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**Title:**

**Double derivations of  $n$ -Lie algebras**

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## DOUBLE DERIVATIONS OF $n$ -LIE ALGEBRAS

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**ABSTRACT.** After introducing double derivations of  $n$ -Lie algebra  $L$  we describe the relationship between the algebra  $\mathcal{D}(L)$  of double derivations and the usual derivation Lie algebra  $\mathcal{D}er(L)$ . In particular, we prove that the inner derivation algebra  $ad(L)$  is an ideal of the double derivation algebra  $\mathcal{D}(L)$ ; we also show that if  $L$  is a perfect  $n$ -Lie algebra with certain constraints on the base field then the centralizer of  $ad(L)$  in  $\mathcal{D}(L)$  is trivial and  $\mathcal{D}(L)$  is centerless. In addition, we obtain that for every perfect  $n$ -Lie algebra  $L$  with zero center, the triple derivations of the derivation algebra  $\mathcal{D}er(L)$  are exactly the derivations of  $\mathcal{D}er(L)$ , and the triple derivations of the inner derivation algebra  $ad(L)$  are precisely the derivations of  $ad(L)$ .

**Keywords:**  $n$ -Lie algebra, double derivation, derivation, inner derivation.

**MSC(2010):** Primary: 7B05; Secondary: 17B30.

### 1. Introduction

The concept of derivations appear in different mathematical fields with many different forms. In algebra systems, derivations are linear mappings satisfying the Leibniz relation. There are several kinds of derivations in the theory of Lie algebras, such as derivations, inner derivations, generalized derivations, triple derivations, and quasi-derivations of Lie algebras [1–5]. They are important in studying both the structures and representations of Lie algebras.

The derivations of  $n$ -Lie algebras are investigated recently with useful results. For instance, the Lie algebra of the automorphism group of an  $n$ -Lie algebra is equal to its derivation algebra, and the derivation algebra of a simple  $n$ -Lie algebra is a semisimple Lie algebra [6–9]. Every module over an  $n$ -Lie algebra  $L$  is also a module over the inner derivation algebra of  $L$ . Finite dimensional irreducible modules of simple  $n$ -Lie algebras over the field of complex numbers are classified using generators and relations [11].

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In this paper, we introduce double derivations of  $n$ -Lie algebras; these derivations are similar to the triple derivations of Lie algebras to some extent [12]. However, the structure of the algebra of double derivations of  $n$ -Lie algebras are different from that of triple derivations of Lie algebras. The double derivations provide a useful tool to study  $n$ -Lie algebras by making use of linear mappings.

In the following section, we begin with some basic notions to be used in the paper, then show that the double derivation algebra of an  $n$ -Lie algebra  $L$  is a Lie subalgebra of  $gl(L)$  and that for a perfect  $n$ -Lie algebra its inner derivation algebra is an ideal of the double derivation algebra. In the final section we mainly study the double derivations of perfect  $n$ -Lie algebras  $L$  with zero center. In addition, we investigate triple derivations of the derivation algebra  $Der(L)$  and of the inner derivation algebra  $ad(L)$ . Our main results about double derivations are stated in Theorems 2.2 and 3.1, and the main results about triple derivations of  $Der(L)$  and  $ad(L)$  are described in Theorems 3.4 and 3.5.

In the following we suppose that  $n$ -Lie algebras are over a field  $\mathcal{F}$  of characteristic ( $char\mathcal{F}$ ) not equal to 2 and not a factor of  $n - 1$ .

### 2. Double derivations of $n$ -Lie algebras

To introduce the concept of double derivation of an  $n$ -Lie algebra we need some notation and basic facts. An  $n$ -Lie algebra  $L$  is a vector space over a field  $\mathcal{F}$  endowed with an  $n$ -ary skew-symmetric multiplication satisfying the  $n$ -Jacobi identity: for all  $x_1, \dots, x_n, y_2, \dots, y_n \in L$ ,

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n].$$

A derivation of an  $n$ -Lie algebra  $L$  is a linear map  $D : L \rightarrow L$ , such that for any elements  $x_1, \dots, x_n$  of  $L$ ,

$$D([x_1, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, D(x_i), \dots, x_n].$$

The set of all derivations of  $L$  is a subalgebra of the general Lie algebra  $gl(L)$ , which is denoted by  $Der(L)$ . The map  $ad(x_1, \dots, x_{n-1}) : L \rightarrow L$ , defined by

$$ad(x_1, \dots, x_{n-1})(x_n) = [x_1, \dots, x_n] \quad \text{for } x_1, \dots, x_n \in L$$

is called a left multiplication, and  $ad(x_1, \dots, x_{n-1})$  is a derivation. The set of all finite linear combinations of left multiplications is an ideal of  $Der(L)$  and is denoted by  $ad(L)$ .

For subspaces  $B_1, \dots, B_n$  of an  $n$ -Lie algebra  $L$ , the symbol  $[B_1, \dots, B_n]$  denotes the subspace spanned by vectors  $[x_1, \dots, x_n]$  for any  $x_i \in B_i$  where  $1 \leq i \leq n$ . The algebra  $[L, \dots, L]$  is called the derived algebra of  $L$ , and is

denoted by  $L^1$ . If  $L^1 = L$ , then  $L$  is called a perfect  $n$ -Lie algebra. The center of an  $n$ -Lie algebra  $L$  is denoted by

$$Z(L) = \{x \in L \mid [x, L, \dots, L] = 0\}.$$

For a subset  $S$  of  $L$ , the centralizer of  $S$  in  $L$  is defined by

$$Z_L(S) = \{x \in L \mid [x, S, L, \dots, L] = 0\}.$$

We are in a position to give the definition of double derivations of  $n$ -Lie algebras.

**Definition 2.1.** Let  $L$  be an  $n$ -Lie algebra over a field  $\mathcal{F}$  with  $n \geq 3$ . A linear map  $D : L \rightarrow L$  is called a *double derivation* of  $L$  if it satisfies the following identity. For any  $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in L$ ,

$$(2.1) \quad \begin{aligned} D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) &= \sum_{i=1}^{n-1} [x_1, \dots, Dx_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\ &\quad + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, Dy_j, \dots, y_n]]. \end{aligned}$$

The vector space spanned by double derivations on  $L$  is denoted by  $\mathcal{D}(L)$ .

From the above definition, the derivations of an  $n$ -Lie algebra  $L$  are double derivations, henceforth we have  $ad(L) \subseteq \mathcal{D}(L)$  and  $Der(L) \subseteq \mathcal{D}(L)$ .

**Theorem 2.2.** For any  $n$ -Lie algebra  $L$ , the algebra  $\mathcal{D}(L)$  is a Lie subalgebra of  $gl(L)$ .

*Proof.* For all  $D_1, D_2 \in \mathcal{D}(L)$  and  $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in L$ , we have

$$\begin{aligned} &D_1 D_2([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\ &= D_1\left(\sum_{i=1}^{n-1} [x_1, \dots, D_2 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]]\right. \\ &\quad \left.+ \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D_2 y_j, \dots, y_n]]\right) \\ &= \sum_{i=1}^{n-1} [x_1, \dots, D_1 D_2 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \end{aligned}$$

$$\begin{aligned}
& + \sum_{s=1}^n \sum_{i=1}^{n-1} [x_1, \dots, D_2 x_i, \dots, x_{n-1}, [y_1, \dots, D_1 y_s, \dots, y_n]] \\
& + \sum_{1 \leq s \neq i \leq n-1} [x_1, \dots, D_1 x_s, \dots, D_2 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
& + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D_1 D_2 y_j, \dots, y_n]] \\
& + \sum_{t=1}^{n-1} \sum_{j=1}^n [x_1, \dots, D_1 x_t, \dots, x_{n-1}, [y_1, \dots, D_2 y_j, \dots, y_n]] \\
& + \sum_{1 \leq t \neq j \leq n} [x_1, \dots, x_{n-1}, [y_1, \dots, D_1 y_t, \dots, D_2 y_j, \dots, y_n]].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& D_2 D_1([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\
& = D_2 \left( \sum_{i=1}^{n-1} [x_1, \dots, D_1 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \right. \\
& \quad \left. + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D_1 y_j, \dots, y_n]] \right) \\
& = \sum_{i=1}^{n-1} [x_1, \dots, D_2 D_1 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
& \quad + \sum_{s=1}^n \sum_{i=1}^{n-1} [x_1, \dots, D_1 x_i, \dots, x_{n-1}, [y_1, \dots, D_2 y_s, \dots, y_n]] \\
& \quad + \sum_{1 \leq s \neq i \leq n-1} [x_1, \dots, D_2 x_s, \dots, D_1 x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
& \quad + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D_2 D_1 y_j, \dots, y_n]] \\
& \quad + \sum_{t=1}^{n-1} \sum_{j=1}^n [x_1, \dots, D_2 x_t, \dots, x_{n-1}, [y_1, \dots, D_1 y_j, \dots, y_n]] \\
& \quad + \sum_{1 \leq t \neq j \leq n} [x_1, \dots, x_{n-1}, [y_1, \dots, D_2 y_t, \dots, D_1 y_j, \dots, y_n]].
\end{aligned}$$

Hence, it implies

$$\begin{aligned}
 & (D_1D_2 - D_2D_1)([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\
 &= \sum_{i=1}^{n-1} [x_1, \dots, (D_1D_2 - D_2D_1)x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
 &\quad + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, (D_1D_2 - D_2D_1)y_j, \dots, y_n]] \\
 &= \sum_{i=1}^{n-1} [x_1, \dots, [D_1, D_2]x_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
 &\quad + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, [D_1, D_2]y_j, \dots, y_n]] \\
 &= [D_1, D_2]([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]).
 \end{aligned}$$

The desired result follows. □

**Theorem 2.3.** *If  $L$  is a perfect  $n$ -Lie algebra, then  $ad(L)$  is an ideal of the Lie algebra  $\mathcal{D}(L)$ .*

*Proof.* Let  $D \in \mathcal{D}(L)$  and  $x_1, \dots, x_{n-1} \in L$ . Since  $L$  is perfect, there exists a finite index set  $I$  and  $x_{i_j} \in L, i \in I, 1 \leq j \leq n$  such that

$$x_1 = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}].$$

For an arbitrary  $z \in L$ , we have

$$\begin{aligned}
 & [D, ad(x_1, \dots, x_{n-1})](z) \\
 &= Dad(x_1, \dots, x_{n-1})(z) - ad(x_1, \dots, x_{n-1})D(z) \\
 &= \sum_{i \in I} Dad([x_{i_1}, \dots, x_{i_n}], x_2, \dots, x_{n-1})(z) - ad(x_1, \dots, x_{n-1})D(z) \\
 &= \sum_{i \in I} D([[x_{i_1}, \dots, x_{i_n}], x_2, \dots, x_{n-1}, z]) - [x_1, \dots, x_{n-1}, D(z)] \\
 &= \sum_{i \in I} \sum_{s=1}^n [[x_{i_1}, \dots, Dx_{i_s}, \dots, x_{i_n}], x_2, \dots, x_{n-1}, z] \\
 &\quad + \sum_{i \in I} \sum_{t=2}^{n-1} [[x_{i_1}, \dots, x_{i_n}], x_2, \dots, Dx_t, \dots, x_{n-1}, z] \\
 &\quad + \sum_{i \in I} [[x_{i_1}, \dots, x_{i_n}], x_2, \dots, x_{n-1}, D(z)] - [x_1, \dots, x_{n-1}, D(z)]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i \in I} \sum_{s=1}^n ad([x_{i_1}, \dots, Dx_{i_s}, \dots, x_{i_n}], x_2, \dots, x_{n-1})(z) \\
 &\quad + \sum_{i \in I} \sum_{t=2}^{n-1} ad([x_{i_1}, \dots, x_{i_n}], x_2, \dots, Dx_t, \dots, x_{n-1})(z).
 \end{aligned}$$

It follows that  $[D, ad(x_1, \dots, x_{n-1})]$  is an inner derivation, and hence  $ad(L)$  is an ideal of  $\mathcal{D}(L)$ .  $\square$

*Remark 2.4.* If  $L$  is not a perfect  $n$ -Lie algebra, then  $ad(L)$  may not be an ideal of  $\mathcal{D}(L)$ . For example, let  $L$  be a 4-dimensional 3-Lie algebra with the multiplication

$$[x_1, x_3, x_4] = x_2, \quad [x_2, x_3, x_4] = x_1,$$

where  $\{x_1, x_2, x_3, x_4\}$  is a basis of  $L$ . For a linear map  $D : L \rightarrow L$ , suppose  $D(x_i) = \sum_{j=1}^4 a_{ij}x_j$ ,  $1 \leq i \leq 4$ . Then the matrix form of  $D$  is  $\sum_{i,j=1}^4 a_{ij}E_{ij}$ , where  $E_{ij}$  is the matrix unit of size  $4 \times 4$  and  $1 \leq i, j \leq 4$ . By a direct computation,

$$\{E_{12} + E_{21}, E_{13}, E_{14}, E_{23}, E_{24}\}$$

is a basis of the inner derivation algebra  $ad(L)$ , and

$$\{E_{11}, E_{22}, E_{12}, E_{21}, E_{13}, E_{14}, E_{23}, E_{24}, E_{33} - E_{44}, E_{34}\}$$

is a basis of  $\mathcal{D}(L)$ . Since  $[E_{12} + E_{21}, E_{12}] = E_{22} - E_{11}$  is not contained in  $ad(L)$ ,  $[ad(L), \mathcal{D}(L)] \not\subseteq ad(L)$ . Therefore,  $ad(L)$  is not an ideal of  $\mathcal{D}(L)$ .

### 3. Double derivations of perfect $n$ -Lie algebras

Let  $L$  be an arbitrary  $n$ -Lie algebra with zero center. For all  $D \in \mathcal{D}er(L)$ , we define a linear map  $\delta_D : L \rightarrow L$  by

$$(3.1) \quad \delta_D(x) = \begin{cases} \sum_{i \in I} \sum_{s=1}^n [x_{i_1}, \dots, Dx_{i_s}, \dots, x_{i_n}] & \text{for all } x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] \in L^1, \\ D(x) & \text{for all } x \in L - L^1. \end{cases}$$

We show that  $\delta_D$  is well defined. In fact, for  $x \in L$ , if

$$x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] = \sum_{j \in J} [y_{j_1}, \dots, y_{j_n}],$$

put  $\alpha = \sum_{i \in I} \sum_{s=1}^n [x_{i_1}, \dots, Dx_{i_s}, \dots, x_{i_n}]$  and  $\beta = \sum_{j \in J} \sum_{t=1}^n [y_{j_1}, \dots, Dy_{j_t}, \dots, y_{j_n}]$ .

Since  $D$  is a double derivation of  $L$ , for all  $z_1, \dots, z_{n-1} \in L$ , we have

$$\begin{aligned}
[z_1, \dots, z_{n-1}, \alpha] &= [z_1, \dots, z_{n-1}, \sum_{i \in I} \sum_{s=1}^n [x_{i_1}, \dots, Dx_{i_s}, \dots, x_{i_n}]] \\
&= D([z_1, \dots, z_{n-1}, x]) - \sum_{p=1}^{n-1} [z_1, \dots, Dz_p, \dots, z_{n-1}, x] \\
&= D([z_1, \dots, z_{n-1}, \sum_{j \in J} [y_{j_1}, \dots, y_{j_n}]]) \\
&\quad - \sum_{p=1}^{n-1} [z_1, \dots, Dz_p, \dots, z_{n-1}, \sum_{j \in J} [y_{j_1}, \dots, y_{j_n}]] \\
&= [z_1, \dots, z_{n-1}, \sum_{j \in J} \sum_{t=1}^n [y_{j_1}, \dots, Dy_{j_t}, \dots, y_{j_n}]] \\
&= [z_1, \dots, z_{n-1}, \beta].
\end{aligned}$$

Thus, we get  $[z_1, \dots, z_{n-1}, \alpha - \beta] = 0$ . This means  $\alpha - \beta \in Z(L)$ , and consequently  $Z(L) = 0$ , so we obtain that  $\alpha = \beta$ . Therefore,  $\delta_D$  is well-defined.

So we obtain a linear map  $\delta : \mathcal{D}(L) \rightarrow \text{End}(L)$  defined by

$$\delta(D) = \delta_D, \quad \forall D \in \mathcal{D}(L).$$

**Theorem 3.1.** *Let  $L$  be a perfect  $n$ -Lie algebra with  $Z(L) = 0$ .*

(a) *If  $D$  is a double derivation of  $L$ , then so is  $\delta_D$ , and for all  $x_1, \dots, x_n \in L$ ,*

$$(3.2) \quad (D - \delta_D)[x_1, \dots, x_n] = -[x_1, \dots, (D - \delta_D)x_i, \dots, x_n], \quad 1 \leq i \leq n;$$

$$(3.3) \quad \delta_{D - \delta_D} = -n(D - \delta_D).$$

(b) *If  $D$  is a double derivation of  $L$ , then  $\delta_D$  is a derivation of  $L$  if and only if  $D$  is a derivation of  $L$ . In particular,  $\delta_D = D$  if  $D$  is a derivation of  $L$ .*

(c) *For all  $x_1, \dots, x_{n-1} \in L$  and any double derivation  $D$  of  $L$ ,*

$$\begin{aligned}
(3.4) \quad [D, ad(x_1, \dots, x_{n-1})] &= ad(\delta_D(x_1), x_2, \dots, x_{n-1}) + \sum_{t=2}^{n-1} ad(x_1, x_2, \dots, Dx_t, \dots, x_{n-1}).
\end{aligned}$$

(d) *The map  $\delta$  is a Lie algebra homomorphism, that is,  $\delta_{[D_1, D_2]} = [\delta_{D_1}, \delta_{D_2}]$ .*



*Proof.* We first prove (a). For  $x_j = \sum_{i \in I_j} [x_{i_{j_1}}, \dots, x_{i_{j_n}}]$  and  $y_s = \sum_{k \in K_s} [y_{k_{s_1}}, \dots, y_{k_{s_n}}]$  in  $L$ , where  $1 \leq j \leq n-1$  and  $1 \leq s \leq n$ , from (3.1) we obtain that

$$\begin{aligned}
& \sum_{j=1}^{n-1} [x_1, \dots, \delta_D x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] + \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, \delta_D y_s, \dots, y_n]] \\
&= \sum_{j=1}^{n-1} [x_1, \dots, \delta_D \sum_{i \in I_j} [x_{i_{j_1}}, \dots, x_{i_{j_n}}], \dots, x_{n-1}, [y_1, \dots, y_n]] \\
&\quad + \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, \delta_D \sum_{k \in K_s} [y_{k_{s_1}}, \dots, y_{k_{s_n}}], \dots, y_n]] \\
&= \sum_{j=1}^{n-1} [x_1, \dots, \sum_{l=1}^n \sum_{i \in I_j} [x_{i_{j_1}}, \dots, D x_{i_{j_l}}, \dots, x_{i_{j_n}}], \dots, x_{n-1}, [y_1, \dots, y_n]] \\
&\quad + \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, \sum_{l=1}^n \sum_{k \in K_s} [y_{k_{s_1}}, \dots, D y_{k_{s_l}}, \dots, y_{k_{s_n}}], \dots, y_n]] \\
&= (n-1)D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
&\quad - (n-2) \sum_{j=1}^{n-1} [x_1, \dots, D x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] \\
&\quad - (n-1)[x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]] \\
&\quad + n[x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]] \\
&\quad - (n-1) \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D y_s, \dots, y_n]] \\
&= (n-1)D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] + [x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]] \\
&\quad - (n-2) \left( \sum_{j=1}^{n-1} [x_1, \dots, D x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] \right. \\
&\quad \left. + \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D y_s, \dots, y_n]] \right) \\
&\quad - \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D y_s, \dots, y_n]] \\
&= D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] - \sum_{s=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, D y_s, \dots, y_n]] \\
&\quad + [x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]] \\
&= \sum_{j=1}^{n-1} [x_1, \dots, D x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] + [x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]] \\
&= \delta_D [x_1, \dots, x_{n-1}, [y_1, \dots, y_n]].
\end{aligned}$$

It follows that  $\delta_D$  is a double derivation of  $L$ .

Now, for any double derivation  $D$  of  $L$  and  $x_i, y_j, z_i \in L$  with  $1 \leq i \leq n-1, 1 \leq j \leq n$ , by (2.1) and (3.1), we have

$$\begin{aligned}
& [z_1, \dots, z_{n-1}, D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]])] \\
&= \sum_{i=1}^{n-1} [z_1, \dots, z_{n-1}, [x_1, \dots, Dx_i, \dots, x_{n-1}, [y_1, \dots, y_n]]] \\
&\quad + \sum_{j=1}^n [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, [y_1, \dots, Dy_j, \dots, y_n]]] \\
&= [z_1, \dots, z_{n-1}, \delta_D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]])] \\
&\quad - [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]]] \\
&\quad + \sum_{j=1}^n [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, [y_1, \dots, Dy_j, \dots, y_n]]] \\
&= [z_1, \dots, z_{n-1}, \delta_D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]])] \\
&\quad - [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, D[y_1, \dots, y_n]]] \\
&\quad + [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, \delta_D([y_1, \dots, y_n])]].
\end{aligned}$$

Thus,

$$\begin{aligned}
& [z_1, \dots, z_{n-1}, (D - \delta_D)([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]])] \\
&= - [z_1, \dots, z_{n-1}, [x_1, \dots, x_{n-1}, (D - \delta_D)[y_1, \dots, y_n]]].
\end{aligned}$$

It follows from  $Z(L) = 0$  and  $[L, \dots, L] = L$  that

$$(D - \delta_D)[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = -[x_1, \dots, x_{n-1}, (D - \delta_D)[y_1, \dots, y_n]],$$

and

$$(D - \delta_D)[x_1, \dots, x_n] = -[x_1, \dots, (D - \delta_D)x_i, \dots, x_n], \quad i = 1, \dots, n.$$

This proves the first part of (a).

Next, we prove the second part of (a). The above discussion shows that if  $D$  is a double derivation of  $L$ , then  $\delta_D$  is also a double derivation of  $L$ . So  $D - \delta_D$  is a double derivation of  $L$ . From (3.2) we find that

$$\delta_{D-\delta_D}[x_1, \dots, x_n] = \sum_{i=1}^n [x_1, \dots, (D - \delta_D)x_i, \dots, x_n] = -n(D - \delta_D)[x_1, \dots, x_n],$$

which is the desired result.

To prove (b), let  $D$  be a derivation of  $L$ . Then by (3.1),  $\delta_D[x_1, \dots, x_n] = D[x_1, \dots, x_n]$  for all  $x_1, \dots, x_n \in L$ . Since  $L$  is a perfect  $n$ -Lie algebra, we get  $\delta_D = D$ .

Now suppose that  $D$  is a double derivation of  $L$  which is not contained in  $\mathcal{D}er(L)$ . The above discussion shows that  $D - \delta_D \neq 0$ . Then there exist  $x_1, \dots, x_n \in L$ , such that  $(D - \delta_D)[x_1, \dots, x_n] \neq 0$ . Suppose that  $x_j = \sum_{i \in I_j} [x_{i_{j_1}}, \dots, x_{i_{j_n}}]$  for  $j = 1, \dots, n$ . We have

$$\begin{aligned} & \sum_{j=1}^n [x_1, \dots, \delta_D x_j, \dots, x_n] \\ &= \sum_{j=1}^n [x_1, \dots, \sum_{i \in I_j} \delta_D [x_{i_{j_1}}, \dots, x_{i_{j_n}}], \dots, x_n] \\ &= \sum_{j=1}^n [x_1, \dots, \sum_{i \in I_j} \sum_{l=1}^n [x_{i_{j_1}}, \dots, D x_{i_{j_l}}, \dots, x_{i_{j_n}}], \dots, x_n] \\ &= nD[x_1, \dots, x_n] - (n-1) \sum_{j=1}^n [x_1, \dots, D x_j, \dots, x_n] \\ &= n(D - \delta_D)[x_1, \dots, x_n] + \delta_D[x_1, \dots, x_n] \\ &\neq \delta_D[x_1, \dots, x_n]. \end{aligned}$$

It follows that  $\delta_D \notin \mathcal{D}er(L)$ . The proof of (b) is complete.

The assertion (c) follows from Theorem 2.3 and (3.1) directly.

Lastly, we prove (d). By (3.1), for any double derivations  $D_1$  and  $D_2$  of  $L$  and  $x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] \in L$ , we have

$$\begin{aligned} [\delta_{D_1}, \delta_{D_2}](x) &= (\delta_{D_1} \delta_{D_2} - \delta_{D_2} \delta_{D_1})(x) \\ &= \delta_{D_1} \left( \sum_{i \in I} \delta_{D_2} [x_{i_1}, \dots, x_{i_n}] \right) - \delta_{D_2} \left( \sum_{i \in I} \delta_{D_1} [x_{i_1}, \dots, x_{i_n}] \right) \\ &= \delta_{D_1} \left( \sum_{i \in I} \sum_{j=1}^n [x_{i_1}, \dots, D_2 x_{i_j}, \dots, x_{i_n}] \right) - \delta_{D_2} \left( \sum_{i \in I} \sum_{j=1}^n [x_{i_1}, \dots, D_1 x_{i_j}, \dots, x_{i_n}] \right) \\ &= \sum_{i \in I} \sum_{1 \leq j \leq n} ([x_{i_1}, \dots, D_1 D_2 x_{i_j}, \dots, x_{i_n}] - [x_{i_1}, \dots, D_2 D_1 x_{i_j}, \dots, x_{i_n}]) \\ &\quad + \sum_{i \in I} \sum_{1 \leq s \neq j \leq n} ([x_{i_1}, \dots, D_1 x_{i_s}, \dots, D_2 x_{i_j}, \dots, x_{i_n}] \\ &\quad - [x_{i_1}, \dots, D_2 x_{i_s}, \dots, D_1 x_{i_j}, \dots, x_{i_n}]) \\ &= \sum_{i \in I} \sum_{j=1}^n ([x_{i_1}, \dots, [D_1, D_2] x_{i_j}, \dots, x_{i_n}] = \delta_{[D_1, D_2]}(x), \end{aligned}$$

and the result follows. The proof is now complete. □

**Theorem 3.2.** *Let  $L$  be a perfect  $n$ -Lie algebra over a field  $\mathcal{F}$ . Then the centralizer of  $ad(L)$  in  $\mathcal{D}(L)$  is trivial. In particular, the center of the Lie algebra  $\mathcal{D}(L)$  is zero.*

*Proof.* Let  $D$  be an element of the centralizer of  $ad(L)$  in  $\mathcal{D}(L)$ . Then,  $[D, ad(x_1, \dots, x_{n-1})] = 0$  for all  $x_1, \dots, x_{n-1} \in L$ . Hence, for arbitrary  $x_1, \dots, x_{n-1}, z \in L$ , we have

$$\begin{aligned} & [D, ad(x_1, \dots, x_{n-1})](z) \\ &= Dad(x_1, \dots, x_{n-1})(z) - ad(x_1, \dots, x_{n-1})D(z) \\ &= D([x_1, \dots, x_{n-1}, z]) - [x_1, \dots, x_{n-1}, Dz] = 0. \\ & \quad D([x_1, \dots, x_{n-1}, z]) \\ &= [x_1, \dots, x_{n-1}, Dz] = [x_1, \dots, Dx_i, \dots, x_{n-1}, z], i = 1, \dots, n-1. \end{aligned}$$

Let  $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in L$ . We always have

$$\begin{aligned} & D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \\ &= \sum_{i=1}^{n-1} [x_1, \dots, Dx_i, \dots, x_{n-1}, [y_1, \dots, y_n]] \\ & \quad + \sum_{j=1}^n [x_1, \dots, x_{n-1}, [y_1, \dots, Dy_j, \dots, y_n]] \\ &= (2n-1)D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]. \end{aligned}$$

We thus obtain that  $(2n-2)D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) = 0$ , that is,

$$D[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = 0.$$

From  $L = [L, \dots, L]$  it follows that  $D = 0$ .  $\square$

A linear map  $D$  of a Lie algebra  $\mathfrak{g}$  is called a triple derivation if it satisfies

$$D([x, [y, z]]) = [Dx, [y, z]] + [x, [Dy, z]] + [x, [y, Dz]]$$

for all  $x, y, z \in \mathfrak{g}$ . The linear Lie algebra generated by the triple derivations of  $\mathfrak{g}$  is the *triple derivation algebra* of  $\mathfrak{g}$  (c.f. [12]).

We are particularly interested in the triple derivations of the Lie algebra  $\mathfrak{g} = \mathcal{D}(L)$  consisting of double derivations as well as in the triple derivations of the Lie algebra  $\mathfrak{g} = ad(L)$  consisting of inner derivations of a perfect  $n$ -Lie algebra  $L$ .

Now, in the following we suppose that  $L$  is a perfect  $n$ -Lie algebra with zero center and  $ad(L)$  is a perfect Lie algebra. Then we have the following results.

**Lemma 3.3.** *Let  $L$  be a perfect  $n$ -Lie algebra with zero center. Then every triple derivation  $D$  of  $\mathcal{D}(L)$  keeps  $ad(L)$  invariant. Furthermore, if  $D(ad(L)) = 0$ , then  $D = 0$ .*

*Proof.* First, we show that  $D(ad(L)) \subseteq ad(L)$ . Since  $ad(L)$  is a perfect Lie algebra, for all  $x_1, \dots, x_{n-1} \in L$ , there exist  $x_{1_i}, \dots, x_{n-1_i}, y_{1_i}, \dots, y_{n-1_i} \in L$ , and  $y_{1_{i_j}}, \dots, y_{n-1_{i_j}}, z_{1_{i_j}}, \dots, z_{n-1_{i_j}} \in L$  such that

$$\begin{aligned} ad(x_1, \dots, x_{n-1}) &= \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})] \\ &= \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), \sum_{j \in J} [ad(y_{1_{ij}}, \dots, y_{n-1_{ij}}), ad(z_{1_{ij}}, \dots, z_{n-1_{ij}})]] \end{aligned}$$

for some finite index sets  $I$  and  $J$ .

Hence, for every triple derivation  $D$  of  $\mathcal{D}(L)$ , we have that

$$\begin{aligned} D(ad(x_1, \dots, x_{n-1})) &= \sum_{i \in I} \sum_{j \in J} D([ad(x_{1_i}, \dots, x_{n-1_i}), [ad(y_{1_{ij}}, \dots, y_{n-1_{ij}}), ad(z_{1_{ij}}, \dots, z_{n-1_{ij}})]]]) \\ &= \sum_{i \in I} \sum_{j \in J} ([Dad(x_{1_i}, \dots, x_{n-1_i}), [ad(y_{1_{ij}}, \dots, y_{n-1_{ij}}), ad(z_{1_{ij}}, \dots, z_{n-1_{ij}})]]] \\ &\quad + [ad(x_{1_i}, \dots, x_{n-1_i}), [Dad(y_{1_{ij}}, \dots, y_{n-1_{ij}}), ad(z_{1_{ij}}, \dots, z_{n-1_{ij}})]]] \\ &\quad + [ad(x_{1_i}, \dots, x_{n-1_i}), [ad(y_{1_{ij}}, \dots, y_{n-1_{ij}}), Dad(z_{1_{ij}}, \dots, z_{n-1_{ij}})]]]). \end{aligned}$$

Thanks to Theorem 2.3, we obtain  $D(ad(L)) \subseteq ad(L)$ .

Now let  $D$  be a triple derivation of  $\mathcal{D}(L)$  which satisfies  $D(ad(L)) = 0$ . Then for each double derivation  $d$  of  $L$ , we have

$$\begin{aligned} [D(d), ad(x_1, \dots, x_{n-1})] &= \sum_{i \in I} [D(d), [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})]] \\ &= \sum_{i \in I} (D([d, [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})]]) \\ &\quad - [d, [D(ad(x_{1_i}, \dots, x_{n-1_i})), ad(y_{1_i}, \dots, y_{n-1_i})]] \\ &\quad - [d, [ad(x_{1_i}, \dots, x_{n-1_i}), D(ad(y_{1_i}, \dots, y_{n-1_i}))]]). \end{aligned}$$

Since  $[d, ad(L)] \subseteq ad(L)$ ,  $D([d, ad(x_1, \dots, x_{n-1})]) = 0$ . We obtain that  $[D(d), ad(x_1, \dots, x_{n-1})] = 0$ . Therefore,  $D(d)$  is in the centralizer of  $ad(L)$  in  $D(L)$ . Using Theorem 3.2 implies  $D(d) = 0$ , and consequently  $D = 0$ .  $\square$

**Theorem 3.4.** *Let  $L$  be a perfect  $n$ -Lie algebra with zero center. Then the triple derivation algebra of  $ad(L)$  coincides with the derivation algebra of  $ad(L)$ .*

*Proof.* Since  $ad(L)$  is a perfect Lie algebra with zero center, by the Theorem in [12], we obtain that the triple derivation algebra of  $ad(L)$  is the same as the derivation algebra of  $ad(L)$ .  $\square$   $\square$

**Theorem 3.5.** *Let  $L$  be a perfect  $n$ -Lie algebra with zero center. Then the triple derivation algebra of  $Der(L)$  is equal to the derivation algebra of  $Der(L)$ .*

*Proof.* If  $D$  is a triple derivation of  $Der(L)$ , then  $D$  is a triple derivation of  $ad(L)$  by Lemma 3.3, and so  $D$  is a derivation of  $ad(L)$  by Theorem 3.4. Note

that for any  $x_1, \dots, x_{n-1} \in L$ , there exist  $x_{1_i}, \dots, x_{n-1_i}, y_{1_i}, \dots, y_{n-1_i} \in L$  such that

$$ad(x_1, \dots, x_{n-1}) = \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})]$$

for some finite index set  $I$ . Then for all  $d_1, d_2 \in \mathcal{D}er(L)$ , we get

$$\begin{aligned} & D([[d_1, d_2], ad(x_1, \dots, x_{n-1})]) \\ &= D([[d_1, d_2], \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})]]) \\ &= [D([d_1, d_2]), \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), ad(y_{1_i}, \dots, y_{n-1_i})]] \\ &\quad + [[d_1, d_2], \sum_{i \in I} [D(ad(x_{1_i}, \dots, x_{n-1_i})), ad(y_{1_i}, \dots, y_{n-1_i})]] \\ &\quad + [[d_1, d_2], \sum_{i \in I} [ad(x_{1_i}, \dots, x_{n-1_i}), D(ad(y_{1_i}, \dots, y_{n-1_i}))]] \\ &= [D([d_1, d_2]), ad(x_1, \dots, x_{n-1})] + [[d_1, d_2], D(ad(x_1, \dots, x_{n-1}))], \end{aligned}$$

which leads to,  $[D([d_1, d_2]) - [D(d_1), d_2] - [d_1, D(d_2)], ad(x_1, \dots, x_{n-1})] = 0$ . From Theorem 3.2 it follows that  $D([d_1, d_2]) = [D(d_1), d_2] + [d_1, D(d_2)]$ , which completes the proof.  $\square$

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