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Author(s):

J.J. Ao

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ON TWO CLASSES OF THIRD ORDER BOUNDARY VALUE PROBLEMS WITH FINITE SPECTRUM

J.J. AO

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Abstract. The spectral analysis of two classes of third order boundary value problems is investigated. For every positive integer m we construct two classes of regular third order boundary value problems with at most $2m+1$ eigenvalues, counting multiplicity. These kinds of finite spectrum results are previously known only for even order boundary value problems.
Keywords: Eigenvalues, finite spectrum, third order boundary value problems.
MSC(2010): Primary: 34B24; Secondary: 34L15, 34L05.

1. Introduction

It is well known [5, 6, 13] that the spectrum of classical self-adjoint boundary value problems is unbounded and therefore infinite. These are problems with a positive leading coefficient p and a positive weight function w. In 1964 Atkinson, in his well known book [5], weakened these conditions to $1/p \geq 0$ and $w \geq 0$ for the second order i.e. Sturm-Liouville(S-L) case and suggested that when $1/p$ and w are identically zero on subintervals of the domain interval (which is allowed by the general theory of differential equations) there may only be a finite number of eigenvalues. But he gave no example to illustrate this. In 2001 Kong, Wu and Zettl [11] constructed, for each positive integer m, regular self-adjoint and non-self-adjoint Sturm-Liouville problems with separated and coupled boundary conditions whose spectrum consists of exactly m eigenvalues. Recently, Ao et al. generalized the finite spectrum results to fourth order boundary value problems [2, 3], and 2nth order boundary value problems [4]. However, these results are only restricted into even order problems, and there is no such finite spectrum results for odd order problems.

As is well known that the odd order differential boundary value problems, such as first order or third order problems arise in a variety of different areas

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of applied mathematics and physics [10, 12]. There are significant many researchers study these problems in various aspects, here we only refer some of them [1, 7, 10, 12].

Motivated by the above results, in this paper, we construct similar finite spectrum problems of order $n = 3$. As far as we know, this is the first such examples for odd order cases. As in [11] our construction is based on the characteristic function whose zeros are the eigenvalues of the problem. But the analysis of this function for $n = 3$ is considerably more complicated than the $n = 2$ case and more generally, in this paper, we discuss two kinds of third order problems which are generated by different equation types.The key to this analysis is still an iterative construction of the characteristic function. At the end of this paper we illustrate our results by two examples.

The paper is organized as follows: Following this Introduction, Section 2 contains some basic notations and preliminaries. The statements and proofs of finite spectrum of the first class of third order problems are given in Section 3. A brief argument of second class of third order problems and two examples are given in Section 4 to illustrate our results.

2. Notation and preliminaries

We consider two classes of third order boundary value problems (BVPs) in this paper. The first is the third order BVP consisting of the equation

$$(py'')' + qy = \lambda wy,\ on\ J = (a, b),\ with\ -\infty < a < b < +\infty \tag{2.1}$$

together with boundary conditions. The second is the third order BVP consisting of the equation

$$(py')'' + qy = \lambda wy,\ on\ J = (a, b),\ with\ -\infty < a < b < +\infty$$

together with boundary conditions. In Sections 2 and 3 we only consider equation (2.1) and detailed analysis for class two will be presented in Section 4.

Here λ is the spectral parameter and the coefficients satisfy the minimal conditions

$$r = 1/p, q, w \in L(J, \mathbb{C}), \tag{2.2}$$

where $L(J, \mathbb{C})$ denotes the complex valued functions which are Lebesgue integrable on J. Condition (2.2) is minimal in the sense that it is necessary and sufficient for all initial value problems of equation (2.1) to have unique solutions on $[a, b]$; see [8, 13].

Under the minimal conditions (2.2) it is convenient to use the system formulation of equation (2.1) and to introduce quasi-derivatives u_j as follows [6, 9]:

Let $u_1 = y,\ u_2 = y',\ u_3 = py''$. Then we have the system representation of (2.1) as follows:

$$u_1' = u_2,\ u_2' = ru_3,\ u_3' = (\lambda w - q)u_1,\ on\ J. \tag{2.3}$$

This can be written in the following matrix form:

$$U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{p} \\ \lambda w - q & 0 & 0 \end{pmatrix} U, \ U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \ \text{on } J.$$

Remark 2.1. Note that condition (2.2) does not restrict the sign of any of the coefficients r, q, w. Also, each of r, q, w is allowed to be identically zero on subintervals of J. If r is identically zero on a subinterval I, then there exists a solution y which is identically zero on I, but one of its quasi-derivatives $u_2 = y',\ u_3 = py''$ may be a nonzero constant function on I.

Definition 2.2. By a trivial solution of equation (2.1) on an interval $I \subset J$ we mean a solution y which is identically zero on I and whose quasi-derivatives $u_2 = y',\ u_3 = py''$ are also identically zero on I.

We consider two point boundary conditions (BCs) of the form

$$AY(a) + BY(b) = 0, \ Y = \begin{pmatrix} y \\ y' \\ py'' \end{pmatrix}, \quad A, B \in M_3(\mathbb{C}), \tag{2.4}$$

where $M_3(\mathbb{C})$ denotes the set of square matrices of order 3 over the complex numbers $\mathbb{C}$.

Lemma 2.3. *Let (2.2) hold and let $\Phi(x, \lambda) = [\phi_{ij}(x, \lambda)]$ denote the fundamental matrix of the system (2.3) determined by the initial condition $\Phi(a, \lambda) = I$. Then a complex number λ is an eigenvalue of the third order problem (2.1), (2.4) if and only if*

$$\Delta(\lambda) = \mathbf{det}[A + B\,\Phi(b, \lambda)] = 0. \tag{2.5}$$

Proof. Suppose $\Delta(\lambda) = 0$. Then $[A + B\,\Phi(b, \lambda)]\,C = 0$ has a nontrivial vector solution. After solving the initial value problem

$$Y' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{p} \\ \lambda w - q & 0 & 0 \end{pmatrix} Y, \ Y = \begin{pmatrix} y \\ y' \\ py'' \end{pmatrix} \ \text{on } J, \ Y(a) = C,$$

we get $Y(b) = \Phi(b, \lambda)Y(a)$ and $[A + B\,\Phi(b, \lambda)]Y(a) = 0$.

From this it follows that the top component of Y, say, y is an eigenfunction of the third order problem (2.1), (2.4); this means λ is an eigenvalue of this problem. Conversely, if λ is an eigenvalue and y an eigenfunction of λ, then $Y = \begin{pmatrix} y \\ y' \\ py'' \end{pmatrix}$ satisfies $Y(b) = \Phi(b, \lambda)Y(a)$ and consequently $[A + B\,\Phi(b, \lambda)]Y(a) = 0$. If $Y(a) = 0$, then y is the trivial solution which contradicts the fact that y is an eigenfunction, so we have $\mathbf{det}[A + B\,\Phi(b, \lambda)] = 0$. □

Next we find a formula for $\Delta(\lambda)$ which highlights its dependence on λ and on the matrices $A,\ B$.

Lemma 2.4. *Let (2.2) hold and let $\Phi(x,\lambda) = [\phi_{ij}(x,\lambda)]$ denote the fundamental matrix of the system (2.3) determined by the initial condition $\Phi(a,\lambda) = I$. Then the characteristic function $\Delta(\lambda) = \mathbf{det}[A + B\Phi(b,\lambda)]$ can be written as*

$$\Delta(\lambda) = \mathbf{det}(A) + \mathbf{det}(B) + \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}\phi_{ij} + \sum_{1\le i,j,k,l\le 4,\ j\neq l} d_{ijkl}\phi_{ij}\phi_{kl}, \tag{2.6}$$

where $c_{ij}, 1 \le i,j \le 4,\ d_{ijkl}, 1 \le i,j,k,l \le 4, j \neq l$ are constants which depend only on the matrices A and B.

Proof. This follows from a tedious but straightforward computation. □

Remark 2.5. For simplicity, in (2.6) we write ϕ_{ij} as ϕ^1, $\phi_{ij}\phi_{kl}$ as ϕ^2, and regardless of their subscripts just to indicate that they are products of one or two functions of λ.

The third order problem (2.1) and (2.4), or equivalently (2.3) and (2.4), is said to be degenerate if in (2.6) either $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for every $\lambda \in \mathbb{C}$.

3. Third order problems with finite spectrum

In this section we assume (2.2) holds and there exists a partition of the interval J

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b, \tag{3.1}$$

for some odd integer $n = 2m+1$, such that

$$r = \frac{1}{p} = 0 \text{ on } [a_{2k}, a_{2k+1}],\ \int_{a_{2k}}^{a_{2k+1}} w \neq 0, \tag{3.2}$$

$$\int_{a_{2k}}^{a_{2k+1}} w(x)x dx \neq 0,\ k = 0,1,\ldots,m;$$

and

$$q = w = 0 \text{ on } [a_{2k+1}, a_{2k+2}],\ \int_{a_{2k+1}}^{a_{2k+2}} r \neq 0, \tag{3.3}$$

$$\int_{a_{2k+1}}^{a_{2k+2}} r(x)x dx \neq 0,\ k = 0,1,\ldots,m-1.$$

Definition 3.1. We say that a third order equation (2.1) is of Atkinson type if the conditions (3.1)-(3.3) hold.

We also need some additional notations which we will state here. Given (3.1)-(3.3), let

$$r_k = \int_{a_{2k+1}}^{a_{2k+2}} r, \ \hat{r}_k = \int_{a_{2k+1}}^{a_{2k+2}} r(x)xdx, \ k = 0, 1, \ldots, m-1; \tag{3.4}$$

$$q_k = \int_{a_{2k}}^{a_{2k+1}} q, \ \hat{q}_k = \int_{a_{2k}}^{a_{2k+1}} q(x)xdx, \ k = 0, 1, \ldots, m;$$

$$w_k = \int_{a_{2k}}^{a_{2k+1}} w, \ \hat{w}_k = \int_{a_{2k}}^{a_{2k+1}} w(x)xdx, \ k = 0, 1, \ldots, m;$$

Following [11] we determine the structure of the principal fundamental matrix of system (2.3) which is basic to our results.

Lemma 3.2. *Let (2.2), (3.1)-(3.3) hold. Let $\Phi(x,\lambda) = [\phi_{ij}(x,\lambda)]$ be the fundamental matrix solution of the system (2.3) determined by the initial condition $\Phi(a,\lambda) = I$ for each $\lambda \in \mathbb{C}$. Let*

$$F_k(x,\lambda,a_k) = \begin{pmatrix} 1 & x-a_k & 0 \\ 0 & 1 & 0 \\ \int_{a_k}^{x}(\lambda w - q)dt & \int_{a_k}^{x}(\lambda w - q)(t-a_k)dt & 1 \end{pmatrix}, \tag{3.5}$$

$$k = 0, 2, \ldots, 2m;$$

$$F_k(x,\lambda,a_k) = \begin{pmatrix} 1 & x-a_k & \int_{a_k}^{x} r(x-t)dt \\ 0 & 1 & \int_{a_k}^{x} rdt \\ 0 & 0 & 1 \end{pmatrix}, k = 1, 3, \ldots, 2m-1.$$

Then for $1 \le k \le 2m+1$ we have

$$\Phi(a_k,\lambda) = F_{k-1}(a_k,\lambda,a_{k-1})\,\Phi(a_{k-1},\lambda). \tag{3.6}$$

And more simpler, if we let

$$T_0 = F_0(a_1,\lambda,a_0), \ T_k = F_{2k}(a_{2k+1},\lambda,a_{2k})\,F_{2k-1}(a_{2k},\lambda,a_{2k-1}), \ k = 1, 2, \ldots, m,$$

then

$$\Phi(a_1,\lambda) = F_0(a_1,\lambda,a_0) = T_0, \ \Phi(a_{2k+1},\lambda) = T_k\,\Phi(a_{2k-1},\lambda), \ k = 1, 2, \ldots, m.$$

Hence we have the following formula

$$\Phi(a_{2k+1},\lambda) = T_k T_{k-1} \cdots T_0, \ k = 0, 1, \ldots, m.$$

Proof. Observe from (2.3) that u_2 is constant on each subinterval $[a_{2k}, a_{2k+1}]$, $k = 0, 1, \ldots, m$, where r is identically zero, and thus on each of these subintervals we have

$$u_2(x) = u_2(a_{2k}), \tag{3.7}$$

$$u_1(x) = u_1(a_{2k}) + u_2(a_{2k})(x - a_{2k}),$$

and

$$u_3(x) = u_3(a_{2k}) + u_1(a_{2k}) \int_{a_{2k}}^{x} (\lambda w - q)dt + u_2(a_{2k}) \int_{a_{2k}}^{x} (\lambda w - q)(t - a_{2k})dt.$$

Similarly, because q and w are identically zero, u_4 is constant on each subinterval $[a_{2k-1}, a_{2k}],\ k = 1, 2, \ldots, m-1$, so we have

$$u_3(x) = u_3(a_{2k-1}), \tag{3.8}$$

$$u_2(x) = u_2(a_{2k-1}) + u_3(a_{2k-1}) \int_{a_{2k-1}}^{x} r dt,$$

and

$$u_1(x) = u_1(a_{2k-1}) + u_2(a_{2k-1})(x - a_{2k-1}) + u_3(a_{2k-1}) \int_{a_{2k-1}}^{x} r(x-t)dt.$$

We see that $u_i(x), i = 1, 2, 3$ are piecewise continuous functions on $[a, b]$. Let $U(x) = [u_1(x), u_2(x), u_3(x)]^T$ on $[a, b]$, and set $U^{(j)}(x, \lambda) = U(x, \lambda; e_j)$, where $e_j,\ j = 1, 2, 3$ are the standard unit vectors, then it is easy to see that $\Phi(x, \lambda) = [U^{(1)}\ U^{(2)}\ U^{(3)}]$. This establishes (3.6). □

Note that $b = a_{2m+1}$. The structure of Φ given in Lemma 3.2 yields the following:

Corollary 3.3. *For the fundamental matrix Φ we have that*

$$\phi_{ij}(b, \lambda) = R_{ij}\lambda^m + \tilde{\phi}_{ij}(\lambda),\ i, j = 1, 2,\ \text{or}\ i = j = 3;$$

$$\phi_{ij}(b, \lambda) = R_{ij}\lambda^{m+1} + \tilde{\phi}_{ij}(\lambda),\ i = 3,\ j = 1, 2; \tag{3.9}$$

$$\phi_{ij}(b, \lambda) = R_{ij}\lambda^{m-1} + \tilde{\phi}_{ij}(\lambda),\ i = 1, 2,\ j = 3,$$

where R_{ij} are constants related to $r_k, \hat{r}_k, k = 0, 1, \ldots m-1,\ w_k, \hat{w}_k, k = 0, 1, \ldots, m$ and the end points $a,\ b,\ \tilde{\phi}_{ij}(\lambda)$ are functions of λ, in which the degrees of λ are smaller then $m, m+1,$ or $m-1$ respectively. For example, $\phi_{11}(b, \lambda) = R_{11}\lambda^m + \tilde{\phi}_{11}(\lambda)$, so the degree of λ in $\tilde{\phi}_{11}(\lambda)$ is smaller then m.

Now we construct regular third order problems with general self-adjoint and non-self-adjoint BCs which have at most $2m+1$ eigenvalues for each $m \in \mathbb{N}$.

Theorem 3.4. *Let $m \in \mathbb{N}$, and let (2.2), (3.1)-(3.3) hold. Then the third order boundary value problem (2.1), (2.4) has at most $2m+1$ eigenvalues.*

Proof. Since $\Delta(\lambda) = \mathbf{det}[A + B\Phi(b, \lambda)]$, where $\Phi(b, \lambda) = [\phi_{ij}(b, \lambda)]$. From Lemma 2.4 and Corollary 3.3 we know that the characteristic function $\Delta(\lambda)$ is a polynomial of λ and has the form of (2.6). We denote the maximum of degree of λ in $\phi_{ij}(b,\lambda)$ by $d_{ij},\ 1 \le i,j \le 3$, by Corollary 3.3 the maximum of degree of λ in the matrix $\Phi(b,\lambda)$ can be written as the following matrix

$$(d_{ij}) = \begin{pmatrix} m & m & m-1 \\ m & m & m-1 \\ m+1 & m+1 & m \end{pmatrix}. \tag{3.10}$$

In term of (2.6) and (3.10), we conclude that the maximum of the degree of λ in $\Delta(\lambda)$ is $2m+1$. Thus from the Fundamental Theorem of Algebra, $\Delta(\lambda)$ has at most $2m+1$ roots. $\square$

The next theorem highlights the fact that every equation of Atkinson type is equivalent to an equation with piecewise polynomial coefficients. But first we give a lemma which is needed later.

Lemma 3.5. *Let $t_1, t_2 \in \mathbb{R}$ and $t_1 \ne t_2$. Then for any $\eta_0, \eta_1 \in \mathbb{R}$, there exists a unique polynomial $P(x) = \tau_1 x + \tau_0$, such that*

$$\int_{t_1}^{t_2} P(x)\,dx = \eta_0, \ \int_{t_1}^{t_2} P(x)\,x\,dx = \eta_1. \tag{3.11}$$

Proof. For the proof see [3]. $\square$

Denote the polynomials constructed in Lemma 3.5 by $\chi(t_1,t_2,\eta_0,\eta_1)$ to highlight their dependence on these parameters. Define piecewise polynomial functions $\bar p(x), \bar q(x)$ and $\bar w(x)$ on J by

$$\bar p(x) = \begin{cases} \chi^{-1}(a_{2k-1}, a_{2k}, r_k, \hat r_k), & x \in [a_{2k-1}, a_{2k}],\ k = 1,2,\ldots,m \\ \infty, & x \in [a_{2k}, a_{2k+1}],\ k = 0,1,\ldots,m; \end{cases}$$

$$\bar q(x) = \begin{cases} \chi(a_{2k}, a_{2k+1}, q_k, \hat q_k), & x \in [a_{2k}, a_{2k+1}],\ k = 0,1,\ldots,m \\ 0, & x \in [a_{2k-1}, a_{2k}],\ k = 1,2,\ldots,m; \end{cases} \tag{3.12}$$

$$\bar w(x) = \begin{cases} \chi(a_{2k}, a_{2k+1}, w_k, \hat w_k), & x \in [a_{2k}, a_{2k+1}],\ k = 0,1,\ldots,m \\ 0, & x \in [a_{2k-1}, a_{2k}],\ k = 1,2,\ldots,m. \end{cases}$$

Then we have the following Theorem.

Theorem 3.6. *Let (2.2) hold. Assume equation (2.1) is of Atkinson type, and let $\bar p(x), \bar q(x)$ and $\bar w(x)$ on J be defined by (3.12), where $r_k, \hat r_k,\ k = 1,2,\ldots,m$, and $q_k, \hat q_k, w_k, \hat w_k,\ k = 0,1,\ldots,m$, are given by (3.4). Then the BVP (2.1), (2.4) has exactly the same eigenvalues as the third order BVP consisting of the equation*

$$(\bar p y'')' + \bar q y = \lambda \bar w y, \quad on \quad J = (a,b), \tag{3.13}$$

and with the same BC (2.4).

Proof. From Lemma 3.5 and (3.12) we know that the BVP (2.1), (2.4) and the BVP (3.13), (2.4) define the same $r_k, \hat{r}_k, k = 1, 2, \dots, n,\ q_k, \hat{q}_k, w_k, \hat{w}_k, k = 0, 1, 2, \dots, n$, and under the same boundary conditions, they define the same characteristic function $\Delta(\lambda)$, hence have the same eigenvalues. □

By Theorem 3.6 and its proof we see that for a fixed BC (2.4) on a given interval J, there is a family of problems of Atkinson type which has exactly the same eigenvalues as the problem (3.13), (2.4), with piecewise polynomial coefficients. Such a family is called the equivalent family of (3.13), (2.4).

4. The second class of problems and examples

In this section we discuss another class of third order BVP consisting of the equation

$$(py')'' + qy = \lambda wy, \ on \ J = (a, b), \ with \ -\infty < a < b < +\infty \tag{4.1}$$

together with boundary conditions. Although this equation is different from equation (2.1), the argument about the finite spectrum results is similar to the one that matches the first class of problems we mentioned in the previous sections. Hence in the following we only claim the different aspects of the problem briefly.

Assume that the conditions such as (2.2), (3.1)-(3.3) and so on still hold. Let $u_1 = y,\ u_2 = py',\ u_3 = (py')'$. Then we have the following system:

$$u_1' = ru_2,\ u_2' = u_3,\ u_3' = (\lambda w - q)u_1, \ on \ J. \tag{4.2}$$

By replacing two point boundary conditions (2.4) into

$$AY(a) + BY(b) = 0,\ Y = \begin{pmatrix} y \\ py' \\ (py')' \end{pmatrix}, \quad A, B \in M_3(\mathbb{C}), \tag{4.3}$$

equations (3.5) turn into

$$F_k(x, \lambda, a_k) = \begin{pmatrix} 1 & 0 & 0 \\ \int_{a_k}^x (\lambda w - q)(x - t)dt & 1 & x - a_k \\ \int_{a_k}^x (\lambda w - q)dt & 0 & 1 \end{pmatrix}, \quad k = 0, 2, \dots, 2m; \tag{4.4}$$

$$F_k(x, \lambda, a_k) = \begin{pmatrix} 1 & \int_{a_k}^x r dt & \int_{a_k}^x r(t - a_k)dt \\ 0 & 1 & x - a_k \\ 0 & 0 & 1 \end{pmatrix}, k = 1, 3, \dots, 2m - 1,$$

and (3.9), (3.10), (3.13) turn into

$$\phi_{ij}(b, \lambda) = R_{ij}\lambda^m + \tilde{\phi}_{ij}(\lambda),\ i, j = 2, 3, \ or \ i = j = 1;$$

$$\phi_{ij}(b, \lambda) = R_{ij}\lambda^{m+1} + \tilde{\phi}_{ij}(\lambda),\ j = 1,\ i = 2, 3; \tag{4.5}$$

$$\phi_{ij}(b,\lambda) = R_{ij}\lambda^{m-1} + \tilde{\phi}_{ij}(\lambda),\ j=2,3,\ i=1,$$

$$(d_{ij}) = \begin{pmatrix} m & m-1 & m-1 \\ m+1 & m & m \\ m+1 & m & m \end{pmatrix}, \tag{4.6}$$

$$(\bar{p}y')'' + \bar{q}y = \lambda\bar{w}y, \quad on \quad J=(a,b), \tag{4.7}$$

respectively.

Then we can conclude that the similar statements in Lemmas and Theorems for the first class of third order BVPs (2.1), (2.4) still hold for the second class of BVPs (4.1), (4.3). All the statements and proofs are similar to the previous statements and proofs, hence is omitted here.

Now we illustrate our results by two examples for each class of problems respectively.

Example 4.1. Consider the third order boundary value problem of first class

$$\begin{cases} (py'')' + qy = \lambda wy,\ on\ J=(-2,5), \\ y(-2) + py''(-2) + y(5) + y'(5) + 2py''(5) = 0, \\ y'(-2) + 2py''(-2) + 2y(5) + 1/2y'(5) - py''(5) = 0, \\ y'(-2) + 3y(5) + 7y'(5) + 3py''(5) = 0. \end{cases} \tag{4.8}$$

Let $m=2$ and $p,\ q,\ w$ are piecewise polynomial functions defined as follows:

(4.9)

$$p(x) = \begin{cases} \infty,\ (-2,0) \\ 1/(2x+1),\ (0,1) \\ \infty,\ (1,3) \\ 1/2,\ (3,4) \\ \infty,\ (4,5), \end{cases} \quad q(x) = \begin{cases} x,\ (-2,0) \\ 0,\ (0,1) \\ 2,\ (1,3) \\ 0,\ (3,4) \\ 1,\ (4,5), \end{cases} \quad w(x) = \begin{cases} 1/2,\ (-2,0) \\ 0,\ (0,1) \\ x-1,\ (1,3) \\ 0,\ (3,4) \\ 2/3,\ (4,5). \end{cases}$$

From the conditions given we know that

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1/2 & -1 \\ 3 & 7 & 3 \end{pmatrix}. \tag{4.10}$$

Then we can deduce that the characteristic function

$$\Delta(\lambda) = -12250/27\lambda^3 - 7463/27\lambda^2 + 67787/36\lambda - 69707/36. \tag{4.11}$$

Hence the third order BVP (4.8), (4.9) has exactly $m+1=3$ eigenvalues

$$\lambda_0 = -2.7159,\ \lambda_1 = 1.0534 + 0.6796i,\ \lambda_2 = 1.0534 - 0.6796i.$$

Example 4.2. Consider the third order boundary value problem

$$
(4.12) \quad \begin{cases} (py')'' + qy = \lambda wy, \ \text{on} \ J = (-2, 5), \\ y(-2) + (py')'(-2) + y(5) + (py')(5) + 2(py')'(5) = 0, \\ (py')(-2) + 2(py')'(-2) + 2y(5) + 1/2(py')(5) - (py')'(5) = 0, \\ (py')(-2) + 3y(5) + 7(py')(5) + 3(py')'(5) = 0. \end{cases}
$$

Still let $m = 2$ and $p, \ q, \ w$ are piecewise polynomial functions defined as in (4.9).

Then we can deduce that the characteristic function

$$
(4.13) \qquad \Delta(\lambda) = -17353/54\lambda^3 + 44467/54\lambda^2 + 43325/72\lambda - 124711/72.
$$

Hence the third order BVP (4.12), (4.9) has exactly $m + 1 = 3$ eigenvalues

$$
\lambda_0 = -1.4216, \ \lambda_1 = 1.5715, \ \lambda_2 = 2.4126.
$$

The graphs of the characteristic functions in Examples 4.1 and 4.2 are displayed in Figures 1 and 2 respectively.

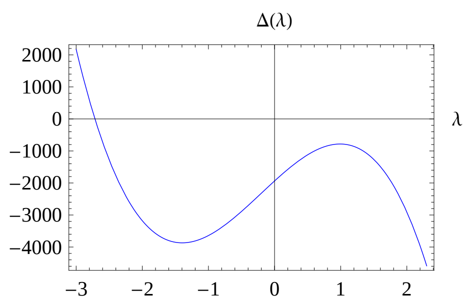

FIGURE 1. Characteristic Function in Example 4.1

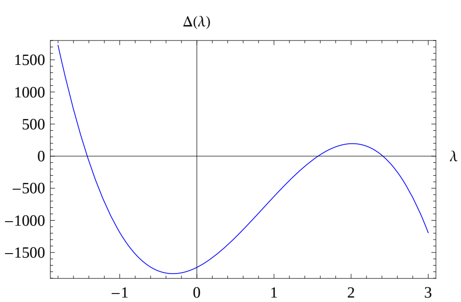

FIGURE 2. Characteristic Function in Example 4.2

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(Ji-jun Ao) School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
and
College of Sciences, Inner Mongolia University of Technology, Hohhot 010051, China.

E-mail address: george_ao78@sohu.com