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AMENABILITY AND WEAK AMENABILITY OF TRIANGULAR BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\mathcal{X}$ be a Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. Forrest and Marcoux have studied the n-weak amenability of triangular Banach algebras. We show that when $\mathcal{A}$ has a bounded approximate identity and $\mathcal{X}$ is essential, then $\mathcal{T}$ is weakly amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable. We also study the amenability of triangular Banach algebras and show that $\mathcal{T}$ is amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are amenable and $\mathcal{X} = \{0\}$.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{X}$ be a Banach $\mathcal{A},\mathcal{B}$-module. That is, $\mathcal{X}$ is a left Banach $\mathcal{A}$-module, a right Banach $\mathcal{B}$-module, $(ax)b = a(xb)$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x \in \mathcal{X}$ and there exists a constant $k0$ such that

$$\| axb \| \leq k \| a \| \| x \| \| b \|.$$

$\mathcal{X}$ is said to be essential provided that for every $x \in \mathcal{X}$ there are $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $y, z \in \mathcal{X}$ such that $x = ay = zb$. Let $\mathcal{X}^*$ be the topological dual of $\mathcal{X}$. Then $\mathcal{X}^*$ is a Banach $\mathcal{B}$, $\mathcal{A}$-module via the following actions

$$\langle x, bx^* \rangle = \langle xb, x^* \rangle, \qquad \langle x, x^*a \rangle = \langle ax, x^* \rangle$$

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for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$.

For $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ we define $xx^* \in \mathcal{A}^*$ and $x^*x \in \mathcal{B}^*$ by

$$\langle a, xx^* \rangle = \langle ax, x^* \rangle, \ \langle b, x^*x \rangle = \langle xb, x^* \rangle \ (a \in \mathcal{A}, \ b \in \mathcal{B}).$$

Similarly for $x \in \mathcal{X}$, $F_2 \in \mathcal{A}^{**}$ and $G_2 \in \mathcal{B}^{**}$ we define $F_2 x \in \mathcal{X}^{**}$ and $xG_2 \in \mathcal{X}^{**}$ via the actions (c.f.[6])

$$\langle x^*, F_2 x \rangle = \langle xx^*, F_2 \rangle, \ \langle x^*, xG_2 \rangle = \langle x^*x, G_2 \rangle \ (x^* \in \mathcal{X}^*).$$

We may continue this process to higher order dual spaces of $\mathcal{X}$, and $\mathcal{X}^{(2n)}$ is a Banach $\mathcal{A}$, $\mathcal{B}$-module, $\mathcal{X}^{(2n-1)}$ is a Banach $\mathcal{B}$, $\mathcal{A}$-module, $\mathcal{A}^{(2n)}\mathcal{X} \subseteq \mathcal{X}^{(2n)}, \mathcal{X}\mathcal{B}^{(2n)} \subseteq \mathcal{X}^{(2n)}, \mathcal{X}\mathcal{X}^{(2n-1)} \subseteq \mathcal{A}^{(2n-1)}$ and $\mathcal{X}^{(2n-1)}\mathcal{X} \subseteq \mathcal{B}^{(2n-1)}$ for all $n > 0$.

A Banach $\mathcal{A}, \mathcal{B}$-module $\mathcal{X}$ is called non-degenerate if $\mathcal{A}x = \{0\}$ implies $x = 0$ and $x\mathcal{B} = \{0\}$ implies $x = 0$ for all $x \in \mathcal{X}$. When $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identities and $\mathcal{X}$ is essential, then $\mathcal{X}$ is a non-degenerate Banach $\mathcal{A}, \mathcal{B}$-module. Also when $\mathcal{X}$ is essential, then $\mathcal{X}^*$ is a non-degenerate Banach $\mathcal{B}, \mathcal{A}$-module.

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A derivation $\delta : \mathcal{A} \longrightarrow \mathcal{X}$ is a linear map such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. The derivation δ is inner if it is of the form $\delta(a) = \delta_x(a) := ax - xa$ for some $x \in \mathcal{X}$. The linear space of all bounded derivations from $\mathcal{A}$ to $\mathcal{X}$ is denoted by $\mathcal{Z}^1(\mathcal{A}, \mathcal{X})$ and the linear subspace of all inner derivations by $\mathcal{N}^1(\mathcal{A}, \mathcal{X})$. The first Hochschild cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$ is defined to be the linear space $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = \mathcal{Z}^1(\mathcal{A}, \mathcal{X})/\mathcal{N}^1(\mathcal{A}, \mathcal{X})$ [15]. A Banach algebra $\mathcal{A}$ is said to be amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ (see, [1], [2], [3], [9], [10], [11], [12], [13]). A Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ ([1], [14], [17], [18], [19]) and $\mathcal{A}$ is called n-weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, where $\mathcal{A}^{(n)}$ is the n-th dual module of $\mathcal{A}$ when $n \geq 1$ and is $\mathcal{A}$ itself when $n = 0$ ([4]).

Forrest and Marcoux in [7] have studied a class of Banach algebras, which is called triangular Banach algebras. They have studied the n-weak amenability of triangular Banach algebras in [8]. They consider the cases where $\mathcal{A}$ and $\mathcal{B}$ have units and $\mathcal{X}$ is unital Banach $\mathcal{A}, \mathcal{B}$-module.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{X}$ be a Banach $\mathcal{A}, \mathcal{B}$-module. We define the corresponding triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ & \mathcal{B} \end{bmatrix}$ with the usual 2×2 matrix operations and obvious interval module actions, and the norm

$$\| \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \| = \| a \| + \| x \| + \| b \|.$$

In this paper $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\mathcal{X}$ is a Banach $\mathcal{A}$,$\mathcal{B}$- module and $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ & \mathcal{B} \end{bmatrix}$ is the corresponding triangular Banach algebra.

2. $(2n-1)$-weak amenability

Forrest and Marcoux in [8] proved the following theorem.

Theorem 2.1. *If for every continuous derivation $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n-1)}$ there exist continuous derivations $\delta_1 : \mathcal{A} \longrightarrow \mathcal{A}^{(2n-1)}$, $\delta_4 : \mathcal{B} \longrightarrow \mathcal{B}^{(2n-1)}$ and an element $\phi_0 \in \mathcal{X}^{(2n-1)}$ such that for all $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in \mathcal{T}$*

$$D(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}) = \begin{bmatrix} \delta_1(a) - x\phi_0 & \phi_0 a - b\phi_0 \\ 0 & \delta_4(b) + \phi_0 x \end{bmatrix},$$

then $\mathcal{H}^1(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq \mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1(\mathcal{B}, \mathcal{B}^{(2n-1)})$.

Proof. See [8, Lemma 3.2, Theorem 3.4 and Theorem 3.7]. □

It is easy to see that module actions on $\mathcal{T}^{(2n-1)}$ and $\mathcal{T}^{(2n)}$ are as follows:

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & G_{2n} \end{bmatrix} = \begin{bmatrix} aF_{2n} & a\phi_{2n} + xG_{2n} \\ 0 & bG_{2n} \end{bmatrix},$$

$$\begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & G_{2n} \end{bmatrix} \cdot \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} F_{2n}a & F_{2n}x + \phi_{2n}b \\ 0 & G_{2n}b \end{bmatrix},$$

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \theta_{2n-1} & \phi_{2n-1} \\ 0 & \varphi_{2n-1} \end{bmatrix} = \begin{bmatrix} a\theta_{2n-1} + x\phi_{2n-1} & b\phi_{2n-1} \\ 0 & b\varphi_{2n-1} \end{bmatrix},$$

$$\begin{bmatrix} \theta_{2n-1} & \phi_{2n-1} \\ 0 & \varphi_{2n-1} \end{bmatrix} \cdot \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} \theta_{2n-1}a & \phi_{2n-1}a \\ 0 & \varphi_{2n-1}b + \phi_{2n-1}x \end{bmatrix}$$

for all $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in \mathcal{T}$, $\begin{bmatrix} \theta_{2n-1} & \phi_{2n-1} \\ 0 & \varphi_{2n-1} \end{bmatrix} \in \mathcal{T}^{(2n-1)}$ and $\begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & G_{2n} \end{bmatrix} \in \mathcal{T}^{(2n)}$.

With simple calculations we can prove the following lemmas which are left to reader.

Lemma 2.2. *Let $\mathcal{A}(\mathcal{B})$ have a bounded approximate identity and let $T:\mathcal{A}\longrightarrow\mathcal{X}^*(T:\mathcal{B}\longrightarrow\mathcal{X}^*)$ be a bounded right $\mathcal{A}$-module (left $\mathcal{B}$-module) homomorphism. Then there is $x_0^*\in\mathcal{X}^*$ such that $T(a)=x_0^*a$ $(T(b)=bx_0^*)$ for all $a\in\mathcal{A}$ $(b\in\mathcal{B})$.*

Lemma 2.3. *Let n be a positive integer and let $D:\mathcal{T}\longrightarrow\mathcal{T}^{(n)}$ be a derivation. Then*

$$\text{(i)}\ \begin{cases}\delta_1:\mathcal{A}\longrightarrow\mathcal{A}^{(n)}\\ \delta_1(a)=\pi_1(D(\begin{bmatrix}a&0\\0&0\end{bmatrix}))\end{cases}\quad \textit{and}\quad \begin{cases}\delta_4:\mathcal{B}\longrightarrow\mathcal{B}^{(n)}\\ \delta_4(b)=\pi_4(D(\begin{bmatrix}0&0\\0&b\end{bmatrix}))\end{cases}$$

are bounded derivations,

$$\text{(ii)}\ \begin{cases}T:\mathcal{A}\longrightarrow\mathcal{X}^{(n)}\\ T(a)=\pi_2(D(\begin{bmatrix}a&0\\0&0\end{bmatrix}))\end{cases}\quad \textit{and}\quad \begin{cases}S:\mathcal{B}\longrightarrow\mathcal{X}^{(n)}\\ S(b)=\pi_2(D(\begin{bmatrix}0&0\\0&b\end{bmatrix}))\end{cases}$$

are right (left) $\mathcal{A}$-module and left (right) $\mathcal{B}$-module homomorphisms, respectively.

Proposition 2.5. *Let $\mathcal{A}$ have a bounded approximate identity and $\mathcal{A}^{(2n-1)},\mathcal{B}^{(2n-1)}$ and $\mathcal{X}^{(2n-1)}$ be non-degenerate. Then $\mathcal{H}^1(\mathcal{T},\mathcal{T}^{(2n-1)})\simeq\mathcal{H}^1(\mathcal{A},\mathcal{A}^{(2n-1)})\oplus\mathcal{H}^1(\mathcal{B},\mathcal{B}^{(2n-1)})$.*

Proof. Let $D:\mathcal{T}\longrightarrow\mathcal{T}^{(2n-1)}$ be a derivation. By Lemmas 2.1, 2.3 and 2.4 there exist derivations $\delta_1:\mathcal{A}\longrightarrow\mathcal{A}^{(2n-1)}$ and $\delta_4:\mathcal{B}\longrightarrow\mathcal{B}^{(2n-1)}$ and $\phi_0\in\mathcal{X}^{(2n-1)}$ such that

$$D(\begin{bmatrix}a&0\\0&0\end{bmatrix})=\begin{bmatrix}\delta_1(a)&\phi_0a\\0&0\end{bmatrix},\quad \pi_4(D(\begin{bmatrix}0&0\\0&b\end{bmatrix}))=\delta_4(b).$$

Let $b\in\mathcal{B}$ and $D(\begin{bmatrix}0&0\\0&b\end{bmatrix})=\begin{bmatrix}\theta&\phi\\0&\delta_4(b)\end{bmatrix}$, then for all $a\in\mathcal{A}$,

$$\begin{bmatrix}\theta&\phi\\0&\delta_4(b)\end{bmatrix}\begin{bmatrix}a&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&b\end{bmatrix}\begin{bmatrix}\delta_1(a)&\phi_0a\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

It follows that $(\phi + b\phi_0)a = 0$ and $\theta a = 0$, $(a \in \mathcal{A})$. Since $\mathcal{X}^{(2n-1)}$ and $\mathcal{A}^{(2n-1)}$ are non-degenerate, we get $\phi = -b\phi_0$, $\theta = 0$ and $D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} 0 & -b\phi_0 \\ 0 & \delta_4(b) \end{bmatrix}$.

Let $a \in \mathcal{A}$, $x \in \mathcal{X}$, $b \in \mathcal{B}$ and $D(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \theta & \phi \\ 0 & \varphi \end{bmatrix}$. We have

$$\begin{bmatrix} \theta & \phi \\ 0 & \varphi \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1(a) & \phi_0 a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently $\phi a = 0$ and $(\theta + x\phi_0)a = 0$. Since $\mathcal{A}^{(2n-1)}$ and $\mathcal{X}^{(2n-1)}$ are non-degenerate, we obtain that $\phi = 0$ and $\theta = -x\phi_0$. A similar calculation shows that $\varphi = \phi_0 x$.

Therefore $D(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}) = \begin{bmatrix} \delta_1(a) - x\phi_0 & \phi_0 a - b\phi_0 \\ 0 & \delta_4(b) + \phi_0 x \end{bmatrix}$ for all $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in \mathcal{T}$ and the result follows from Lemma 2.2. □

Corollary 2.6. *Let $\mathcal{A}$ have a bounded approximate identity, $\mathcal{B}$ be a Banach algebra such that $\mathcal{B}^2 = \mathcal{B}$ and $\mathcal{X}$ be an essential Banach $\mathcal{A},\mathcal{B}$-module. Then*

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^*) \simeq \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) \oplus \mathcal{H}^1(\mathcal{B}, \mathcal{B}^*).$$

Dales, Ghahramani and Gronbaek [4, proposition, 1.3] have shown that if $\mathcal{A}$ is a weakly amenable Banach algebra, then $\mathcal{A}^2$, the linear span of products of elements in $\mathcal{A}$, is dense in $\mathcal{A}$ (c.f. [10], [11], [12]). Hence $\mathcal{A}^*$ is non-degenerate.

Corollary 2.7. *Let $\mathcal{A}$ or $\mathcal{B}$ have a bounded approximate identity and let $\mathcal{X}$ be essential. Then $\mathcal{T}$ is weakly amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable.*

Proof. It is easy to see that $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{X}^*$ are non-degenerate. □

Theorem 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identities. Let $n \geq 0$, $\mathcal{X}$ and $\mathcal{X}^{(2n)}$ be essential Banach $\mathcal{A},\mathcal{B}$-modules. Then*

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^{(2n+1)}) \simeq \mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(2n+1)}) \oplus \mathcal{H}^1(\mathcal{B}, \mathcal{B}^{(2n+1)}).$$

Proof. Without loss of generality, we can assume that $\{e_\alpha\}$, $\{f_\alpha\}$ and $\{\begin{bmatrix} e_\alpha & 0 \\ 0 & f_\alpha \end{bmatrix}\}$ be bounded approximate identities of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$, respectively. By Lemmas 2.1, 2.3 and 2.4 there exist derivations $\delta_1 : \mathcal{A} \longrightarrow \mathcal{A}^{(2n+1)}$ and $\delta_4 : \mathcal{B} \longrightarrow \mathcal{B}^{(2n+1)}$, $\phi_0 \in \mathcal{X}^{(2n+1)}$ and $\psi_0 \in \mathcal{X}^{(2n+1)}$ such that

$$D(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \delta_1(a) & \phi_0 a \\ 0 & 0 \end{bmatrix}, D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} 0 & b\psi_0 \\ 0 & \delta_4(b) \end{bmatrix} (a \in \mathcal{A},\ b \in \mathcal{B}).$$

By [3, Proposition, 2.9.7], we have $\psi_0 = -\phi_0$ and therefore

$$D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} 0 & -b\phi_0 \\ 0 & \delta_4(b) \end{bmatrix}.$$

Let $x = ay = zb$ be an arbitrary element of $\mathcal{X}$ and let

$$D(\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \theta_{2n+1} & \phi_{2n+1} \\ 0 & \psi_{2n+1} \end{bmatrix}, D(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \theta'_{2n+1} & \phi'_{2n+1} \\ 0 & \psi'_{2n+1} \end{bmatrix}.$$

Then

$$D(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} a\theta'_{2n+1} & 0 \\ 0 & \phi_0 x \end{bmatrix} = \begin{bmatrix} -x\phi_0 & 0 \\ 0 & \psi_{2n+1} b \end{bmatrix} = \begin{bmatrix} -x\phi_0 & 0 \\ 0 & \phi_0 x \end{bmatrix}.$$

Therefore $D(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}) = \begin{bmatrix} \delta_1(a) - x\phi_0 & \phi_0 a - b\phi_0 \\ 0 & \delta_4(b) + \phi_0 x \end{bmatrix}$ and the proof is completed by Lemma 2.2 □

3. $(2n)$-weak amenability

In [8] Forrest and Marcoux defined the following sets. For each positive integer n, we denote the centralizer of $\mathcal{A}$ in $\mathcal{A}^{(2n)}$ as

$$\mathcal{Z}_{\mathcal{A}}(\mathcal{A}^{(2n)}) = \{F_{2n} \in \mathcal{A}^{(2n)} \mid F_{2n}a = aF_{2n} \text{ for all } a \in \mathcal{A}\}.$$

For $F_{2n} \in \mathcal{A}^{(2n)}$ and $G_{2n} \in \mathcal{B}^{(2n)}$ we consider the map $\rho_{F_{2n},G_{2n}} : \mathcal{X} \longrightarrow \mathcal{X}^{(2n)}$ defined by $x \longmapsto F_{2n}x - xG_{2n}$. The set

$$\mathcal{ZR}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}) = \{\rho_{F_{2n},G_{2n}} : \mathcal{X} \longrightarrow \mathcal{X}^{(2n)} \mid F_{2n} \in \mathcal{Z}_{\mathcal{A}}(\mathcal{A}^{(2n)}), G_{2n} \in \mathcal{Z}_{\mathcal{B}}(\mathcal{B}^{(2n)})\}$$

is called central Rosenblum operators on $\mathcal{X}$ with coefficient in $\mathcal{X}^{(2n)}$. We also have $\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}) = \{\phi : \mathcal{X} \longrightarrow \mathcal{X}^{(2n)} \mid \phi$ is left $\mathcal{A}$-module and right $\mathcal{B}$-module homomorphism $\}$.

Forrest and Marcoux [8] proved the following theorem.

Theorem 3.1. *Let n be a positive integer and let $\mathcal{A}$ and $\mathcal{B}$ be $(2n)$-weakly amenable. Let for every continuous derivation $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$, there exist derivations $\delta_1 : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ and $\delta_4 : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$, an element $\phi_0 \in \mathcal{X}^{(2n)}$ and a continuous map $\rho : \mathcal{X} \longrightarrow \mathcal{X}^{(2n)}$ such that $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} \delta_1(a) & a\phi_0 - \phi_0 b + \rho(x) \\ 0 & \delta_4(b) \end{bmatrix}$, $\rho(ax) = \delta_1(a)x + a\rho(x)$ and $\rho(xb) = \rho(x)b + x\delta_4(b)$. Then*

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}) / \mathcal{ZR}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}).$$

Now we have the following theorem.

Theorem 3.2. *Let $\mathcal{A}$ or $\mathcal{B}$ have a bounded approximate identity, and $\mathcal{A}^{(2n)}$, $\mathcal{B}^{(2n)}$ and $\mathcal{X}^{(2n)}$ be non-degenerate. If $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly amenable, then*

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}) / \mathcal{Z}R_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}).$$

Proof. Without loss of generality we may assume that $\mathcal{A}$ has a bounded approximate identity. Let $D : \mathcal{T} \longrightarrow \mathcal{T}^{(2n)}$ be a derivation. It is easy to see that $\pi_4(D(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix})) = 0$ for all $a \in \mathcal{A}$. By Lemmas 2.1, 2.3 and 2.4 there exist derivations $\delta_1 : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ and $\delta_4 : \mathcal{B} \longrightarrow \mathcal{B}^{(2n)}$ and $\phi_0 \in \mathcal{X}^{(2n)}$ such that $D(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \delta_1(a) & a\phi_0 \\ 0 & 0 \end{bmatrix}$ and $\pi_4(D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix})) = \delta_4(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.

For $b \in \mathcal{B}$ let $D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & \delta_4(b) \end{bmatrix}$. Then for all $a \in \mathcal{A}$ we have

$$\begin{bmatrix} \delta_1(a) & a\phi_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & \delta_4(b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\mathcal{A}^{(2n)}$ and $\mathcal{X}^{(2n)}$ are non-degenerate, $F_{2n} = 0$, $\phi = -\phi_0 b$ and

$$D(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) = \begin{bmatrix} 0 & -\phi_0 b \\ 0 & \delta_4(b) \end{bmatrix}.$$

For $x \in \mathcal{X}$, let $D(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & G_{2n} \end{bmatrix}$. Then for each a in $\mathcal{A}$, b in $\mathcal{B}$

$$\begin{bmatrix} 0 & \phi_0 b \\ 0 & \delta_4(b) \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & G_{2n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} F_{2n} & \phi_{2n} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1(a) & a\phi_0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore $F_{2n} = G_{2n} = 0$. We define $\rho : \mathcal{X} \longrightarrow \mathcal{X}^{(2n)}$ by $x \longmapsto \pi_2(D(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}))$. A simple calculation shows that $\rho(ax) = \delta_1(a)x + a\rho(x)$ and $\rho(xb) = \rho(x)b + x\delta_4(b)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. By Theorem 3.1 the proof is completed. □

Let $\mathcal{A}$ be a Banach algebra. We consider the triangular Banach algebra

$$\mathcal{T} = \mathcal{T}_2 \otimes \mathcal{A} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix},$$

where $\mathcal{T}_2$ denotes the algebra of 2×2 upper triangular matrices.

Proposition 3.3. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. If $\mathcal{A}$ is $(2n)$-weakly amenable and $\mathcal{A}^{(2n)}$ is non-degenerate, then $\mathcal{T} = \mathcal{T}_2 \otimes \mathcal{A}$ is $(2n)$-weakly amenable.*

Proof. By Theorem 3.2 it is sufficient to show that $\mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)}) \simeq Z_{\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)})$. Let (e_α) be a bounded approximate identity of $\mathcal{A}$ and let $\phi : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ be a $\mathcal{A}$-module homomorphism. There exist $E \in \mathcal{A}^{(2n)}$ and a subnet $\{\phi(e_\beta)\}$ of $\{\phi(e_\alpha)\}$ such that $\phi(e_\beta) \longrightarrow E$ in the weak* topology. A simple calculation shows that for every a in $\mathcal{A}$; $\phi(a) = aE = Ea$. Therefore $E \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)})$ and $\phi = \rho_{E,0} \in Z_{\mathcal{A}}, \mathcal{A}(\mathcal{A}, \mathcal{A}^{(2n)})$. □

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identities. Let n be a positive integer, $\mathcal{X}$ and $\mathcal{X}^{(2n-1)}$ be essential Banach modules. If $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly amenable, then*

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^{(2n)}) \simeq \mathrm{Hom}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}) / \mathcal{ZR}_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \mathcal{X}^{(2n)}).$$

Lemma 3.5. *Suppose that $\mathcal{T}$ is 2-weakly amenable. Then there exist $F_0 \in Z_{\mathcal{A}}(\mathcal{A}^{**})$ and $G_0 \in Z_{\mathcal{B}}(\mathcal{B}^{**})$ such that for every x in $\mathcal{X}$; $\hat{x} = xG_0 - F_0x$ where $\hat{x}$ is the canonical image of x in $\mathcal{X}^{**}$.*

Proof. It is easy to see that $D : \mathcal{T} \longrightarrow \mathcal{T}^{**}$ defind by $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \hat{x} \\ 0 & 0 \end{bmatrix}$ is a continuous derivation. Therefore there are $F_0 \in \mathcal{A}^{**}, G_0 \in \mathcal{B}^{**}$ and $x_0^{**} \in \mathcal{A}^{**}$ such that $D = \delta_{\begin{bmatrix} F_0 & x_0^{**} \\ 0 & G_0 \end{bmatrix}}$. So that for every $a \in \mathcal{A}, b \in \mathcal{B}$ and $x \in \mathcal{X}$ we have

$$\begin{aligned} \begin{bmatrix} 0 & \hat{x} \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} F_0 & x_0^{**} \\ 0 & G_0 \end{bmatrix} - \begin{bmatrix} F_0 & x_0^{**} \\ 0 & G_0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} aF_0 - F_0a & ax_0^{**} + xG_0 - F_0x - x_0^{**}b \\ 0 & bG_0 - G_0b \end{bmatrix}. \end{aligned}$$

Hence $F_0 \in Z_{\mathcal{A}}(\mathcal{A}^{**})$, $G_0 \in Z_{\mathcal{B}}(\mathcal{B}^{**})$ and for every x in $\mathcal{X}$, $\hat{x} = xG_0 - F_0x$. □

Proposition 3.6. *Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A}^{(m)} \\ & \mathcal{A} \end{bmatrix}$ for some nonnegative integer m. Suppose that $\mathcal{T}$ is $(2n)$-weakly amenable for some positive integer n. Then $\mathcal{A}$ has a bounded approximate identity.*

Proof. $\mathcal{T}$ is 2-weakly amenable by [4, Proposition, 1.2]. So there exist $F_0, G_0 \in Z_{\mathcal{A}}(\mathcal{A}^{**})$ such that $\hat{x} = xG_0 - F_0x$ $(x \in \mathcal{A}^{(m)})$ by Lemma 3.5. If m is odd then for all $a^* \in \mathcal{A}^*$; $a^* = a^*(G_0 - F_0) = (G_0 - F_0)a^*$, and if m is even then for all $a \in \mathcal{A}$; $\hat{a} = a(F_0 - G_0) = (F_0 - G_0)a$. So in both cases it is easy to see that $G_0 - F_0$ is a mixed unit for $\mathcal{A}^{**}$ and hence $\mathcal{A}$ has a bounded approximate identity. □

It is well known that for a Banach algebra $\mathcal{A}$ its second dual $\mathcal{A}^{**}$ is a Banach algebra when equipped with the first or second Arens products (for more details see [6]). Recall that a Banach algebra $\mathcal{A}$ is called a dual Banach algebra if there is a closed submodule $\mathcal{X}$ of $\mathcal{A}^*$such that $\mathcal{A} = \mathcal{X}^*$ (see [19, 4.4.1]).

Proposition 3.7. *Let $\mathcal{A}$ be a second dual of a Banach algebra or a dual Banach algebra and* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A}^{(2m)} \\ & \mathcal{A} \end{bmatrix}$ *for some positive integer m. Suppose that $\mathcal{T}$ is $(2n)$-weakly amenable for some positive integer n. Then $\mathcal{A}$ has an identity.*

Proof. Without loss of generality, we may assume that $\mathcal{A}$ is a dual Banach algebra or the second dual of a Banach algebra with the first Arens product. $\mathcal{T}$ is 2-weakly amenable by [4, Proposition, 1.2]. So by Lemma 3.5 there exist $F_0, G_0 \in Z_{\mathcal{A}}(\mathcal{A}^{(**)})$ such that for all $x \in \mathcal{A}^{(2m)}$; $\hat{x} = xG_0 - F_0x$. Therefore $\hat{a} = a(G_0 - F_0) = (G_0 - F_0)a$ for all a in $\mathcal{A}$. Suppose that $\pi : \mathcal{X} \longrightarrow \mathcal{X}^{(**)}$ is the canonical embedding, where $\mathcal{X}$ is the predual of $\mathcal{A}$. Put $e = \pi^*(G_0 - F_0)$. For $a \in \mathcal{A}$ we have

$$\begin{aligned} \langle x, ea\rangle &= \langle \pi(ax), G_0 - F_0\rangle \\ &= \langle a\pi(x), G_0 - F_0\rangle \\ &= \langle \pi(x), \hat{a}\rangle \\ &= \langle x, a\rangle \ (x \in \mathcal{X}). \end{aligned}$$

So e is a right identity for $\mathcal{A}$. Now if $\mathcal{A}$ is a dual Banach algebra, similarly e is a left identity for $\mathcal{A}$, and if $\mathcal{A}$ is the second dual of Banach algebra $\mathcal{B}$, then for $a \in \mathcal{A}$ there exists net $\{b_\alpha\}$ in $\mathcal{B}$ such that $b_\alpha \longrightarrow a$ in the weak* topology.

$$\begin{aligned} \langle x, ae\rangle &= \langle ex, a\rangle = \lim_\alpha \langle b_\alpha, ex\rangle \\ &= \lim_\alpha \langle \pi(xb_\alpha), G_0 - F_0\rangle = \lim_\alpha \langle \pi(xb_\alpha), G_0 - F_0\rangle \\ &= \lim_\alpha \langle \pi(x), \hat{\hat{b}}\rangle = \lim_\alpha \langle x, \hat{b}\rangle = \langle x, a\rangle \qquad (x \in \mathcal{X}). \end{aligned}$$

Therefore e is a right identity for $\mathcal{A}$ and so $\mathcal{A}$ has an identity. □

Corolollary 3.8. *Let $\mathcal{A}$ be the second dual of a Banach algebra or a dual Banach algebra and let* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix}$. *Suppose that $\mathcal{T}$ is $(2n)$-weakly amenable for some positive integer n. Then $\mathcal{T}$ is $(2n)$-weakly amenable for all positive integer n.*

4. Amenability of the triangular Banach algebra $\mathcal{T}$

In this section we give a necessary and sufficient condition for the amenability of $\mathcal{T}$.

Theorem 4.1. *If* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ *has a bounded approximate identity, then* $\mathcal{A}$ *and* $\mathcal{B}$ *have bounded approximate identities and* $\mathcal{X}$ *is neo-unital.*

Proof. Let $\left\{ \begin{bmatrix} a_\alpha & x_\alpha \\ 0 & b_\alpha \end{bmatrix} \right\}$ be a bounded approximate identity for $\mathcal{T}$. For any $a \in \mathcal{A}$, we have

$$\begin{bmatrix} a_\alpha & x_\alpha \\ 0 & b_\alpha \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_\alpha a & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $a_\alpha a \longrightarrow a$. Similarly $a a_\alpha \longrightarrow a$ and thus $\{a_\alpha\}$ is a bounded approximate identity for $\mathcal{A}$. Similarly $\{b_\alpha\}$ is a bounded approximate identity for $\mathcal{B}$. For any $x \in X$,

$$\begin{bmatrix} a_\alpha & x_\alpha \\ 0 & b_\alpha \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_\alpha x \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

so that $a_\alpha x \longrightarrow x$ and thus by Cohn factorization theorem $\mathcal{X} = \mathcal{A}.\mathcal{X}$ and similarly $\mathcal{X} = \mathcal{X}.B$. □

Now we prove the main theorem of this section.

Theorem 4.2. $\mathcal{T}$ *is amenable if and only if both* $\mathcal{A}, \mathcal{B}$ *are amenable and* $\mathcal{X} = 0$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be amenable and $\mathcal{X} = 0$. Since $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} \simeq \mathcal{A}$, $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} \simeq \mathcal{B}$, the closed ideal $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}$ of $\mathcal{T}$ and the quotient algebra $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}$ are amenable and thus $\mathcal{T} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix}$ is amenable. For the converse, suppose that $\mathcal{T}$ is amenable. Since $\begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & 0 \end{bmatrix}$ is a closed ideal of T, the quotient algebra

$\begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & 0 \end{bmatrix}$ is amenable. On the other hand $\begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ $/ \begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & 0 \end{bmatrix} \simeq B$, thus $\mathcal{B}$ is amenable. Similarly, since $\begin{bmatrix} 0 & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix}$ is a closed ideal of $\mathcal{T}$ and $\begin{bmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} 0 & \mathcal{X} \\ 0 & \mathcal{B} \end{bmatrix} \simeq \mathcal{A}$, the Banach algebra $\mathcal{A}$ is amenable. Since $\mathcal{T}$ is amenable and $\begin{bmatrix} 0 & \mathcal{X} \\ 0 & 0 \end{bmatrix}$ is a closed ideal of $\mathcal{T}$ which is complemented in $\mathcal{T}$, $\begin{bmatrix} 0 & \mathcal{X} \\ 0 & 0 \end{bmatrix}$ is amenable and thus it has a bounded approximate identity. However this is not possible unless $\mathcal{X} = 0$. □

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