THE MAXIMUM NUMBER OF EDGES IN A STRONGLY MULTIPLICATIVE GRAPH

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Abstract. We derive a formula for the maximum number of edges in a strongly multiplicative graph as a function of its order.

Recently, L.W. Beineke and S.M. Hegde [3] introduced the notion of a strongly multiplicative graph.

Definition (Beineke, Hegde [3]). A graph with $n$ vertices is said to be strongly multiplicative if its vertices can be labeled $1, 2, \ldots, n$, so that the values on the edges, obtained as the product of labels of the end vertices, all are distinct.

An interesting problem is to obtain a formula for the maximum number of edges $\lambda(n)$ for a strongly multiplicative graph of order $n$. In [3], Beineke and Hegde gave an upper bound for $\lambda(n)$. In [2], C. Adiga et al. obtained a sharper upper bound for $\lambda(n)$. Then in [1], Adiga et al. established a formula for $\lambda(n)$ in terms of the divisor function. We quote their result now.

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**Theorem** (Adiga et al. [1]).

\[
\lambda(n) = \sum_{k=1}^{n(n-1)} g(k),
\]

where

\[
g(k) = \min\{1, f(k)\},
\]

\[
f(k) = \begin{cases} 
\frac{d(k)}{\sqrt{k}} & \text{if } 1 \leq k \leq n, \\
\frac{d(k)}{\sqrt{k}} - d_n(k) & \text{if } n < k \leq n(n-1)
\end{cases}
\]

and where \(d(k)\) denotes the number of distinct divisors of \(k\), \(\lfloor x \rfloor\) denotes the largest integer less than or equal to \(x\), and \(d_n(k)\) denotes the number of divisors of \(k\) greater than \(n\).

In this note we derive the following formula for \(\lambda(n)\).

**Theorem.**

\[
\lambda(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n} \sum_{k=1}^{m-1} \frac{\theta(m, k)}{[\sqrt{mk} - 1] - k + 1},
\]

where

\[
\theta(m, k) = \sum_{s=k+1}^{\lfloor\sqrt{mk}\rfloor} \left\lfloor \frac{mk}{s} \right\rfloor.
\]

**Proof.** Let \(\delta(n) = \lambda(n) - \lambda(n-1)\). Then

\[
\lambda(n) = \sum_{m=2}^{n} \delta(m). \tag{2.1}
\]

Thus, in view of (2.1) it is enough to obtain a formula for \(\delta(m)\). Consider the array of products
The maximum number of edges in a strongly multiplicative graph

\[\begin{array}{cccc}
1.2 & 1.3 & 1.4 & \ldots \\
2.3 & 2.4 & \ldots & 1.(n-1) \\
3.4 & \ldots & 3.(n-1) & 1.n \\
\end{array}\]

\[\begin{array}{cccc}
2.3 & 2.4 & \ldots & 2.(n-1) \\
3.4 & \ldots & 3.(n-1) & 2.n \\
(n-2).(n-1) & (n-2).n & (n-1).n. \\
\end{array}\]

Let \(A_k\) denote the set of all elements of the \(k^{th}\) row. We count the number of terms in the last column which appear in other rows. If \(k.n\) is divisible by \(s\) \((k \leq s - 1)\), then there exists an \(m < n\) such that \(k.n = s.m\), and hence \(k.n\) repeats in the \(s^{th}\) row, i.e. \(k.n \in A_s\). Observe that \(k.n\) may belong to \(A_s\), where \(s\) is the largest integer such that \(k + 1 < s < \sqrt{kn}\). Thus the number of repetition of \(k.n\) in these rows is

\[\theta(n, k) = \sum_{s=k+1}^{[\sqrt{kn}-1]} \left\lfloor \frac{nk}{s} \right\rfloor\]

By the definition of \(\theta(n, k)\), it is clear that

\[0 \leq \theta(n, k) \leq \left\lfloor \sqrt{kn} \right\rfloor - k + 1.\]

Thus

\[\left\lfloor -\frac{\theta(n, k)}{\left\lfloor \sqrt{kn} - 1 \right\rfloor - k + 1} \right\rfloor = \begin{cases} 
-1 & \text{if } kn \in \bigcup_{s=k+1}^{\sqrt{kn}-1} A_s, \\
0 & \text{otherwise}. 
\end{cases}\]

This implies that

\[\delta(n) = (n-1) + \sum_{k=1}^{n-1} \left\lfloor -\frac{\theta(n, k)}{\left\lfloor \sqrt{kn} - 1 \right\rfloor - k + 1} \right\rfloor. \quad (2.2)\]

On using (2.2) in (2.1), we complete the proof. \(\square\)
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