtt{z}$ as functions but their defining programs have different codes: $\#\big(\mathtt{comp}(\mathtt{z}; \mathtt{s})\big) = 5 \cdot 7^2$ and $\#(\mathtt{z}) = 1$. Let ν_n be the primitive recursive function with code n, if n is a code of such a function; if n is not a code for a primitive recursive function (such as $n = 3$ or $n = 10$) then let ν_n be the constant zero function $\mathtt{z}$. So, $\nu_0,\ \nu_1,\ \nu_2,\ \cdots$ lists all the primitive recursive functions. We show the existence of a unary function that dominates all the functions ν_i's in the above list. The following theorem seems to have been first formulated by Paul du Bois-Reymond ([1, 2]) in his study of the eventual dominance in the space $\mathbb{N}^{\mathbb{N}}$.

Theorem 5.1. *For a list of functions $f_1, f_2, f_3, \cdots : \mathbb{N} \to \mathbb{N}$, there exists a unary function $\mathbb{N} \to \mathbb{N}$ that dominates them all.*

Proof. Define the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as $f(n, m) = \max_{(i \leqslant n)} f_i(m)$ and let g be defined by the following diagram where $\mathtt{s}$ is the successor function:

$$\begin{tikzcd}
\mathbb{N} \times \mathbb{N} \arrow[r, "f"] & \mathbb{N} \arrow[d, "\mathtt{s}"] \\
\mathbb{N} \arrow[u, "\triangle_{\mathbb{N}}"] \arrow[r, "g"'] & \mathbb{N}.
\end{tikzcd}$$

In fact, the function $g : \mathbb{N} \to \mathbb{N}$ is defined as $g(x) = \max_{(i \leqslant x)} f_i(x) + 1$. Since the successor function does not have any fixed–point, the function g is not equal to any of f_i's. Moreover, g dominates all the f_i's, since for any $m \in \mathbb{N}$ and any $x \geqslant m$ by the definition of g we have $g(x) > \max_{(i \leqslant x)} f_i(x) \geqslant f_m(x)$. $\square$

For dominating the primitive recursive functions (some of which are not unary) we can consider their unarized version: let $\rho_0, \rho_1, \rho_2, \cdots$ be the list of unary functions $\mathbb{N} \to \mathbb{N}$ defined as $\rho_i(x) = \nu_i(x, \ldots, x)$. Whence $\rho_0, \rho_1, \rho_2, \cdots$ lists all the unary primitive recursive functions, and the construction of Theorem 5.1 produces a unary function which dominates all the unary primitive recursive functions. Let us note that the function g obtained in the proof of Theorem 5.1 is computable (intuitively) and so recursive (by Church's Thesis); one can show directly that the above function g is recursive (without appealing to Church's Thesis) by some detailed work through Recursion Theory (cf. e.g. [6]).

6. Conclusions

There are many interesting questions and suggestions for further research at the end of [18] which motivated the research presented in this paper; most of the questions have not been answered, yet. The proposed schema, i.e., the diagram of the proof of Theorem 1.1,

$$\begin{tikzcd} B \times B \arrow[r, "f"] & D \arrow[d, "\alpha"] \\ B \arrow[u, "\triangle_B"] \arrow[r, "g"'] & D \end{tikzcd}$$

can be used as a criterion for testing whether an argument is diagonal or not. What makes that argument (of the non–existence of a fixed–point for $\alpha : D \to D$) *diagonal* is the diagonal function $\triangle_B : B \to B\times B$. In most of our arguments we assumed $D = \mathbf{2} = \{0, 1\}$ and $\alpha = \mathtt{neg}$ by which the proof was constructed by diagonalizing out of the function $f : B \times B \to D$. Only in Theorem 5.1 we considered $D = \mathbb{N}$ and $\alpha = \mathtt{s}$ (the successor function) which was used for generating a dominating function. We could have used the diagonalizing out argument by setting $D = \mathbf{2} = \{0, 1\}$ and $\alpha = \mathtt{neg}$ for the function $\tilde{f} : \mathbb{N}\times\mathbb{N} \to \mathbf{2}$, defined by

$$\tilde{f}(n, m) = \begin{cases} 0 & \text{if } f_n(m) = 0, \\ 1 & \text{if } f_n(m) \neq 0. \end{cases}$$

Then the constructed function $\tilde{g} : \mathbb{N} \to \mathbf{2}$ by $\tilde{g}(n) = \mathtt{neg}(\tilde{f}(n, n))$ differs from all the functions f_i's (because $\tilde{g}(i) \neq f_i(i)$ for all i). So, this way one could construct a non–primitive recursive (but recursive) function, though this function does not dominate all the primitive recursive functions.

For other exciting questions and examples of theorems or paradoxes, which seem to be self–referential, we refer the reader to the last section of [18]. It will be nice to see some of those proposals or other more phenomena fit in the above universal diagonal schema.

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