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A NOTE ON BLOW-UP IN PARABOLIC EQUATIONS WITH LOCAL AND LOCALIZED SOURCES

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(Communicated by Asadollah Aghajani)

Abstract. This note deals with the systems of parabolic equations with local and localized sources involving n components. We obtained the exponent regions, where $k \in \{1, 2, \cdots, n\}$ components may blow up simultaneously while the other $(n-k)$ ones still remain bounded under suitable initial data. It is proved that different initial data can lead to different blow-up phenomena even in the same exponent regions, and moreover, different blow-up mechanism leads to different blow-up rates and blow-up sets.

Keywords: Non-simultaneous blow-up, simultaneous blow-up, blow-up rate, blow-up set.
MSC(2010): Primary: 35K15; Secondary: 35K55, 35B40, 35B33.

1. Introduction

In this note, we consider the following system of n parabolic equations,

$$\begin{cases} (u_i)_t = \Delta u_i + u_i^{p_i} + u_{i+1}^{q_{i+1}}(0,t), \ (x,t) \in B_R \times (0,T), \\ u_i = 0, \ (x,t) \in \partial B_R \times (0,T), \\ u_i(x,0) = u_{i,0}(x), \ i = 1,2,\cdots,n, \ n \geq 2, \ x \in B_R, \\ u_{n+1} := u_1, \ p_{n+1} := p_1, \ q_{n+1} := q_1, \end{cases} \tag{1.1}$$

where $B_R = \{x \in \mathbf{R}^N \,|\, |x| < R\}$; exponents $p_i, \ q_i \geq 0$ $(i = 1, 2, \cdots, n)$; $u_{1,0}(x), \ u_{2,0}(x), \cdots, \ u_{n,0}(x) \geq, \not\equiv 0$ are radially non-increasing, which satisfy the compatibility conditions. Let T be the maximal existence time of solutions. The existence and uniqueness of local solutions to system (1.1) is well known (see [4]).

For the scalar cases of (1.1), Okada and Fukuda [8] completed the classifications for total and single point blow-up solutions, also with the blow-up rate

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estimates. Liu, Li and Gao [7] also studied the scalar problem of (1.1) with the inner sources replaced by $k_1u^p(x,t)+k_2u^q(x_0,t)$, and obtained uniform blow-up profiles on all compact subsets of the domain for global blow-up solutions.

Recently, Zheng and Wang [20] discussed the special system of (1.1) with $n = 2$

$$\begin{cases} u_t = \Delta u + u^{p_1} + v^{q_2}(0,t), & (x,t) \in B_R \times (0,T), \\ v_t = \Delta v + v^{p_2} + u^{q_1}(0,t), & (x,t) \in B_R \times (0,T) \end{cases} \tag{1.2}$$

with $p_1, p_2, q_1, q_2 > 1$. For solutions radially symmetric, radially non-increasing in space and nondecreasing in time, total versus single point blow-up were considered. Moreover, four kinds of simultaneous blow-up rates were established. The parabolic equations in (1.2) with local sources

$$\begin{cases} u_t = \Delta u + u^{p_1} + v^{q_2}, & (x,t) \in \mathbf{R}^N \times (0,T), \\ v_t = \Delta v + v^{p_2} + u^{q_1}, & (x,t) \in \mathbf{R}^N \times (0,T) \end{cases} \tag{1.3}$$

were studied by Souplet and Tayachi [16] with p_1, p_2, q_1, $q_2 > 1$. The optimal classifications on non-simultaneous blow-up are given there. Rossi and Souplet studied equations (1.3) in [13] with null Dirichlet conditions. The phenomena for coexistence of non-simultaneous and simultaneous blow-up have been firstly observed in the exponent region $p_1 > q_1 + 1$, $p_2 > q_2 + 1$.

The n-componential parabolic systems, like (1.1), come from chemical reactions, heat transfer, population dynamics, etc, which describe the phenomena in real-life world more exactly than parabolic systems with two components, and are worthy to be considered (see, for example, [3, 10, 18]). The components $u_1, u_2, \dots, u_n$ represent, for example, the concentrations of the chemical reactants, the temperatures of the materials during heat propagations, the densities of the biological populations during migrations, where the nonlinear reactions in dynamical systems take place both at a local site and at a single point. For a more detail information, the readers can refer to the books [9, 14].

In work [18], Wang discussed the n-componential parabolic problem

$$(u_i)_t = \Delta u_i + u_{i+1}^{q_{i+1}}, \ i = 1, 2, \cdots, n, \quad (x,t) \in \Omega \times (0,T), \tag{1.4}$$

subject to null Dirichlet conditions, where Ω is a general bounded domain of $\mathbf{R}^N$; $u_{n+1} := u_1$, $q_{n+1} := q_1$. It was found out that any blow-up must be simultaneous. If $\prod_{i=1}^n q_i > 1$ and $(u_i)_t \geq 0$, then there exist positive constants C and c such that

$$c(T-t)^{-\lambda_i} \leq \max_{\Omega} u_i(\cdot,t) \leq C(T-t)^{-\lambda_i}, \quad i = 1, 2, \cdots, n \tag{1.5}$$

with

$$\lambda_i = \frac{1 + q_i + \sum_{l=i+1}^{n+i-2} q_i \cdots q_l}{\prod_{i=1}^n q_i - 1}.$$

The corresponding Cauchy problem of equations (1.4) was discussed by Fila and Quittner [3]. They obtained that there exists some positive constant C such that

$$u_i(x,t) \le C(T-t)^{-\lambda_i}, \quad i=1,2,\cdots,n$$

provided $\max\{\lambda_1,\, \lambda_2, \cdots,\, \lambda_n\} > N/2$.

Pedersen and Lin [10] discussed the localized n-componential parabolic system

$$(1.6)\quad (u_i)_t = \Delta u_i + u_{i+1}^{q_{i+1}}(x_0,t),\ i=1,2,\cdots,n,\ (x,t)\in\Omega\times(0,T),$$

subject to null Dirichlet conditions, where x_0 is a point in Ω; $q_i \ge 1$, $i = 1,2,\cdots,n$; $u_{n+1} := u_1$, $q_{n+1} := q_1$. The simultaneous blow-up rate (1.5) was obtained. Moreover, boundary layer estimates were considered.

There are also some good works for the parabolic systems (see [12–19] etc.).

For system (1.1), one can find that all of the components could blow up by themselves and influence their neighbors through the coupled localized sources. In the present paper, the non-simultaneous blow-up of n components means that at least one component of the n ones remain bounded, while some others blow up simultaneously, which is much more complex than that of the systems with only two components.

In the next section, two main results are given, which will be proved in Sections 3 and 4, respectively.

2. Main results

It can be checked from works [10,18] that the positive solutions of system (1.1) blow up for large initial data, if

$$\max\Big\{p_1,p_2,\cdots,p_n,\prod_{i=1}^{n} q_i\Big\} > 1.$$

In the sequel, we only consider blow-up phenomena for n components of solutions to system (1.1) with $T<+\infty$. Denote $\xi_i := \xi_{i+n}$ for $i\le 0$, and a set of initial data as follows,

$$(2.1)\qquad \mathbb{V}_0 = \Big\{\Delta u_{i,0} + u_{i,0}^{p_i} + u_{i+1,0}^{q_{i+1}}(0) \ge 0,\ i=1,2,\cdots,n\Big\}.$$

Hence, by the comparison principle, we have

$$U_i(t) = u_i(0,t) = \max\Big\{u_i(y,\tau)\mid (y,\tau)\in[0,R]\times[0,t]\Big\}, \quad 1\le i\le n.$$

The notation $U_i(t) \sim (T-t)^{-\beta_i}$ represents that there exist two positive constants c_i and C_i such that

$$c_i(T-t)^{-\beta_i} \le U_i(t) \le C_i(T-t)^{-\beta_i}.$$

Now, we give the main results. The first theorem shows the non-simultaneous blow-up phenomena, where one component of the n ones blows up by itself and can provide sufficient help to the blow-up of its neighbors.

Theorem 2.1. *Let $i \in \{1, 2, \cdots, n\}$ and $k \in \{0, 1, \cdots, n-2\}$. Assume that*

$$\beta_i = \frac{1}{p_i - 1} > 0, \quad \beta_j := q_{j+1}\beta_{j+1} - 1 > 0,$$

$$p_j < \frac{\beta_j + 1}{\beta_j}, \ j = i-1, i-2, \cdots, i-k, \quad q_{i-k}\beta_{i-k} < 1.$$

Then there exist suitable initial data in $\mathbb{V}_0$ such that $u_{i-k}, u_{i-k+1}, \cdots, u_i$ blow up simultaneously while the other $(n-k-1)$ components remain bounded up to the blow-up time T, and the blow-up rates are

$$\begin{aligned} &\big(U_{i-k}(t), U_{i-k+1}(t), \cdots, U_i(t)\big) \\ &\sim \left((T-t)^{-\beta_{i-k}}, (T-t)^{-\beta_{i-k+1}}, \cdots, (T-t)^{-\beta_i}\right). \end{aligned} \tag{2.2}$$

Moreover, u_i is single point blow-up and $u_{i-k}, u_{i-k+1}, \cdots, u_{i-1}$ are total blow-up.

In Theorem 2.1, the condition $p_i > 1$ implies that u_i can blow up by itself. Regardless of whether u_{i-1} owing the blow-up capability by itself or not, the conditions $p_{i-1} < (\beta_{i-1} + 1)/\beta_{i-1}$ and $\beta_{i-1} > 0$ guarantee that the role of $u_i^{q_i}(0, t)$ is much stronger than $u_{i-1}^{p_{i-1}}(x, t)$ under suitable requirements of the initial data, which means that the blow-up of u_{i-1} is dominated by the localized term $u_i^{q_i}(0, t)$. And then u_{i-1} provides sufficient help to the blow-up of u_{i-2}. By the same way, such phenomena happen up to u_{i-k}. But, due to $q_{i-k}\beta_{i-k} < 1$, u_{i-k} can not lead to the blow-up of u_{i-k-1}.

By Theorem 2.1, one can find that

Corollary 2.2. *There exist suitable initial data such that only u_i, $i \in \{1, 2, \cdots, n\}$ blows up with the other $n-1$ ones still remain bounded if and only if $q_i + 1 < p_i$.*

In fact, the sufficient condition is just the subcase $k = 0$ of Theorem 2.1. The necessity can be obtained by the similar methods used in [6, Theorem 2.2]. □

By Corollary 2.2, one can obtain another interesting result as follows,

Corollary 2.3. *Any blow-up must be the case for at least two components blowing up simultaneously if and only if*

$$p_i \leq q_i + 1, \quad i = 1, 2, \cdots, n.$$

One can check that, for $i_1 + 1 = i_2$ and $n = 2$, the necessary and sufficient conditions in Corollaries 2.2 and 2.3 are just in [20, Theorems 2.2, 2.3, Corollary 2.1], and are compatible with [16, Theorem 1].

The second theorem gives that there exist suitable initial data such that any two components of the n ones can blow up simultaneously by themselves, and either of them can provide sufficient help to the blow-up of some other components.

Theorem 2.4. *Let* $i_1, i_2 \in \{1, 2, \cdots, n\}$, $i_1 < i_2$, $k_1 \in \{0, 1, \cdots, n+i_1-i_2-1\}$, *and* $k_2 \in \{0, 1, \cdots, i_2 - i_1 - 1\}$. *Assume that*

$$\begin{aligned}
&\alpha_{i_1} = \frac{1}{p_{i_1} - 1} > 0, \quad \alpha_\mu := q_{\mu+1}\alpha_{\mu+1} - 1 > 0, \\
&p_\mu \le 1, \ \mu = i_1 - 1, i_1 - 2, \cdots, i_1 - k_1, \quad q_{i_1-k_1}\alpha_{i_1-k_1} < 1, \\
&\alpha_{i_2} = \frac{1}{p_{i_2} - 1} > 0, \quad \alpha_\nu := q_{\nu+1}\alpha_{\nu+1} - 1 > 0, \\
&p_\nu \le 1, \ \nu = i_2 - 1, i_2 - 2, \cdots, i_2 - k_2, \quad q_{i_2-k_2}\alpha_{i_2-k_2} < 1.
\end{aligned}$$

There exist suitable initial data in $\mathbb{V}_0$ *for small* R *such that* u_j, $j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1; i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2$ *blow up simultaneously while the others remain bounded with blow-up rates*

$$U_j(t) \sim (T - t)^{-\alpha_j},$$

$j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1; i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2$. *Moreover,* u_{i_1} *and* u_{i_2} *are single point blow-up, while* u_j, $j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1 - 1; i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2 - 1$ *are total blow-up.*

By Theorem 2.4, if $i_1 + 1 = i_2 - k_2$ and $i_2 + 1 = n + i_1 - k_1$, simultaneous blow-up happens. In fact, $k_1 + k_2 + 2 = n$.

It can be checked that the simultaneous blow-up components in Theorem 2.4 can be divided into two groups:

$$\begin{aligned}
&u_j, \quad j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1, \\
\text{and} \quad &u_j, \quad j = i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2.
\end{aligned}$$

Three kinds of phenomena are involved as follows,

(i) only u_{i_1} and u_{i_2} blow up simultaneously, i.e. $k_1 = k_2 = 0$. It is interesting that the positions of u_{i_1} and u_{i_2} are arbitrary due to the different values of i_1 and i_2;

(ii) for $k_1 \neq 0$ and $k_2 = 0$ (or $k_1 = 0$ and $k_2 \neq 0$), u_{i_1} and u_{i_2} can blow up by themselves and only u_{i_1} (or u_{i_2}) can provide sufficient help to the blow-up of u_j, $j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1 - 1$ (or u_j, $j = i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2 - 1$);

(iii) for $k_1 \neq 0$ and $k_2 \neq 0$, both u_{i_1} and u_{i_2} can blow up by themselves and can provide sufficient help to the blow-up of u_j, $j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1 - 1$ and u_j, $j = i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2 - 1$, respectively.

Combining Theorem 2.1 with Theorem 2.4, one can find out that the exponent regions of Theorem 2.4 are the coexistence regions, that is, Theorem 2.4 guarantees that there exist suitable initial data such that both u_j,

$j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1$ and u_j, $j = i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2$ blow up simultaneously; By Theorem 2.1, there also exist initial data such that either u_j, $j = i_1 - k_1, i_1 - k_1 + 1, \cdots, i_1$ or u_j, $j = i_2 - k_2, i_2 - k_2 + 1, \cdots, i_2$ blow up. One can find out that results are compatible with [13, Theorem 1.1], and just in [20, Theorem 2.4] if $i_1 + 1 = i_2$ and $n = 2$.

3. **Proof of Theorem 2.1**

Let $\phi(x,t) = e^{-\lambda t}\varphi(x)$, where φ and λ are the first eigenfunction and the first eigenvalue of

$$-\Delta\varphi = \lambda\varphi,\ x \in B_R, \quad \text{and} \quad \varphi = 0,\ x \in \partial B_R,$$

normalized by $\|\varphi(\cdot)\|_\infty = 1$, respectively. In order to prove Theorem 2.1, we introduce two lemmas. The first lemma gives some important upper estimates of solutions.

Lemma 3.1. If $p_m > 1$ ($m \in \{1, 2, \cdots, n\}$) and $T < +\infty$, then

$$U_m(t) \le \tilde{C}_m (T - t)^{-\frac{1}{p_m - 1}} \tag{3.1}$$

with $\tilde{C}_m = [(p_m - 1)\eta\phi(0,T)]^{-\frac{1}{p_m-1}}$ for the initial data in $\mathbb{V}_0$ satisfying that

$$\Delta u_{m,0} + (1 - \eta\varphi)(u_{m,0}^{p_m} + u_{m+1,0}^{q_{m+1}}(0)) \ge 0, \quad \eta \in (0,1).$$

Proof. Construct function

$$I_m(x,t) = (u_m)_t(x,t) - \eta\phi(x,t)(u_m^{p_m}(x,t) + u_{m+1}^{q_{m+1}}(0,t)).$$

It can be checked that

$$\begin{aligned}(I_m)_t - \Delta I_m - p_m u_m^{p_m-1} I_m \ge\, & (1 - \eta\phi) q_{m+1} u_{m+1}^{q_{m+1}-1}(0,t)(u_{m+1})_t(0,t) \\ & + 2\eta p_m u_m^{p_m-1} \nabla u_m \cdot \nabla\phi \\ & + \eta\phi p_m (p_m - 1) u_m^{p_m-1} |\nabla u_m|^2 \\ \ge\, & 0, \quad (x,t) \in B_R \times (0,T),\end{aligned}$$

and

$$I_m(x,t) = 0, \quad (x,t) \in \partial B_R \times (0,T),$$

$$I_m(x,0) = \Delta u_{m,0}(x) + (1 - \eta\varphi(x))(u_{m,0}^{p_m}(x) + u_{m+1,0}^{q_{m+1}}(0)) \ge 0, \quad x \in B_R.$$

By the comparison principle, we obtain that

$$(u_m)_t(x,t) \ge \eta\phi(x,t)(u_m^{p_m}(x,t) + u_{m+1}^{q_{m+1}}(0,t)), \quad (x,t) \in B_R \times (0,T). \tag{3.2}$$

Then (3.1) can be obtained by integrating the above inequality (3.2). □

The second one shows some important relationships among different components.

Lemma 3.2. *Let $i \in \{1, 2, \cdots, n\}$ and $k \in \{1, 2, \cdots, n-2\}$. Assume that*

$$\beta_i = \frac{1}{p_i - 1} > 0, \quad \beta_j := q_{j+1}\beta_{j+1} - 1 > 0,$$

$$p_j < \frac{\beta_j + 1}{\beta_j}, \; j = i-1, i-2, \cdots, i-k,$$

and u_l, $l \in \{1, 2, \cdots, n\} \backslash \{i, i-1, \cdots, i-k\}$ are bounded. Then there exist positive constants $C_{i-1}, C_{i-2}, \cdots, C_{i-k}$, independent of t, such that

$$U_i^{\frac{1}{\beta_i}}(t) \le C_{i-1} U_{i-1}^{\frac{1}{\beta_{i-1}}}(t) \le \cdots \le C_{i-k} U_{i-k}^{\frac{1}{\beta_{i-k}}}(t), \quad t \in (0, T). \tag{3.3}$$

Proof. Without loss of generality, we only prove the case for $i = n$.

Firstly, we prove the inequality

$$U_n^{1/\beta_n}(t) \le C_{n-1} U_{n-1}^{1/\beta_{n-1}}(t), \quad t \in (0, T).$$

If the above inequality does not hold, then there would exist some $t_m \to T$ such that

$$U_{n-1}(t_m) U_n^{-\frac{\beta_{n-1}}{\beta_n}}(t_m) \to 0 \quad \text{as } m \to +\infty.$$

It implies that, $U_n(t_m) \to +\infty$ as $t_m \to T$. Let $\lambda_m = (U_n(t_m))^{-1/(2\beta_n)}$, then $\lambda_m \to 0$ as $m \to +\infty$.

Scale (u_n, u_{n-1}) to $(\varphi_n^{\lambda_m}, \varphi_{n-1}^{\lambda_m})$ as follows,

$$\varphi_\mu^{\lambda_m}(y, s) = \lambda_m^{2\beta_\mu} u_\mu(\lambda_m y, \lambda_m^2 s + t_m), \quad \mu = n, n-1$$

for $(y, s) \in \bar{B}_{\lambda_m} \times (-t_m/\lambda_m^2, (T - t_m)/\lambda_m^2)$ with $B_{\lambda_m} = \{y \in \mathbf{R}^N \mid \lambda_m y \in B_R\}$.

For $s \in (-t_m/\lambda_m^2, 0]$, we have

$$0 \le \varphi_n^{\lambda_m} \le 1, \; \varphi_n^{\lambda_m}(0,0) = 1; \quad 0 \le \varphi_{n-1}^{\lambda_m} \le \left(U_n^{-\frac{\beta_{n-1}}{\beta_n}} U_{n-1}\right)(t_m) \to 0, \tag{3.4}$$

as $m \to +\infty$. Moreover, $(\varphi_n^{\lambda_m}, \varphi_{n-1}^{\lambda_m})$ solves that

$$\begin{cases} (\varphi_n)_s = \Delta\varphi_n + \lambda_m^{2+2\beta_n - 2p_n\beta_n} \varphi_n^{p_n} + \lambda_m^{2+2\beta_n} \Phi_1, \\ (\varphi_{n-1})_s = \Delta\varphi_{n-1} + \lambda_m^{2+2\beta_{n-1} - 2p_{n-1}\beta_{n-1}} \varphi_{n-1}^{p_{n-1}} + \lambda_m^{2+2\beta_{n-1} - 2q_n\beta_n} \varphi_n^{q_n}(0, s) \end{cases} \tag{3.5}$$

with bounded $\Phi_1 = u_1^{q_1}(\lambda_m y, \lambda_m^2 s + t_m)$. All the powers of λ_m in (3.5) are nonnegative, and hence the four coefficients tend to 0 or 1 as $m \to +\infty$. By the known Schauder's estimates, we can find a subsequence converging uniformly on compact subsets of $\mathbf{R}^N \times (-\infty, 0]$ to $(\varphi_n, \varphi_{n-1})$, which satisfies that

$$(\varphi_n)_s = \Delta\varphi_n + \varphi_n^{p_n}, \quad (\varphi_{n-1})_s = \Delta\varphi_{n-1} + \varphi_n^{q_n}, \quad (y, s) \in \mathbf{R}^N \times (-\infty, 0].$$

We get $\varphi_{n-1} \equiv 0$, $\varphi_n(0,0) = 1$ from (3.4). This is a contradiction.

Secondly, we prove the following inequality

$$C_{n-1} U_{n-1}^{1/\beta_{n-1}}(t) \le C_{n-2} U_{n-2}^{1/\beta_{n-2}}(t), \quad t \in (0, T).$$

If the latter equality does not hold, then there would exist another sequence $t_m \to T$ such that

$$U_{n-2}(t_m)U_{n-1}^{-\beta_{n-2}/\beta_{n-1}}(t_m) \to 0 \quad \text{as } m \to +\infty.$$

Clearly, it follows that $U_{n-1}(t_m) \to +\infty$ as $t_m \to T$. Let $\lambda_m = (U_{n-1}(t_m))^{-1/(2\beta_{n-1})}$. Similarly, scale u_ν to

$$\psi_\nu^{\lambda_m}(y,s) = \lambda_m^{2\beta_\nu} u_\nu(\lambda_m y, \lambda_m^2 s + t_m), \quad \nu = n, n-1, n-2$$

for $(y,s) \in \bar{B}_{\lambda_m} \times (-t_m/\lambda_m^2, (T-t_m)/\lambda_m^2)$. For $s \in (-t_m/\lambda_m^2, 0]$, we have

$$0 \le \psi_n^{\lambda_m} \le C_{n-1}^{\beta_n}, \quad 0 \le \psi_{n-1}^{\lambda_m} \le 1, \quad \psi_{n-1}^{\lambda_m}(0,0) = 1;$$

$$0 \le \psi_{n-2}^{\lambda_m} \le \big(U_{n-1}^{-\frac{\beta_{n-2}}{\beta_{n-1}}} U_{n-2}\big)(t_m) \to 0, \quad m \to +\infty.$$

We also get a contradiction, similarly to the proof of the first part.

By the same methods, one can check that the other inequalities of (3.3) hold. □

Proof of Theorem 2.1. Let $T < +\infty$. Without loss of generality, we only prove the case $i = n$ with $k = 1$. The proof is made up of five steps as follows,

Step 1. *the upper estimates for u_{n-1} and u_n.*

Take $u_{i,0}(0) = \xi_i > 0$, $i = 1, 2, \cdots, n-1$ and choose constants S_i, $i = 1, 2, \cdots, n-1$, satisfying

$$S_{n-1} > \left\{ \left[\frac{2}{\varepsilon(p_n - 1)}\right]^{\frac{q_n}{p_{n-1}}} \frac{1}{\beta_{n-1}} \right\}^{p_{n-1}},$$

$$S_i > \xi_i^{p_i}, \quad i = 1, 2, \cdots, n-2.$$

Choose the initial data in $\mathbb{V}_0$ such that T satisfies

$$\phi(0,T) < \frac{1}{2},$$

$$S_{n-1} \ge \Big[\xi_{n-1}T^{\beta_{n-1}} + \beta_{n-1}^{-1}\big(S_{n-1}T^{q_n\beta_n - p_{n-1}\beta_{n-1}} + \tilde{C}_n^{q_n}\big)\Big]^{p_{n-1}},$$

$$S_{n-2} \ge \Big(\xi_{n-2} + S_{n-2}T + \frac{1}{1 - q_{n-1}\beta_{n-1}} S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} T^{1 - q_{n-1}\beta_{n-1}}\Big)^{p_{n-2}},$$

$$S_i \ge \Big(\xi_i + S_i T + S_{i+1}^{\frac{q_{i+1}}{p_{i+1}}} T\Big)^{p_i}, \quad i = 1, 2, \cdots, n-3.$$

Consider the auxiliary problem

$$\begin{cases} (\bar{z}_{n-1})_t = \Delta \bar{z}_{n-1} + S_{n-1}(T-t)^{-p_{n-1}\beta_{n-1}} \\ \qquad\qquad + \tilde{C}_n^{q_n}(T-t)^{-q_n\beta_n}, \ (x,t) \in B_R \times (0,T), \\ \bar{z}_{n-1}(x,t) = 0, \ (x,t) \in \partial B_R \times (0,T), \\ \bar{z}_{n-1}(x,0) = u_{n-1,0}(x), \ x \in B_R. \end{cases}$$

By Green's identity and $p_{n-1} < (\beta_{n-1}+1)/\beta_{n-1}$, we obtain

$$\begin{aligned}\bar{z}_{n-1} &\leq \xi_{n-1} + S_{n-1}\int_0^t (T-\tau)^{-p_{n-1}\beta_{n-1}}d\tau + \tilde{C}_n^{q_n}\int_0^t (T-\tau)^{-q_n\beta_n}d\tau \\ &\leq S_{n-1}^{\frac{1}{p_{n-1}}}(T-t)^{-\beta_{n-1}}.\end{aligned}$$

Then $\bar{z}_{n-1}$ satisfies

$$\begin{cases}(\bar{z}_{n-1})_t \geq \Delta\bar{z}_{n-1} + \bar{z}_{n-1}^{p_{n-1}} + \tilde{C}_n^{q_n}(T-t)^{-q_n\beta_n}, & (x,t)\in B_R\times(0,T),\\ \bar{z}_{n-1}(x,t) = 0, & (x,t)\in\partial B_R\times(0,T),\\ \bar{z}_{n-1}(x,0) = u_{n-1,0}(x), & x\in B_R.\end{cases}$$

By Lemma 3.1, we have $u_n \leq \tilde{C}_n(T-t)^{-\beta_n}$ for $p_n > 1$. Then u_{n-1} satisfies

$$\begin{cases}(u_{n-1})_t \leq \Delta u_{n-1} + u_{n-1}^{p_{n-1}} + \tilde{C}_n^{q_n}(T-t)^{-q_n\beta_n}, & (x,t)\in B_R\times(0,T),\\ u_{n-1}(x,t) = 0, & (x,t)\in\partial B_R\times(0,T),\\ u_{n-1}(x,0) = u_{n-1,0}(x), & x\in B_R.\end{cases}$$

By the comparison principle, we have

$$u_{n-1} \leq \bar{z}_{n-1} \leq S_{n-1}^{\frac{1}{p_{n-1}}}(T-t)^{-\beta_{n-1}}, \quad (x,t)\in\bar{B}_R\times(0,T).$$

Step 2. $u_1, u_2, \cdots, u_{n-2}$ *remain bounded up to* T.

Consider the auxiliary problem

$$\begin{cases}(\bar{z}_{n-2})_t = \Delta\bar{z}_{n-2} + S_{n-2} + S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}}(T-t)^{-q_{n-1}\beta_{n-1}}, & (x,t)\in B_R\times(0,T),\\ \bar{z}_{n-2}(x,t) = 0, & (x,t)\in\partial B_R\times(0,T),\\ \bar{z}_{n-2}(x,0) = \bar{z}_{n-2,0}(x), & x\in B_R.\end{cases}$$

Using Green's identity and the inequality $q_{n-1}\beta_{n-1} < 1$, we have $\bar{z}_{n-2} \leq S_{n-2}^{\frac{1}{p_{n-2}}}$ in $B_R\times(0,T)$. It follows that

$$(\bar{z}_{n-2})_t \geq \Delta\bar{z}_{n-2} + \bar{z}_{n-2}^{p_{n-2}} + S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}}(T-t)^{-q_{n-1}\beta_{n-1}}, \quad (x,t)\in B_R\times(0,T).$$

Using the upper estimate of u_{n-1}, we obtain

$$(u_{n-2})_t \leq \Delta u_{n-2} + u_{n-2}^{p_{n-2}} + S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}}(T-t)^{-q_{n-1}\beta_{n-1}}, \quad (x,t)\in B_R\times(0,T).$$

Then by the comparison principle we get

$$u_{n-2} \leq \bar{z}_{n-2} \leq S_{n-2}^{\frac{1}{p_{n-2}}}, \quad (x,t)\in\bar{B}_R\times(0,T).$$

Next to prove the boundedness of $u_1, u_2, \cdots, u_{n-3}$. Take u_{n-3} for example. Consider the following auxiliary problem

$$\begin{cases} (\bar{z}_{n-3})_t = \Delta \bar{z}_{n-3} + S_{n-3} + S_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}, & (x,t) \in B_R \times (0,+\infty), \\ \bar{z}_{n-3}(x,t) = 0, & (x,t) \in \partial B_R \times (0,+\infty), \\ \bar{z}_{n-3}(x,0) = u_{n-3,0}(x), & x \in B_R. \end{cases}$$

By Green's identity, $\bar{z}_{n-3} \le S_{n-3}^{\frac{1}{p_{n-3}}}$ in $B_R \times (0,T)$. Hence, there is

$$(\bar{z}_{n-3})_t \ge \Delta \bar{z}_{n-3} + \bar{z}_{n-3}^{p_{n-3}} + S_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}, \quad (x,t) \in B_R \times (0,T).$$

For $u_{n-2} \le S_{n-2}^{\frac{1}{p_{n-2}}}$, u_{n-3} satisfies

$$(u_{n-3})_t \le \Delta u_{n-3} + u_{n-3}^{p_{n-3}} + S_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}, \quad (x,t) \in B_R \times (0,T).$$

Consequently, we have

$$u_{n-3} \le \bar{z}_{n-3} \le S_{n-3}^{\frac{1}{p_{n-3}}}, \quad (x,t) \in \bar{B}_R \times (0,T).$$

Step 3. *the lower estimate for u_{n-1}.*

By Lemma 3.2, one can obtain that

$$U_n^{1/\beta_n} \le C_{n-1} U_{n-1}^{1/\beta_{n-1}}, \quad t \in (0,T).$$

So u_{n-1} must blow up. By Green's identity,

$$U_{n-1}(t) \le U_{n-1}(z) + U_{n-1}^{p_{n-1}}(t)(T-z) + C_{n-1}^{q_n\beta_n} U_{n-1}^{\frac{q_n\beta_n}{\beta_{n-1}}}(t)(T-z). \tag{3.6}$$

Take z such that

$$U_{n-1}(z) = \frac{1}{2} U_{n-1}(t) > 1.$$

By $p_{n-1} < (\beta_{n-1}+1)/\beta_{n-1}$, the inequality (3.6) gives $U_{n-1}(z) \ge c(T-z)^{-\beta_{n-1}}$ for $z \in (0,T)$.

Step 4. $U_n(t) \ge c(T-t)^{-\beta_n}$, $t \in (0,T)$.

Otherwise, there would exist some $\varepsilon_j \to 0$ and $t_j \to T$ such that $U_n(t_j) < \varepsilon_j (T-t_j)^{-\beta_n}$. By Green's identity, we have

$$U_{n-1}(t) \le U_{n-1}(z) + U_{n-1}^{p_{n-1}}(t)(t-z) + U_n^{q_n}(t)(t-z).$$

It can be proved that there exist some $z \in (0,t)$ and $M > 0$ such that

$$U_{n-1}(z) = \frac{1}{2} U_{n-1}(t) > 1$$

and $t - z \le M(T-t)$ as t near T. Then we arrive at

$$U_{n-1}(t) \le C U_{n-1}^{p_{n-1}}(t)(T-t) + C U_n^{q_n}(t)(T-t).$$

By the blow-up rate estimates for $U_{n-1}(t)$ and taking $t = t_j$, we have

$$c(T-t_j)^{-\beta_{n-1}} \le C(T-t_j)^{-p_{n-1}\beta_{n-1}+1} + C\varepsilon_j^{q_n}(T-t_j)^{-q_n\beta_n+1}.$$

It requires that

$$\beta_{n-1} \le p_{n-1}\beta_{n-1} - 1, \quad \text{or} \quad \beta_{n-1} < q_n\beta_n - 1.$$

But $\beta_{n-1} > p_{n-1}\beta_{n-1} - 1$ and $\beta_{n-1} = q_n\beta_n - 1$, which is a contradiction.

Step 5. *Total versus single point blow-up.*

For any $m \in \{1, 2, \cdots, n\}$, u_m satisfies

$$\begin{aligned} u_m(x,t) &\ge \phi(x,t)F_m(t) \\ &= \phi(x,t)\int_0^t u_{m+1}^{q_{m+1}}(0,\tau)d\tau, \quad (x,t) \in B_R \times [0,T). \end{aligned} \tag{3.7}$$

In fact, if we set

$$J_m(x,t) = u_m(x,t) - \phi(x,t)F_m(t),$$

then it is easy to check that

$$\begin{aligned} &(J_m)_t - \Delta J_m \ge u_m^{p_m} + (1-\phi(x,t))u_{m+1}^{q_{m+1}}(0,t) \ge 0, \quad (x,t) \in B_R \times [0,T), \\ &J_m = 0, \quad (x,t) \in \partial B_R \times [0,T), \\ &J_m(x,0) \ge 0, \quad x \in B_R. \end{aligned}$$

By the comparison principle, one can obtain (3.7).

By $U_n(t) \ge c(T-t)^{-\beta_n}$ and (3.7), we obtain

$$u_{n-1}(x,t) \ge \phi(0,T)\int_0^t (T-\tau)^{-q_n\beta_n}d\tau.$$

Due to $q_n\beta_n > 1$, $u_{n-1}(x,t)$ blows up everywhere in B_R, i.e., $u_{n-1}(x,t)$ is total blow-up.

Now, we prove that u_n is single point blow-up. If not, there would exist a blow-up point x_0, $|x_0| = r_0 \ne 0$. So $u_n(x,t)$ blows up in the whole interval $[0, r_0]$. For bounded u_1, $F_n(t) < +\infty$. Then there exists some $t_1 \in [0,T)$ such that

$$u_n(x,t) - F_n(t) > 0, \quad (x,t) \in K_0 \times [t_1, T), \tag{3.8}$$

where $K_0 = \{x \in B_R \mid \delta_1 < x_j < \eta_1,\ j = 1, 2, \cdots, N,\ 0 < \delta_1 < \eta_1 < r_0N^{-1/2}\}$.

Define function

$$J(x,t) = (u_n)_{x_1} + C(x)(u_n - F_n(t))^{p^*}, \quad (x,t) \in K_0 \times [t_1, T),$$

where

$$1 < p^* < p_n, \quad C(x) = \varepsilon\prod_{j=1}^N \sin(\mu_0(x_j - \delta_1)), \quad \mu_0 = \frac{\pi}{\eta_1 - \delta_1}$$

with $\varepsilon > 0$ to be determined (see [8, 20]).

By computation and the comparison principle, one can obtain that $J(x,t) \leq 0$ with small ε for $(x,t) \in \bar{K}_0 \times [t^*, T)$, that is,

$$-(u_n)_{x_1}(u_n - F_n(t))^{-p^*} \geq C(x), \quad (x,t) \in \bar{K}_0 \times [t^*, T). \tag{3.9}$$

Fix $(a_2, a_3, \cdots, a_N) \in \mathbf{R}^{N-1}$, and take $a = (\delta_1, a_2, \cdots, a_N)$, $a^* = (\eta_1, a_2, \cdots, a_N)$. Integrating (3.9) from a to a^*, we obtain a contradiction as follows,

$$0 < \int_{\delta_1}^{\eta_1} C(x)dx_1 < \frac{1}{p^*-1}(u_n(a^*,t) - F(t))^{1-p^*}, \quad p^* \in (1, p_n). \qquad \square$$

4. Proof of Theorem 2.4

For convenience, we only prove the subcase for

$$k_1 = 1,\ k_2 = 0,\ i_1 = n-2,\ i_2 = n.$$

So Theorem 2.4 turns into

Theorem 4.1. *Assume that*

$$\alpha_n = \frac{1}{p_n - 1} > 0, \quad q_n\alpha_n < 1, \quad \alpha_{n-2} = \frac{1}{p_{n-2}-1} > 0,$$
$$\alpha_{n-3} = q_{n-2}\alpha_{n-2} - 1 > 0, \quad q_{n-3}\alpha_{n-3} < 1, \quad p_{n-3} \leq 1.$$

Then there exist suitable initial data for small R such that only u_{n-3}, u_{n-2}, u_n blow up simultaneously, and the blow-up rates are

$$\big(U_{n-3}(t), U_{n-2}(t), U_n(t)\big) \sim \big((T-t)^{-\alpha_{n-3}}, (T-t)^{-\alpha_{n-2}}, (T-t)^{-\alpha_n}\big).$$

Moreover, u_{n-2} and u_n are single point blow-up while u_{n-3} is total blow-up.

Construct two subsets of $\mathbb{V}_0$ as follows,

$$\mathbb{V}_1 = \Big\{ \big(u_{1,0}, u_{2,0}, \cdots, u_{n,0}\big) \in \mathbb{V}_0 \ \Big|$$
$$\Delta u_{i,0} + (1 - \eta\varphi)(u_{i,0}^{p_i} + u_{i+1,0}^{q_{i+1}}(0)) \geq 0,\ i = n-2, n\Big\}, \tag{4.1}$$

$$\mathbb{V}_2 = \left\{\left(\breve{u}_{1,0}, \breve{u}_{2,0}, \cdots, \breve{u}_{n-3,0}, \frac{\breve{u}_{n-1,0}}{(1-\lambda_1)\lambda_2}, \frac{\breve{u}_{n-2,0}}{\lambda_1}, \frac{\breve{u}_{n,0}}{(1-\lambda_1)(1-\lambda_2)}\right) \ \right|$$
$$\lambda_1, \lambda_2 \in (0,1), \quad \big(\breve{u}_{1,0}, \breve{u}_{2,0}, \cdots, \breve{u}_{n,0}\big) \in \mathbb{V}_1\Big\}. \tag{4.2}$$

Lemma 4.2. *Under the conditions of Theorem 4.1, there exists some $\bar{\lambda}_1 \in (1/2, 1)$ such that, for any $\lambda_2 \in (0,1)$, non-simultaneous blow-up happens with u_j, $j = 1, 2, \cdots, n-4, n-1$ remaining bounded for the initial data in $\mathbb{V}_2$.*

Proof. Consider the auxiliary problem

$$\begin{cases} (\underline{u}_{n-2})_t = \Delta \underline{u}_{n-2} + \underline{u}_{n-2}^{p_{n-2}}, & (x,t) \in B_R \times (0, \underline{T}_{n-2}), \\ \underline{u}_{n-2}(x,t) = 0, & (x,t) \in \partial B_R \times (0, \underline{T}_{n-2}), \\ \underline{u}_{n-2}(x,0) = \underline{u}_{n-2,0}(x), & x \in B_R, \end{cases} \tag{4.3}$$

where $\underline{u}_{n-2,0} = \breve{u}_{n-2,0}/(1-\lambda_1)$ is radially symmetric with λ_1 to be determined. Take

$$M_j > \breve{u}_{j,0}^{p_j}(0), \quad j = 1, 2, \cdots, n-4,$$
$$M_{n-1} > \left[2\breve{u}_{n-1,0}(0)\right]^{p_{n-1}},$$
$$M_{n-3} > \left(\alpha_{n-3}^{-1}\tilde{C}_{n-2}^{q_{n-2}}\right)^{p_{n-3}}.$$

Due to (4.3), there must exist some $\bar{\lambda}_1 \in (1/2, 1)$ such that, if $\lambda_1 = \bar{\lambda}_1$, then $\underline{T}_{n-2}$ satisfies that

(4.4)
$$\left[\breve{u}_{n-3,0}(0)\underline{T}_{n-2}^{\alpha_{n-3}} + \alpha_{n-3}^{-1}\left(M_{n-3}\underline{T}_{n-2}^{1+\alpha_{n-3}-p_{n-3}\alpha_{n-3}} + \tilde{C}_{n-2}^{q_{n-2}}\right)\right]^{p_{n-3}} \le M_{n-3},$$

(4.5)
$$\left[\breve{u}_{n-4,0}(0) + M_{n-4}\underline{T}_{n-2} + \frac{\underline{T}_{n-2}^{1-q_{n-3}\alpha_{n-3}}}{1-q_{n-3}\alpha_{n-3}} M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}\right]^{p_{n-4}} \le M_{n-4},$$

(4.6)
$$\left[\breve{u}_{j,0}(0) + M_j\underline{T}_{n-2} + M_{j+1}^{\frac{q_{j+1}}{p_{j+1}}}\underline{T}_{n-2}\right]^{p_j} \le M_j, \quad j = 1, 2, \cdots, n-5,$$

(4.7)
$$\left[2\breve{u}_{n-1,0}(0) + M_{n-1}\underline{T}_{n-2} + \frac{\tilde{C}_n^{q_n}}{1-q_n\alpha_n}\underline{T}_{n-2}^{1-q_n\alpha_n}\right]^{p_{n-1}} \le M_{n-1}.$$

Since

$$u_{n-2,0}(x) = \frac{\breve{u}_{n-2,0}(x)}{(1-\lambda_1)\lambda_2} \ge \frac{\breve{u}_{n-2,0}(x)}{1-\lambda_1} = \underline{u}_{n-2,0}(x),$$

the blow-up time T of (1.1) satisfies that $T \le \underline{T}_{n-2}$. In addition, T satisfies that (4.4)–(4.7) instead of $\underline{T}_{n-2}$.

Consider the following problem

$$(4.8) \qquad \begin{cases} (\bar{u}_{n-3})_t = \Delta\bar{u}_{n-3} + M_{n-3}(T-t)^{-p_{n-3}\alpha_{n-3}} \\ \qquad +\tilde{C}_{n-2}^{q_{n-2}}(T-t)^{-q_{n-2}\alpha_{n-2}}, \ (x,t) \in B_R \times (0,T), \\ \bar{u}_{n-3}(x,t) = 0, \ (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-3}(x,0) = \breve{u}_{n-3,0}(x), \ x \in B_R. \end{cases}$$

From the inequality $p_{n-3} \le 1$ we get

$$\begin{aligned} \bar{u}_{n-3} &\le \left[\breve{u}_{n-3,0}(0)T^{\alpha_{n-3}} + \alpha_{n-3}^{-1}\left(M_{n-3}T^{1+\alpha_{n-3}-p_{n-3}\alpha_{n-3}} + \tilde{C}_{n-2}^{q_{n-2}}\right)\right](T-t)^{-\alpha_{n-3}} \\ &\le M_{n-3}^{\frac{1}{p_{n-3}}}(T-t)^{-\alpha_{n-3}}. \end{aligned}$$

Then, due to (4.8), we have

$$(\bar{u}_{n-3})_t \ge \Delta \bar{u}_{n-3} + \bar{u}_{n-3}^{p_{n-3}} + \tilde{C}_{n-2}^{q_{n-2}}(T-t)^{-q_{n-2}\alpha_{n-2}}, \quad (x,t) \in B_R \times (0,T).$$

By Lemma 3.1, u_{n-3} satisfies

$$(u_{n-3})_t \le \Delta u_{n-3} + u_{n-3}^{p_{n-3}} + \tilde{C}_{n-2}^{q_{n-2}}(T-t)^{-q_{n-2}\alpha_{n-2}}, \quad (x,t) \in B_R \times (0,T).$$

By comparison principle, we get

$$u_{n-3} \le \bar{u}_{n-3} \le M_{n-3}^{1/p_{n-3}}(T-t)^{-\alpha_{n-3}}, \quad (x,t) \in B_R \times (0,T).$$

Consider the auxiliary problem

$$\begin{cases} (\bar{u}_{n-4})_t = \Delta \bar{u}_{n-4} + M_{n-4} + M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}(T-t)^{-q_{n-3}\alpha_{n-3}}, & (x,t) \in B_R \times (0,T), \\ \bar{u}_{n-4}(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-4}(x,0) = \breve{u}_{n-4,0}(x), & x \in B_R. \end{cases}$$

Using Green's identity we get

$$\bar{u}_{n-4} \le \breve{u}_{n-4,0}(0) + M_{n-4}T + \frac{T^{1-q_{n-3}\alpha_{n-3}}}{1-q_{n-3}\alpha_{n-3}} M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} \le M_{n-4}^{\frac{1}{p_{n-4}}}, \quad t \in (0,T).$$

Consequently we have

$$(\bar{u}_{n-4})_t \ge \Delta \bar{u}_{n-4} + \bar{u}_{n-4}^{p_{n-4}} + M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}(T-t)^{-q_{n-3}\alpha_{n-3}}, \quad (x,t) \in B_R \times (0,T).$$

By (1.1) and $u_{n-3} \le M_{n-3}^{1/p_{n-3}}(T-t)^{-\alpha_{n-3}}$, we obtain

$$u_{n-4} \le \bar{u}_{n-4} \le M_{n-4}^{1/p_{n-4}}, \quad (x,t) \in B_R \times (0,T).$$

The boundedness of u_j $(j = 1, 2, \cdots, n-5)$ can be proved similarly.

Consider the following problem

$$\begin{cases} (\bar{u}_{n-1})_t = \Delta \bar{u}_{n-1} + M_{n-1} + \tilde{C}_n^{q_n}(T-t)^{-q_n\alpha_n}, & (x,t) \in B_R \times (0,T), \\ \bar{u}_{n-1}(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-1}(x,0) = \dfrac{\breve{u}_{n-1,0}(x)}{\lambda_1}, & x \in B_R. \end{cases}$$

For $q_n\alpha_n < 1$, we have $\bar{u}_{n-1} \le M_{n-1}^{\frac{1}{p_{n-1}}}$. Then

$$(\bar{u}_{n-1})_t \ge \Delta \bar{u}_{n-1} + \bar{u}_{n-1}^{p_{n-1}} + \tilde{C}_n^{q_n}(T-t)^{-q_n\alpha_n}, \quad (x,t) \in B_R \times (0,T).$$

By Lemma 3.1, $U_n(t) \le \tilde{C}_n(T-t)^{-\alpha_n}$. So u_{n-1} satisfies the inequality

$$(u_{n-1})_t \le \Delta u_{n-1} + u_{n-1}^{p_{n-1}} + \tilde{C}_n^{q_n}(T-t)^{-q_n\alpha_n}, \quad (x,t) \in B_R \times (0,T).$$

Using the comparison principle,

$$u_{n-1} \le \bar{u}_{n-1} \le M_{n-1}^{1/p_{n-1}}, \quad (x,t) \in B_R \times (0,T). \qquad \square$$

Lemma 4.3. *Assume that the conditions of Theorem 4.1 hold. For the fixed $\lambda_1 = \bar{\lambda}_1$ in Lemma 4.2, there exists some $\lambda_2' \in (0, 1/2)$ such that only u_{n-3}, u_{n-2} blow up with the initial data in $\mathbb{V}_2$, satisfying $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \lambda_2'$.*

Proof. Consider the auxiliary problem

$$\text{(4.9)} \quad \begin{cases} (\bar{u}_n)_t = \Delta \bar{u}_n + M_n + M_1^{\frac{q_1}{p_1}}, & (x,t) \in B_R \times (0, +\infty), \\ \bar{u}_n(x,t) = 0, & (x,t) \in \partial B_R \times (0, +\infty), \\ \bar{u}_n(x,0) = 2\breve{u}_{n,0}(x), & x \in B_R, \end{cases}$$

where M_1 is defined in Lemma 4.2, and $M_n > [2\breve{u}_{n,0}(0)/(1-\bar{\lambda}_1)]^{p_n}$.

Assume the initial data of auxiliary problem (4.3) satisfies that $\underline{u}_{n-2,0} = \breve{u}_{n-2,0}/[(1-\bar{\lambda}_1)\lambda_2]$ with λ_2 to be determined. For (4.3), there exists some $\lambda_2' \in (0, 1/2)$ such that, if $\lambda_2 = \lambda_2'$, then $\underline{T}_{n-2}$ satisfies the following inequality

$$M_n \geq \Big(\frac{2\breve{u}_{n,0}(0)}{1-\bar{\lambda}_1} + M_n \underline{T}_{n-2} + M_1^{\frac{q_1}{p_1}} \underline{T}_{n-2} \Big)^{p_n}.$$

We have $\underline{u}_{n-2} \leq u_{n-2}$, and hence $T \leq \underline{T}_{n-2}$. Considering system (4.9) in $[0, T)$, we have

$$\bar{u}_n \leq \frac{2\breve{u}_{n,0}(0)}{1-\bar{\lambda}_1} + M_n T + M_1^{\frac{q_1}{p_1}} T \leq M_n^{\frac{1}{p_n}}.$$

Consequently

$$(\bar{u}_n)_t \geq \Delta \bar{u}_n + \bar{u}_n^{p_n} + M_1^{\frac{q_1}{p_1}}, \quad (x,t) \in B_R \times (0, T).$$

By $u_1 \leq M_1^{1/p_1}$, we obtain that

$$u_n \leq \bar{u}_n \leq M_n^{1/p_n}, \quad (x,t) \in B_R \times (0, T).$$

We claim that only u_{n-2} and u_{n-3} blow up simultaneously. By Lemma 3.2, one can obtain

$$U_{n-2}^{1/\beta_{n-2}}(t) \leq C U_{n-3}^{1/\beta_{n-3}}(t), \quad t \in (0, T).$$

Hence, u_{n-3} blows up if u_{n-2} blows up. On the other hand, if u_{n-3} blows up, then u_{n-2} must be the blow-up component. Otherwise, u_{n-3} would remain bounded for $p_{n-3} \leq 1$. □

Lemma 4.4. *Assume that the conditions of Theorem 4.1 hold. For the fixed $\lambda_1 = \bar{\lambda}_1$ in Lemma 4.2, there exists some $\lambda_2'' \in (1/2, 1)$ such that only u_n blows up with the initial data in $\mathbb{V}_2$, where $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \lambda_2''$.*

Proof. Consider the auxiliary problem

$$\begin{cases} (\underline{u}_n)_t = \Delta \underline{u}_n + \underline{u}_n^{p_n}, & (x,t) \in B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,t) = 0, & (x,t) \in \partial B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,0) = \dfrac{\breve{u}_{n,0}(x)}{(1-\bar{\lambda}_1)(1-\lambda_2)}, & x \in B_R \end{cases}$$

with λ_2 to be determined. Take

$$M_{n-2}=\left[\frac{2\breve{u}_{n-2,0}(0)}{(1-\bar{\lambda}_1)}\right]^{p_{n-2}}.$$

There exists some $\lambda_2''\in(1/2,1)$ such that, if $\lambda_2=\lambda_2''$, then $\underline{T}_n$ satisfies

$$\frac{2\breve{u}_{n-2,0}(0)}{1-\bar{\lambda}_1}+M_{n-2}\underline{T}_n+M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}}\underline{T}_n\le M_{n-2}^{\frac{1}{p_{n-2}}}$$

with M_{n-1} defined in Lemma 4.2.

Choose the initial data in $\mathbb{V}_2$ with $\lambda_1=\bar{\lambda}_1$ and $\lambda_2=\lambda_2''$, then $\underline{u}_n\le u_n$ and $T\le\underline{T}_n$, and hence

$$\frac{2\breve{u}_{n-2,0}(0)}{1-\bar{\lambda}_1}+M_{n-2}T+M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}}T< M_{n-2}^{\frac{1}{p_{n-2}}}.$$

Consider the auxiliary problem

$$(4.10)\qquad\begin{cases}(\bar{u}_{n-2})_t=\Delta\bar{u}_{n-2}\\ \qquad +M_{n-2}+M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}},\ (x,t)\in B_R\times(0,+\infty),\\ \bar{u}_{n-2}(x,t)=0,\ (x,t)\in\partial B_R\times(0,+\infty),\\ \bar{u}_{n-2}(x,0)=\dfrac{2\breve{u}_{n-2,0}(x)}{1-\bar{\lambda}_1},\ x\in B_R.\end{cases}$$

Consider system (4.10) in $[0,T)$. By Green's identity, $\bar{u}_{n-2}\le M_{n-2}^{\frac{1}{p_{n-2}}}$. Then

$$(\bar{u}_{n-2})_t\ge\Delta\bar{u}_{n-2}+\bar{u}_{n-2}^{p_{n-2}}+M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}},\quad (x,t)\in B_R\times(0,T).$$

By $u_{n-1}\le M_{n-1}^{1/p_{n-1}}$, u_{n-2} satisfies

$$(u_{n-2})_t\le\Delta u_{n-2}+u_{n-2}^{p_{n-2}}+M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}},\quad (x,t)\in B_R\times(0,T).$$

So by the comparison principle, u_{n-2} is bounded. For $p_{n-3}\le 1$, u_{n-3} is also bounded. So u_n is the blow-up component. $\square$

Lemma 4.5. (i) *There exists some small R such that the initial data set in $\mathbb{V}_0$ satisfying u_{n-3},u_{n-2} blowing up simultaneously at time T while the others remaining bounded is open in L^∞-topology.*

(ii) *The initial data set in $\mathbb{V}_0$ such that u_n blows up at time T while the others remain bounded is open in L^∞-topology.*

Proof. We only prove (i). Case (ii) can be proved, similarly.

Assume that $(u_1,u_2,\cdots,u_n)$ is the blow-up solution of (1.1) with

$$(u_{1,0},u_{2,0},\cdots,u_{n,0})\in\mathbb{V}_1p,$$

satisfying that u_{n-3}, u_{n-2} blow up simultaneously at time T while the others remain bounded. Let

$$0 < 2\xi \le u_j(0,t) \le M, \quad j = 1,2,\cdots,n-4,n-1,n.$$

It suffices to prove that there exists a neighborhood of $(u_{1,0}, u_{2,0}, \cdots, u_{n,0})$ in $\mathbb{V}_1$ such that every solution $(\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_n)$, coming from it, satisfies that $\hat{u}_{n-3}, \hat{u}_{n-2}$ blow up simultaneously in finite time while the others remain bounded. Take constants

$$S_j > (2M + 2\xi)^{p_j}, \quad j = 1,2,\cdots,n-4,n-1,n.$$

Let $(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n)$ solve the following system

$$\begin{cases} (\tilde{u}_j)_t = \Delta\tilde{u}_j + \tilde{u}_j^{p_j} + \tilde{u}_{j+1}^{q_{j+1}}(0,t), & (x,t) \in B_R \times (0,T_0), \\ \tilde{u}_j(x,t) = 0, & (x,t) \in \partial B_R \times (0,T_0), \\ \tilde{u}_j(x,0) = \tilde{u}_{j,0}(x),\ j = 1,2,\cdots,n, & x \in B_R \end{cases}$$

with $(\tilde{u}_{1,0}, \tilde{u}_{2,0}, \cdots, \tilde{u}_{n,0})$ to be determined in $\mathbb{V}_0$.

Denote

$$\begin{aligned} \mathcal{N}(u_{1,0}, u_{2,0}, \cdots, u_{n,0}) = \Big\{ & (\tilde{u}_{1,0}, \tilde{u}_{2,0}, \cdots, \tilde{u}_{n,0}) \in \mathbb{V}_0 | \\ & \|\tilde{u}_{j,0}(x) - u_j(x, T-\varepsilon_0)\|_\infty < \xi, \quad j = 1,2,\cdots,n, \\ & (\tilde{u}_{1,0}, \tilde{u}_{2,0}, \cdots, \tilde{u}_{n,0}) \\ & p = (\hat{u}_1(x,T-\varepsilon_0), \hat{u}_2(x,T-\varepsilon_0), \cdots, \hat{u}_n(x,T-\varepsilon_0)), \\ & (\hat{u}_{1,0}, \hat{u}_{2,0}, \cdots, \hat{u}_{n,0}) \in \mathbb{V}_1 \Big\}. \end{aligned}$$

For fixed $\xi > 0$, there exists some $\varepsilon_0 > 0$ such that if

$$(\tilde{u}_{1,0}, \tilde{u}_{2,0}, \cdots, \tilde{u}_{n,0}) \in \mathcal{N}(u_{1,0}, u_{2,0}, \cdots, u_{n,0}),$$

then

$$\tilde{u}_{n-2}(0,t) \le [(p_{n-2}-1)\eta\phi(0,T_0)]^{-\frac{1}{p_{n-2}-1}} (T_0 - t)^{-\frac{1}{p_{n-2}-1}},$$

and T_0 satisfies

$$\begin{aligned} & \eta_0 + T_0 < 1, \\ & S_{n-4} \ge \Big(2M + 2\xi + S_{n-4}T_0 + \frac{1}{1 - q_{n-3}\alpha_{n-3}} S_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} T_0^{1 - q_{n-3}\alpha_{n-3}}\Big)^{p_{n-4}}, \\ & S_j \ge \Big(2M + 2\xi + S_jT_0 + S_{j+1}^{\frac{q_{j+1}}{p_{j+1}}} T_0\Big)^{p_j}, \quad j = 1,2,\cdots,n-5,n-1,n, \\ & S_{n-3} = \Big\{\alpha_{n-3}^{-1}(1 - \eta_0 - T_0)^{-1}[(p_{n-2}-1)\eta\phi(0,T_0)]^{-\frac{q_{n-2}}{p_{n-2}-1}}\Big\}^{p_{n-3}}, \end{aligned}$$

where $\eta_0 = \displaystyle\int_{B_R} \Gamma dy < 1$ for small R.

Due to the Green's identity and the jump relation, if $\tilde{u}_{n-3,0}(0) \geq 1$, we obtain

$$\tilde{u}_{n-3}(0,t) \leq S_{n-3}^{\frac{1}{p_{n-3}}}(T_0-t)^{-\alpha_{n-3}}.$$

Consider the following problem

$$\begin{cases} (\bar{u}_{n-4})_t = \Delta\bar{u}_{n-4} + S_{n-4} + S_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}(T_0-t)^{-q_{n-3}\alpha_{n-3}},\ (x,t) \in B_R \times (0,T_0), \\ \bar{u}_{n-4}(x,t) = 0,\ (x,t) \in \partial B_R \times (0,T_0), \\ \bar{u}_{n-4}(x,0) = \bar{u}_{n-4,0}(x),\ x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-4,0}(0) = 2\tilde{u}_{n-4,0}(0)$; $\bar{u}_{n-4,0}(x) \geq \tilde{u}_{n-4,0}(x)$ in B_R and

$$\Delta\bar{u}_{n-4,0} + S_{n-4} + S_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}T_0^{-q_{n-3}\alpha_{n-3}} \geq 0 \quad \text{in } B_R.$$

By Green's identity, we have $\bar{u}_{n-4} \leq S_{n-4}^{\frac{1}{p_{n-4}}}$; consequently,

$$(\bar{u}_{n-4})_t \geq \Delta\bar{u}_{n-4} + \bar{u}_{n-4}^{p_{n-4}} + +S_{n-3}^{\frac{q_{n-3}}{p_{n-3}}}(T_0-t)^{-q_{n-3}\alpha_{n-3}}, \quad (x,t) \in B_R \times (0,T_0).$$

Then, by the comparison principle, we have

$$\tilde{u}_{n-4} \leq \bar{u}_{n-4} \leq S_{n-4}^{1/p_{n-4}}, \quad (x,t) \in B_R \times (0,T).$$

Next, consider another system

$$\begin{cases} (\bar{u}_{n-5})_t = \Delta\bar{u}_{n-5} + S_{n-5} + S_{n-4}^{\frac{q_{n-4}}{p_{n-4}}}, & (x,t) \in B_R \times (0,+\infty), \\ \bar{u}_{n-5}(x,t) = 0, & (x,t) \in \partial B_R \times (0,+\infty), \\ \bar{u}_{n-5}(x,0) = \bar{u}_{n-5,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-5,0}(0) = 2\tilde{u}_{n-5,0}(0)$; $\bar{u}_{n-5,0}(x) \geq \tilde{u}_{n-5,0}(x)$ in B_R and

$$\Delta\bar{u}_{n-5,0} + S_{n-5} + S_{n-4}^{\frac{q_{n-4}}{p_{n-4}}} \geq 0, \quad (x,t) \in B_R \times (0,T).$$

We also obtain that

$$\tilde{u}_{n-5} \leq \bar{u}_{n-5} \leq S_{n-5}^{1/p_{n-5}}, \quad (x,t) \in B_R \times (0,T_0).$$

Similarly, $\tilde{u}_j$, $j = n-6, n-7, \cdots, 1, n, n-1$ remain bounded. By the methods used in Lemma 4.3, we obtain that $\tilde{u}_{n-3}$ and $\tilde{u}_{n-2}$ blow up simultaneously at time T_0.

According to the continuity with respect to initial data for bounded solutions, there must exist a neighborhood $\mathbb{N}_1(\subset \mathbb{V}_0)$ of $(u_{1,0}, u_{2,0}, \cdots, u_{n,0})$ such that every solution $(\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_n)$, starting from $\mathbb{N}_1$, will enter the set $\mathcal{N}$ at time $T - \varepsilon_0$, and then keeps the property that $\hat{u}_{n-3}, \hat{u}_{n-2}$ blow up simultaneously while the others still remain bounded. □

Proof of Theorem 4.1. Lemma 4.2 says that there exists some $\bar{\lambda}_1 \in (1/2, 1)$ such that any initial data in $\mathbb{V}_2$ satisfying $\lambda_1 = \bar{\lambda}_1$ develops the non-simultaneous blow-up solution with u_j, $j = 1, 2, \cdots, n-4, n-1$ remaining bounded. We know from Lemma 4.3 that there exists some $\lambda_2' \in (0, 1/2)$ such that the solution of (1.1) with the initial data in $\mathbb{V}_2$ satisfying $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \lambda_2'$ blows up non-simultaneously, where u_{n-3}, u_{n-2} blow up simultaneously and the others are bounded. Lemma 4.4 guarantees that there exists some $\lambda_2'' \in (1/2, 1)$ such that u_n blows up alone with the initial data in $\mathbb{V}_2$ where $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \lambda_2''$. Clearly, the sets of the initial data in $\mathbb{V}_2$ such that only u_{n-3}, u_{n-2} blow up simultaneously and that u_n blows up alone are all open by Lemma 4.5. Notice that $\mathbb{V}_2$ is connected. So there must exist suitable initial data (suitable $\bar{\lambda}_2 \in (\lambda_2', \lambda_2'')$) such that u_n, u_{n-3} and u_{n-2} blow up simultaneously while the others remain bounded. The blow-up rates and sets can be obtained by the methods to establish the ones in Theorem 2.1. $\square$

By the proof of Theorem 4.1, one can check that, if $k_1 = k_2 = 0$, the results of Theorem 2.4 still holds without the restriction on the radius R.

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