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ORDER-TYPE EXISTENCE THEOREM FOR SECOND ORDER NONLOCAL PROBLEMS AT RESONANCE

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Abstract. This paper gives an abstract order-type existence theorem for second order nonlocal boundary value problems at resonance and obtain existence criteria for at least two positive solutions, where f is a continuous function. Our results generalize or extend related results in the literature and give a positive answer to the question raised in the literature. An example is given to illustrate the new results.
Keywords: Order-type existence theorem, nonlocal problems, at resonance.
MSC(2010): Primary: 34B15; Secondary: 34C25.

1. Introduction

In the past few years, there have been many works related to the existence of positive solutions for second-order nonlocal boundary value problems at resonance for example in the papers [1, 7–12, 16, 19]. Recently, Bai and Fang [1] established the existence of positive solutions of the second order differential equation

$$\begin{cases} (p(t)x'(t))' = f(t, x(t), x'(t)), & t \in (0, 1), \\ x'(0) = 0, \quad x(1) = x(\eta), \end{cases}$$

by using a fixed point index theorem for semi-linear A-proper maps due to Cremins [4]. Applying Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima [14], Infante and Zima [9] obtained the existence of positive solution for problem

$$\begin{cases} -x''(t) = f(t, x(t)), & t \in (0, 1), \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \end{cases} \tag{1.1}$$

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with $\sum\limits_{i=1}^{m-2} \alpha_i = 1$. A similar approach was used in [2, 7, 11, 12, 16–20].

In a recent paper [7], Franco, Infante and Zima studied the existence of positive solutions of the following second order nonlocal boundary value problems (BVP for short)

$$\begin{cases} -x''(t) = f(t, x(t)), \quad t \in (0,1), \\ x'(0) = 0, \quad a_1 x(1) + a_2 x'(1) = \alpha[x], \end{cases} \tag{1.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $a_1 > 0$, $a_2 \geq 0$ and $\alpha[x]$ is a linear functional on $C[0,1]$ given by

$$\alpha[x] = \int_0^1 x(s) dA(s),$$

involving a Riemann-Stieltjes integral, in particular A has bounded variation and dA is a positive measure. If $a_1 = 1$, $a_2 = 0$ and $\alpha[x] = \sum\limits_{i=1}^{m-2} \alpha_i x(\eta_i), m > 2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, then BVP (1.2) is the direct extension of (1.1).

Note that the BVP (1.2) does have a unique solution when $f \equiv 0$. If instead of $\alpha[u]$ we write 1, then we denote by $\gamma(t)$ such a solution. It can be verified that $\gamma(t) \equiv \frac{1}{a_1}$. Throughout the paper we assume that $\alpha[\gamma] = 1$. We note that $\alpha[\gamma] = 1$ means that the nonlocal boundary value problem (1.2) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} -x''(t) = 0, \quad t \in (0,1), \\ x'(0) = 0, \quad a_1 x(1) + a_2 x'(1) = \alpha[x], \end{cases}$$

has $x(t) \equiv c, t \in [0,1], c \in \mathbb{R}$, as a nontrivial solution.

Let us denote by $k(t,s)$ the Green's function of the BVP (1.2) with $\alpha[u] \equiv 0$ and $f \equiv 0$. We have

$$k(t,s) = \begin{cases} 1 - t + \frac{a_2}{a_1}, & 0 \leq s \leq t \leq 1, \\ 1 - s + \frac{a_2}{a_1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

For simplicity of notation, we set

$$K_A(s) = \int_0^1 k(t,s) dA(t), \ s \in [0,1],$$

$$K(s) = \int_0^1 k(t,s) dt, \ s \in [0,1],$$

and

$$G(t,s) = \frac{M}{a_1} K_A(s) + k(t,s) - K(s), \ t, s \in [0,1],$$

where M is a positive constant. Observe that $K_A(s) = \int_0^1 k(t,s) dA(t) \geq 0$ for $s \in [0,1]$ and K_A is not identically zero on $[0,1]$.

In 2011, Franco, Infante and Zima [7] showed the following excellent result.

Theorem 1.1 ([7, Theorem 2.3])**.** *Assume that there exist constants $\beta > 0$, $\kappa > 0$ and $R > 0$ such that:*

(C_1) $\kappa M \max\{K_A(s) : s \in [0,1]\} \leq a_1$,
(C_2) $G(t,s) \geq 0$ *for* $t, s \in [0,1]$,
(C_3) $1 - \kappa G(t,s) \geq 0$ *for* $t, s \in [0,1]$,
(C_4) $f(t,x) > -\kappa x$, *for all* $(t,x) \in [0,1] \times [0,R]$,
(C_5) $f(t,R) < 0$ *for all* $t \in [0,1]$,
(C_6) *there exist* $r \in (0,R), t_0 \in [0,1], a > 0, \beta \in (0,1)$ *and continuous functions* $g : [0,1] \to [0,\infty)$, $h : (0,r] \to [0,\infty)$ *such that* $f(t,x) \geq g(t)h(x)$ *for all* $t \in [0,1]$ *and* $x \in (0,r]$, *and* $h(x)/x^a$ *is non-increasing on* $(0,r]$ *with*

$$\frac{h(r)}{r}\int_0^1 G(t_0,s)g(s)ds \geq \frac{1-\beta}{\beta^a}.$$

Then the BVP (1.2) *has a positive solution on* $[0,1]$.

Quoting the sentence in [7, Remark 2.4],

"..., that the conditions (C_5) *and* (C_6) *cannot be used to prove the existence of multiple positive solutions, due to an incompatibility of such requirements when nesting several* Ω_i*'s."*

Seeing such a fact, we can not but ask "**Whether or not we can obtain a conclusion concerning the multiplicity of positive solutions of (1.2)?**". Inspired by the above-mentioned result, we attempt to establish the existence results of multiple positive solutions for boundary value problem (1.2), which is a generalization of Theorem 1.1 and give a positive answer to the question stated above.

This paper is organized as follows. The preliminary definitions and an order-type existence theorem are in Section 2. Section 3 is devoted to the proof of main results, followed by an example, in Section 4, to demonstrate the validity of our main results.

2. Some background definitions and preliminaries

For the convenience of the readers, we present here the necessary definitions and an order-type existence theorem based on the partial order method combined with the properties of the fixed point index. It is worth pointing out that the partial order method can be applied to the fixed point results. For example, see [5,6] and the references therein.

Definition 2.1. Let X be a Banach space. A nonempty convex closed set $K \subset X$ is said to be a cone provided that

(i) $ax \in K$ for all $x \in K$ and all $a \geq 0$, and
(ii) $x, -x \in K$ implies $x = \theta$, where θ denotes the zero element.

Note that every cone $K \subset X$ induces an ordering in X given by

$$x \leq y \Leftrightarrow y - x \in K.$$

Definition 2.2. Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$ where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y.

Definition 2.3. A map $T : D \subset X \to Y$ is called approximation-proper (abbreviated A-proper) at a point $y \in Y$ with respect to Γ, if $T_n \equiv Q_n T|_{D \cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \to y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x \in D$, and $Tx = y$. T is said to be A-proper on a set Ω if it is A-proper at all points of Ω.

We will assume that $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero, that is, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. As a consequence of this property X and Y may be expressed as direct sums; $X = X_0 \bigoplus X_1$, $Y = Y_0 \bigoplus Y_1$ with continuous linear projections $P : X \to \operatorname{Ker} L = X_0$ and $Q : Y \to Y_0$. The restriction of L to $\operatorname{dom} L \cap X_1$, denoted L_1, is a bijection onto $\operatorname{Im} L = Y_1$ with continuous inverse $L_1^{-1} : Y_1 \to \operatorname{dom} L \cap X_1$. Since X_0 and Y_0 have the same finite dimension, there exists a continuous bijection $J : Y_0 \to X_0$. It is known (see [13]) that the coincidence equation

$$Lx = Nx$$

is equivalent to

$$x = (P + JQN)x + (L_1^{-1}(I - Q)N)x.$$

In order to prove the existence and multiplicity of positive solutions of (1.2) we will apply the following abstract result due to Chu and Wang.

Lemma 2.4 (Chu and Wang [3])**.** *If $L : \operatorname{dom} L \to Y$ is Fredholm of index zero, and let $L - \lambda N$ be A-proper for $\lambda \in [0, 1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I - Q)N$ maps K to K. Suppose Ω_1 and Ω_2 are two bounded open sets in X such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $\Omega_2 \cap K \cap \operatorname{dom} L \neq \emptyset$. If one of the two conditions*

- (C$_1$) *$(P + JQN)x + L_1^{-1}(I - Q)Nx \not\geq x$, for all $x \in \partial\Omega_1 \cap K$ and $(P + JQN)x + L_1^{-1}(I - Q)Nx \not\leq x$, for all $x \in \partial\Omega_2 \cap K$ and*
- (C$_2$) *$(P + JQN)x + L_1^{-1}(I - Q)Nx \not\leq x$, for all $x \in \partial\Omega_1 \cap K$ and $(P + JQN)x + L_1^{-1}(I - Q)Nx \not\geq x$, for all $x \in \partial\Omega_2 \cap K$*

is satisfied, then there exists $x \in (\overline{\Omega}_2 \backslash \Omega_1) \cap K$ such that $Lx = Nx$.

3. Main results

The goal of this section is to apply Lemma 2.4 to discuss the existence and multiplicity of positive solutions for the BVP (1.2). In order to abbreviate our discussion, throughout this paper, we suppose that the following assumptions hold.

(H_1) There are positive solutions μ_1, μ_2 such that $\mu_1 \leq G(t,s) \leq \mu_2$ for $t, s \in [0,1]$.

(H_2) There is κ such that $f(t,x) \geq -\kappa x,\ for\ all\ t \in [0,1],\ x \geq 0$, where κ satisfies

$$\kappa \leq \frac{1}{\mu_1 + \mu_2}. \tag{3.1}$$

In applications below, we take $X = Y = C[0,1]$ with the supremum norm $\|\cdot\|$ and define

$$K = \Big\{x \in X : x(t) \geq 0,\ x(t) \geq \sigma\|x\|, t \in [0,1]\Big\},$$

where $\sigma = \frac{\mu_1}{\mu_2}$. Thus $0 < \sigma \leq 1$. One may readily verify that K is a cone in X.

We define

$$\begin{aligned}
&\mathrm{dom}L = X,\\
&L : \mathrm{dom}L \to Y, \quad (Lx)(t) = x(t) - \gamma(t)\alpha[x],\\
&N : X \to Y, \quad (Nx)(t) = \int_0^1 k(t,s)f(s,x(s))ds.
\end{aligned}$$

Note that it can be checked (see [7]) that BVP (1.2) can be written as

$$Lx = Nx, \qquad x \in K.$$

It is easy to check that

$$\begin{aligned}
\mathrm{Ker}L &= \{x \in X : x(t) - \gamma(t)\alpha[x] = 0, t \in [0,1]\}\\
&= \{x \in X : x(t) \equiv c,\ t \in [0,1], c \in \mathbb{R}\},\\
\mathrm{Im}L &= \{y \in Y : \alpha[y] = 0\},\\
\dim\ \mathrm{Ker}L &= \mathrm{codim}\ \mathrm{Im}L = 1,
\end{aligned}$$

so that L is a Fredholm operator of index zero.

Next, define the projections $P : X \to X$ by

$$Px(t) = \int_0^1 x(s)ds,$$

and $Q : Y \to Y$ by

$$Qy(t) = \gamma(t)\alpha[y],\ t \in [0,1].$$

Furthermore, we define the isomorphism $J : \mathrm{Im}Q \to \mathrm{Im}P$ as $Jy = My$. We are easy to verify that the inverse operator $L_1^{-1} : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ of $L|_{\mathrm{dom}L\cap\mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is $(L_1^{-1}y)(t) = y(t) - \int_0^1 y(s)ds,\ t \in [0,1]$.

Lemma 3.1. *Let* (H_1)–(H_2) *be satisfied, then* $P + JQN + L_1^{-1}(I-Q)N$ *is a positive operator, that is,*

$$(P + JQN + L_1^{-1}(I-Q)N)(K) \subset K.$$

Proof. Let $x \in K$. From conditions (H_1)–(H_2) it follows that

$$\begin{aligned}
& (P+JQN)x(t) + L_1^{-1}(I-Q)Nx(t) \\
= \;& (P+JQN)x(t) + L_1^{-1}Nx(t) - L_1^{-1}QNx(t) \\
= \;& (P+JQN)x(t) + L_1^{-1}Nx(t) - \gamma(t)\alpha[Nx] + \int_0^1 \gamma(s)\alpha[Nx]ds \\
= \;& (P+JQN)x(t) + L_1^{-1}Nx(t) \\
= \;& \int_0^1 x(s)ds + M\gamma(t)\int_0^1 \Big[\int_0^1 k(s,\tau)f(\tau,x(\tau))d\tau\Big]dA(s) + \\
& \int_0^1 k(t,s)f(s,x(s))ds - \int_0^1 \Big[\int_0^1 k(s,\tau)f(\tau,x(\tau))d\tau\Big]ds \\
= \;& \int_0^1 x(s)ds + M\gamma(t)\int_0^1 K_A(s)f(s,x(s))ds + \\
& \int_0^1 k(t,s)f(s,x(s))ds - \int_0^1 K(s)f(s,x(s))ds \\
= \;& \int_0^1 x(s)ds + \int_0^1 G(t,s)f(s,x(s))ds \\
\geq \;& \int_0^1 (1-\kappa G(t,s))x(s)ds \geq \int_0^1 (1-\kappa\mu_2)x(s)ds \geq 0.
\end{aligned}$$

Now we are ready to prove

$$(P+JQN)x(t) + L_1^{-1}(I-Q)Nx(t) \geq \sigma\|(P+JQN)x + L_1^{-1}(I-Q)Nx\|.$$

Using conditions (H_1)–(H_2), we can obtain that

$$\begin{aligned}
& \|(P+JQN)x + L_1^{-1}(I-Q)Nx\| \\
= \;& \max_{t\in[0,1]} \Big[\int_0^1 x(s)ds + \int_0^1 G(t,s)f(s,x(s))ds\Big] \\
= \;& \max_{t\in[0,1]} \Big[\int_0^1 (1-\kappa G(t,s))x(s)ds + \int_0^1 G(t,s)(f(s,x(s)) + \kappa x(s))ds\Big] \\
\leq \;& (1-\kappa\mu_1)\int_0^1 x(s)ds + \mu_2\int_0^1 (f(s,x(s)) + \kappa x(s))ds.
\end{aligned}$$

From the last inequality, we have from conditions (H_1)–(H_2)

$$\begin{aligned}
&\min_{t\in[0,1]}\Big[(P+JQN)x(t)+L_1^{-1}(I-Q)Nx(t)\Big]\\
=\ &\min_{t\in[0,1]}\Big[\int_0^1 x(s)ds+\int_0^1 G(t,s)f(s,x(s))ds\Big]\\
=\ &\min_{t\in[0,1]}\Big[\int_0^1 (1-\kappa G(t,s))x(s)ds+\int_0^1 G(t,s)(f(s,x(s))+\kappa x(s))ds\Big]\\
\geq\ &(1-\kappa\mu_2)\int_0^1 x(s)ds+\mu_1\int_0^1 (f(s,x(s))+\kappa x(s))ds\\
=\ &\frac{\mu_1}{\mu_2}\Big[\mu_2(1-\kappa\mu_2)/\mu_1\int_0^1 x(s)ds+\mu_2\int_0^1 (f(s,x(s))+\kappa x(s))ds\Big]\\
\geq\ &\sigma\Big[(1-\kappa\mu_1)\int_0^1 x(s)ds+\mu_2\int_0^1 (f(s,x(s))+\kappa x(s))ds\Big]\\
=\ &\sigma\|(P+JQN)x+L_1^{-1}(I-Q)Nx\|.
\end{aligned}$$

Therefore, $(P+JQN)x+L_1^{-1}(I-Q)Nx\in K$. □

Now we can state and prove our main results.

Theorem 3.2. *Under assumptions* (H_1)–(H_2), *if moreover there exist two positive numbers* a, b, *such that* $a<b$ *and*

(H_3) $f(t,b)<0,\ t\in[0,1]$,
(H_4) $f(t,x)>0,\ (t,x)\in[0,1]\times[0,a]$,

then the BVP (1.2) *has at least one positive solution* $x^*\in K$ *with* $a\leq\|x^*\|\leq b$.

Proof. First, we note that L, as so defined, is Fredholm of index zero, N is compact by Arzelà-Ascoli theorem and thus $L-\lambda N$ is A-proper for $\lambda\in[0,1]$ by [15, Lemma 2(a)].

To apply Lemma 2.4, we should define two open bounded subsets Ω_1,Ω_2 of X so that Lemma 2.4 holds.

Let

$$\Omega_1=\{x\in X:\|x\|<a\},\ \Omega_2=\{x\in X:\|x\|<b\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2.$$

Next, we show that

$$(P+JQN)x+L_1^{-1}(I-Q)Nx\ngeq x,\ \mathit{for\ any}\ x\in\partial\Omega_2\cap K. \tag{3.2}$$

To wit, suppose that there exists $x_1\in\partial\Omega_2\cap K$ such that

$$(P+JQN)x_1+L_1^{-1}(I-Q)Nx_1\geq x_1.$$

Using [4, Proposition 1], we know that the linear operator $L+J^{-1}P$ is inversely positive. Thus it is enough to prove that $Nx_1 \geq Lx_1$. Following the same reasonings of equality (2.4) in [7] one proves that

$$-x_1''(t) \leq f(t, x_1(t)), \quad t \in [0,1].$$

Let $t_1 \in [0,1]$ be such that $x_1(t_1) = b$. It follows from the boundary conditions and the resonance assumption that $t_1 \in (0,1)$. This gives

$$0 \leq -x_1''(t_1) \leq f(t_1, x_1(t_1)) = f(t_1, b),$$

which contradicts (H_3). Therefore (3.2) holds.

Finally, we claim that

$$(3.3) \qquad (P+JQN)x + L_1^{-1}(I-Q)Nx \not\leq x, \ \textit{for any } x \in \partial\Omega_1 \cap K.$$

In fact, if not, there exists $x_2 \in \partial\Omega_1 \cap K$, such that

$$(P+JQN)x_2 + L_1^{-1}(I-Q)Nx_2 \leq x_2.$$

For any $x_2 \in \partial\Omega_1 \cap K$, we have $\|x_2\| = a$. By condition (H_4) we have

$$\begin{aligned} x_2(t) &\geq (P+JQN)x_2(t) + L_1^{-1}(I-Q)Nx_2(t) \\ &= \int_0^1 x_2(s)ds + \int_0^1 G(t,s)f(s,x_2(s))ds \\ &> \int_0^1 x_2(s)ds, \end{aligned}$$

which is a contradiction. As a result (3.3) is verified.

Thus all conditions of Lemma 2.4 are satisfied and there exists $x^* \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $Lx^* = Nx^*$ and the assertion follows. Thus $x^* \in K$ and $a \leq \|x^*\| \leq b$. □

Next we will establish the existence of two positive solutions to BVP (1.2) by using Lemma 2.4.

Theorem 3.3. *Under assumptions* (H_1)–(H_2)*, if moreover there exist constants a, b and c, such that $0 < a < \sigma b < c$ and*

(H_3)$'$ $f(t,a) < 0, \ t \in [0,1]$,
(H_4)$'$ $f(t,x) > 0, \ (t,x) \in [0,1] \times [\sigma b, b]$,
(H_5)$'$ $f(t,c) < 0, \ t \in [0,1]$,

then the BVP (1.2) has at least two positive solutions $x^*, x^{**} \in K$ with

$$a \leq \|x^*\| < b < \|x^{**}\| \leq c.$$

Proof. We construct the sets $\Omega_a = \{x \in X : \|x\| < a\}$, $\Omega_b = \{x \in X : \|x\| < b\}$ and $\Omega_c = \{x \in X : \|x\| < c\}$ in order to apply Lemma 2.4.

Let $x \in K$ with $\|x\| = b$, we have $\sigma b = \sigma\|x\| \leq x(t) \leq b$. It follows from ($H_4$)$'$ that, by using the similar method to get (3.3), we can get

$$(3.4) \qquad (P+JQN)x + L_1^{-1}(I-Q)Nx \not\leq x, \ \textit{for any } x \in \partial\Omega_b \cap K.$$

It follows from $(\mathrm{H}_3)'$ and $(\mathrm{H}_5)'$ that, by using the same method to get (3.2), we can get

$$(P+JQN)x+L_1^{-1}(I-Q)Nx \ngeq x, \ \textit{for any } x \in \partial\Omega_a \cap K, \tag{3.5}$$

$$(P+JQN)x+L_1^{-1}(I-Q)Nx \ngeq x, \ \textit{for any } x \in \partial\Omega_c \cap K. \tag{3.6}$$

Now, (3.4), (3.5) and the first part of Lemma 2.4 guarantee that there exists $x^* \in K \cap (\overline{\Omega}_b \backslash \Omega_a)$ such that $Lx^* = Nx^*$. By (3.4), (3.6) and the second part of Lemma 2.4 guarantee that there exists $x^{**} \in K \cap (\overline{\Omega}_c \backslash \Omega_b)$ such that $Lx^{**} = Nx^{**}$. Thus $a \le \|x^*\| < b < \|x^{**}\| \le c$. □

Theorem 3.4. *Under assumptions* (H_1)–(H_2)*, if moreover there exist constants* a, b *and* c*, such that* $0 < a < b < \sigma c < c$ *and*

$(\mathrm{H}_3)''$ $f(t,x) > 0,\ (t,x) \in [0,1] \times [0,a]$,
$(\mathrm{H}_4)''$ $f(t,b) < 0,\ t \in [0,1]$,
$(\mathrm{H}_5)''$ $f(t,x) > 0,\ (t,x) \in [0,1] \times [\sigma c, c]$,

then the BVP (1.2) *has at least two positive solutions* $x^*, x^{**} \in K$ *with*

$$a \le \|x^*\| < b < \|x^{**}\| \le c.$$

Proof. We omit the details because they are much similar to that in the proof of Theorem 3.3. □

Remark 3.5. It must be noticed that the condition (3.1) in (H_2) is crucial for our proof. It seems to the authors that the study of (1.2) when (3.1) does not hold should involve techniques different from those employed in this paper.

4. An example

To illustrate how our main results can be used in practice, we present here an example.

Consider the resonant problem

$$\begin{cases} -x''(t) = f(t,x), \quad 0 < t < 1, \\ x'(0) = 0, \ x(1) + x'(1) = \displaystyle\int_0^1 2tx(t)dt, \end{cases} \tag{4.1}$$

where $a_1 = a_2 = 1$, and

$$f(t,x) = \begin{cases} \frac{1}{4}(t+\frac{1}{2})(x^2-2x), & 0 \le x \le 36, \\ \frac{151}{4}(t+\frac{1}{2})(x-38)(x-\frac{6048}{151}), & x \ge 36. \end{cases}$$

In this case $\gamma(t) \equiv 1$,

$$k(t,s) = \begin{cases} 2-t, & 0 \le s \le t \le 1, \\ 2-s, & 0 \le t \le s \le 1, \end{cases}$$

$$K_A(s) = \alpha[k(\cdot,s)] = \int_0^1 2tk(t,s)dt = \frac{1}{3}(4-s^3),$$

$$K(s) = \int_0^1 k(t,s)dt = \frac{1}{2}(3 - s^2),$$

and

$$G(t,s) = \frac{M}{3}(4 - s^3) + k(t,s) - \frac{1}{2}(3 - s^2).$$

For $M = 1/2$, after direct computations, we get

$$G(t,s) = \begin{cases} \frac{7}{6} - \frac{s^3}{6} + \frac{s^2}{2} - t, & 0 \le s \le t \le 1, \\ \frac{7}{6} - \frac{s^3}{6} + \frac{s^2}{2} - s, & 0 \le t \le s \le 1, \end{cases}$$

and $\frac{1}{6} \le G(t,s) \le \frac{7}{6}$ for $t, s \in [0,1]$. So $\mu_1 = \frac{1}{6}, \mu_2 = \frac{7}{6}, \kappa = \frac{1}{2}, \sigma = \frac{1}{7}$ and we have that condition (H_1) holds.

Since $f(t,x) + \frac{1}{2}x$ is nonnegative on $[0,1] \times [0, +\infty)$, the condition ($\mathrm{H}_2$) holds.

If we take $a = 1, b = 36$ and $c = 40$, then it is easy to check the following conditions

(1) $f(t,1) = -\frac{1}{4}(t + \frac{1}{2}) < 0$, for all $t \in [0,1]$,
(2) $f(t,x) > 0$, for all $(t,x) \in [0,1] \times [3,36]$,
(3) $f(t,40) = -4(t + \frac{1}{2}) < 0$, for all $t \in [0,1]$.

Thus all the conditions of Theorem 3.3 are satisfied. The resonant problem (4.1) has at least two positive solutions x_1 and x_2 such that

$$1 \le \|x_1\| < 36 < \|x_2\| \le 40.$$

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References

[1] C. Bai and J. Fang, Existence of positive solutions for three-point boundary value problems at resonance, *J. Math. Anal. Appl.* **291** (2004), no. 2, 538–549.

[2] Y. Chen and X. Tang, Positive solutions of fractional differential equations at resonance on the half-line, *Bound. Value Probl.* **2012** (2012), Article no. 64, 13 pages.

[3] J. Chu and F. Wang, An order-type existence theorem and applications to periodic problems, *Bound. Value Probl.* **2013** (2013), Article no. 37, 12 pages.

[4] C.T. Cremins, A fixed-point index and existence theorems for semilinear equations in cones, *Nonlinear Anal.* **46** (2001), no. 6, 789–806.

[5] Y. Cui and J. Sun, Fixed point theorems for a class of nonlinear operators in Hilbert spaces with lattice structure and application, *Fixed Point Theory Appl.* **2013** (2013), Article no. 345, 9 pages.

[6] Y. Cui and X. Zhang, Fixed points for discontinuous monotone operators, *Fixed Point Theory Appl.* **2010** (2010), Article ID 926209, 11 pages.

[7] D. Franco, G. Infante and M. Zima, Second order nonlocal boundary value problems at resonance, *Math. Nachr.* **284** (2011), no. 7, 875–884.
[8] X. Han, Positive solutions for a three-point boundary value problem at resonance, *J. Math. Anal. Appl.* **336** (2007), no. 1, 556–568.
[9] G. Infante and M. Zima, Positive solutions of multi-point boundary value problems at resonance, *Nonlinear Anal.* **69** (2008), no. 8, 2458–2465.
[10] S. Liang and L. Mu, Multiplicity of positive solutions for singular three-point boundary value problems at resonance, *Nonlinear Anal.* **71** (2009), no. 7-8, 2497–2505.
[11] Y. Liu and C. Shen, On the existence of positive solution for a kind of multi-point boundary value problem at resonance, *Nonlinear Anal.* **72** (2010), no. 11, 4211–4220.
[12] Y. Liu and C. Shen, Positive solutions for second order four-point boundary value problems at resonance, *Topol. Methods Nonlinear Anal.* **38** (2011), no. 1, 1–15.
[13] J. Mawhin, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Differential Equations* **12** (1972) 610–636.
[14] D. O'Regan and M. Zima, Leggett-Williams norm-type theorems for coincidences, *Arch. Math. (Basel)* **87** (2006), no. 3, 233–244.
[15] W.V. Petryshyn, Using degree theory for densely defined A-proper maps in the solvability of semilinear equations with unbounded and noninvertible linear part, *Nonlinear Anal.* **4** (1980), no. 2, 259–281.
[16] A. Yang and W. Ge, Existence of positive solutions for self-adjoint boundary value problems with integral boundary conditions at resonance, *Electron. J. Differential Equations* **2011** (2011), no. 11, 8 pages.
[17] A. Yang and H. Wang, Positive solutions of two-point boundary value problems of nonlinear fractional differential equation at resonance, *Electron. J. Qual. Theory Differ. Equ.* **2011** (2011), no. 71, 15 pages.
[18] H. Zhang and J. Sun, Positive solutions of third-order nonlocal boundary value problems at resonance, *Bound. Value Probl.* **2012** (2012), Article ID 102, 10 pages.
[19] M. Zima, Existence of positive solutions for a second-order three-point boundary value problem at resonance, in: Dynamic Systems and Applications, Vol. 5, pp. 527–532, Dynamic, Atlanta, GA, 2008.
[20] M. Zima, Positive solutions for first-order boundary value problems at resonance, *Commun. Appl. Anal.* **13** (2009), no. 4, 671–680.

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