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THE ASSOCIATED MEASURE ON LOCALLY COMPACT COCOMMUTATIVE KPC-HYPERGROUPS

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Abstract. We study harmonic analysis on cocommutative KPC-hypergroups, which is a generalization of DJS-hypergroups, introduced by Kalyuzhnyi, Podkolzin and Chapovsky. We prove that there is a relationship between the associated measures μ and $\gamma\mu$, where μ is a Radon measure on KPC-hypergroup Q and γ is a character on Q.
Keywords: Cocommutative hypergroups, DJS-hypergroups, KPC-hypergroups, positive definite measures.
MSC(2010): Primary: 28C15; Secondary: 20N20.

1. **Introduction**

Hypergroups were introduced in a series of papers by Dunkle [3], Jewett [4], and Spector [7] in the 70's (we refer to this definition of hypergroups as DJS-hypergroups). For more details about DJS-hypergroups we refer to [1].

In 2010, Kalyuzhnyi, Podkolzin, and Chapovsky [5] introduced new axioms for hypergroups. This is an extension of DJS-hypergroups, on the one hand, and generalizes a normal hypercomplex system with a basis unity to the nonunimodular case, on the other. We refer to this notion as KPC-hypergroup. They studied harmonic analysis on these hypergroups and showed that there is an example of a compact KPC-hypergroup related to the generalized Tchebycheff polynomials, which is not a DJS-hypergroup [5]. Medghalchi and Tabatabaie [6] have studied periodicity on locally compact commutative DJS-hypergroups.

In this paper we study harmonic analysis on locally compact cocommutative KPC-hypergroups. In Section 2, we recall the definition and basic properties of KPC-hypergroups. Periodicity of locally compact KPC-hypergroups and our

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main theorem is presented in Section 3. We show that there is a relationship between the associated measures of μ and $\gamma\mu$, where μ is a Radon measure on KPC-hypergroup Q and γ is a character of Q.

Let Q be a locally compact cocommutative KPC-hypergroup. We denote the set of all complex Radon measures on Q, bounded measures, compact supported measures and positive measures by $M(Q)$, $M_b(Q)$, $M_c(Q)$ and $M^+(Q)$, respectively. We denote the spaces of all complex-valued bounded continuous functions and continuous functions with compact supports by $C_b(Q)$ and $C_c(Q)$, respectively.

2. **Cocommutative KPC-hypergroups**

In this section we recall the definition and basic properties of locally compact KPC-hypergroups and study positive definite functions and measures on them.

Definition 2.1. Let Q be a locally compact second countable Hausdorff space with an involutive homeomorphism $\star : Q \longrightarrow Q$ and let $e \in Q$ satisfy $e^* = e$. Suppose the following conditions hold.

(H_1) There is a $\mathbb{C}$-linear mapping $\Delta : C_b(Q) \to C_b(Q \times Q)$ such that
i. Δ is coassociative, that is,

$$(\Delta \times id) \circ \Delta = (id \times \Delta) \circ \Delta; \tag{2.1}$$

ii. Δ is positive, that is, $\Delta f \geq 0$ for all $f \in C_b(Q)$ such that $f \geq 0$;
iii. Δ preserves the identity, that is, $(\Delta 1)(p, q) = 1$ for all $p, q \in Q$;
iv. For all $f, g \in C_c(Q)$ we have $(1 \otimes f).(\Delta g) \in C_c(Q \times Q)$ and $(f \otimes 1).(\Delta g) \in C_c(Q \times Q)$.

(H_2) The homomorphism $\epsilon : C_b(Q) \to \mathbb{C}$ defined on $C_b(Q)$ by $\epsilon(f) = f(e)$ satisfies the counit property, that is,

$$(\epsilon \times id) \circ \Delta = (id \times \epsilon) \circ \Delta = id, \tag{2.2}$$

in other words, $(\Delta f)(e, p) = (\Delta f)(p, e) = f(p)$ for all $p \in Q$.

(H_3) The function f^- defined by $f^-(q) = f(q^\star)$ for $f \in C_b(Q)$ satisfies

$$(\Delta f^-)(p, q) = (\Delta f)(q^\star, p^\star). \tag{2.3}$$

(H_4) there is a positive measure m on Q, such that supp $m = Q$, and

$$\int_Q (\Delta f)(p, q)g(q)dm(q) = \int_Q f(q)(\Delta g)(p^\star, q)dm(q) \tag{2.4}$$

for all $f \in C_b(Q)$ and $g \in C_c(Q)$, or $f \in C_c(Q)$ and $g \in C_b(Q)$, where $p \in Q$; (such a measure m will be called a left Haar measure on Q.)

Then $(Q, \star, e, \Delta, m)$, or simply Q, is called a locally compact KPC-hypergroup.

Notation. In the above definition, we have used the following notations:

$$[[(\Delta \times id) \circ \Delta(f)](p,q,r)](.) := \Delta(\Delta f(p,\cdot))(q,r),$$
$$[[(id \times \Delta) \circ \Delta(f)](p,q,r)](.) := \Delta(\Delta f(\cdot,q))(p,r),$$
$$[(\epsilon \times id) \circ \Delta(f)](p) := \epsilon(\Delta f(p,\cdot)) = \Delta f(p,e),$$
$$[(id \times \epsilon) \circ \Delta(f)](p) := \epsilon(\Delta f(\cdot,p)) = \Delta f(e,p),$$
$$(f \otimes 1)(p,q).(\Delta g)(p,q) := f(p)1(q).\Delta g(p,q),$$
$$(1 \otimes f)(p,q).(\Delta g)(p,q) := 1(p)f(q).\Delta g(p,q).$$

A KPC-hypergroup Q is called cocommutative if $\Delta f(p,q) = \Delta f(q,p)$, for all $f \in C_b(Q)$ and all $p,q \in Q$ and it is called Hermitian if $q^\star = q$ for all $q \in Q$; By (H_3), every Hermitian hypergroup is cocommutative.

Throughout this paper Q is a locally compact cocommutative KPC-hypergroup and m is a left Haar measure on Q.

Definition 2.2. Let $\mu, \nu \in M(Q)$ be such that the linear functional $\mu * \nu$ defined by

$$(2.5) \qquad (\mu * \nu)(f) = \int_{Q^2} \Delta(f)(p,q)d\mu(p)d\nu(q), \qquad (f \in C_c(Q))$$

is a measure. Then the measures μ and ν are called convolvable. In particular, we have $(\delta_p * \delta_q)(f) = (\Delta f)(p,q)$, where $p,q \in Q$.

If $\mu, \nu \in M_b(Q)$, then μ and ν are convolvable. [5, Lemma 3.3]

Definition 2.3. The convolution of $f, g \in C_c(Q)$ is denoted by $f * g$ and is defined by $(fm) * (gm) = (f * g)m$, where the convolution of measures is given by (2.5). For each $f, g \in C_c(Q)$, we have $f * g \in C_c(Q)$, and by [5],

$$(2.6) \qquad (f * g)(q) = \int_Q f(p)(\Delta g)(p^\star, q)dm(p).$$

Similarly, we define $f * g$ for $f, g \in C_b(Q)$.

Remark 2.4. If m is a left Haar measure and $p \in Q$, then $m * \delta_p$ is a left Haar measure. Since a left Haar measure is unique up to strictly positive scalar mutiples, $m * \delta_{p^\star} = \delta(p)m$ for a positive number $\delta(p)$. (This number does not depend on the left Haar measure m). The function $\delta : Q \longrightarrow \mathbb{C}$ is called the modular function of the locally compact KPC-hypergroup Q.

Definition 2.5. A measure $\mu \in M(Q)$ is called positive definite if for any $g \in C_c(Q)$, we have $\int g * g^\star d\mu \geq 0$, where $g^\star(p) = \overline{g(p^\star)}\delta(p^\star)$, and δ is the modular function. The set of all positive definite measures on Q is denoted by $M^p(Q)$. The set of all bounded positive definite measures is denoted by $M_b^p(Q)$.

A function $f \in C_b(Q)$ is called positive definite if for any $g \in C_c(Q)$, we have $\int f(g * g^\star) dm \geq 0$. We denote the set of all positive definite functions by $P(Q)$.

For each $\mu \in M(Q)$, we define μ^- by $\int_Q f(t) d\mu^-(t) = \int f(t^\star) d\mu(t)$.

Definition 2.6. The convolution of $f \in C_c(Q)$ and $\mu \in M_b(Q)$ is defined by

$$(\mu * f)(q) := \int_Q \Delta f(p^\star, q) d\mu(p), \qquad (q \in Q) \tag{2.7}$$

if the integral exists.

Definition 2.7. A measure $\mu \in M(Q)$ is called shift-bounded if $\mu * f \in C_b(Q)$ for all $f \in C_c(Q)$, and weakly shift-bounded if $\mu * f * \tilde{f} \in C_b(Q)$ for all $f \in C_c(Q)$.

Lemma 2.8. *i. For any $p \in Q$, we have $\delta(p) = 1$, where δ is the modular function of Q.*

ii. For each $f, g \in C_c(Q)$, $f * g = g * f$.

Proof. i. Let $f \in C_c(Q)$ and $p \in Q$. Then

$$\begin{aligned}
\delta(p) m(f) &= (m * \delta_{p^\star})(f) \quad \text{(by 2.5)} \\
&= \int_Q \int_Q \Delta f(q, t) dm(q) d\delta_{p^\star}(t) \\
&= \int_Q \Delta f(q, p^\star) dm(q) \quad (Q \text{ is cocommutative}) \\
&= \int_Q \Delta f(p^\star, q) 1(q) dm(q) \quad \text{(by 2.4)} \\
&= \int_Q \Delta 1(p, q) f(q) dm(q) \quad (\text{by } H_1\ iii) \\
&= \int_Q f(q) dm(q) = m(f).
\end{aligned}$$

Thus $\delta(p) = 1$ for any $p \in Q$.
ii. If $f, g \in C_c(Q)$, for any $q \in Q$, we have

$$\begin{aligned}
(f * g)(q) &= \int_Q f(p)\Delta g(p^\star, q) dm(p) \quad \text{(by 2.3)} \\
&= \int_Q f(p)\Delta g^-(q^\star, p) dm(p) \quad \text{(by 2.4)} \\
&= \int_Q \Delta f(q, p) g^-(p) dm(p) \quad (Q \text{ is cocommutative}) \\
&= \int_Q \Delta f(p, q) g(p^\star) dm(p) \quad (p := p^\star) \\
&= \int_Q \Delta f(p^\star, q) g(p)\delta(p^\star) dm(p) \quad (\text{by } i) \\
&= \int_Q \Delta f(p^\star, q) g(p) dm(p) \quad \text{(by 2.6)} \\
&= (g * f)(q).
\end{aligned}$$

□

The proof of the following proposition is different from the case of DJS-hypergroups.

Proposition 2.9. *If $f \in C_c(Q)$ and $\mu \in M_b(Q)$, then $\mu * f \in C(Q)$.*

Proof. Since Q is a second countable space, we can use sequences for the proof. Let $(q_n)_{n=1}^{\infty}$ be a sequence in Q, such that $q_n \to q$. We should show $(\mu * f)(q_n) \to (\mu * f)(q)$. By [5, Lemma 3.1], Δ is a continuous mapping from $C_b(Q)$ to $C_b(Q \times Q)$. Thus for any $f \in C_c(Q)$, $\Delta f(p^\star, q_n) \to \Delta f(p^\star, q)$. Also, since $\Delta f(., q) \in C_b(Q)$, we have

$$|\int |\Delta f(p^\star, q)| d\mu(p)| \leq \sup_{p \in Q} |\Delta f(p^\star, q)|.||\mu|| < \infty.$$

So by the dominated convergence theorem, we have

$$\int \Delta f(p^\star, q_n) d\mu(p) \to \int \Delta f(p^\star, q) d\mu(p).$$

Therefore $(\mu * f)(q_n) \to (\mu * f)(q)$, and hence $\mu * f \in C(Q)$. □

Proposition 2.10. *If μ is a bounded positive definite measure on Q and $f \in C_c(Q)$, then $\mu * f * \tilde{f} \in P_b(Q)$.*

Proof. Let $\mu \in M_b(Q)$ and $f \in C_c(Q)$. By [5, Lemma 5.2] we have $f * \tilde{f} \in C_c(Q)$. So by Proposition 2.8, $\mu * (f * \tilde{f}) \in C(Q)$. By [5, corollary 5.3], $C_c(Q)$ is an involutive algebra with the multiplication and involution defined by (2.6) and $f^\star(p) = \bar{f}(p^\star)\delta(q^\star)$, respectively. Define $g := f * \tilde{f}$. Then

$$\begin{aligned}
|(\mu * g)(q)| &= |\int \Delta g(p^\star, q) d\mu(p)| \\
&\leq \int |\Delta g(p^\star, q)| d|\mu|(p) \\
&\leq ||\Delta g|| \int d|\mu|(p) \\
&\leq ||\Delta|| ||g|| |\mu|(Q) \quad (5,\ \text{Lemma } 3.1) \\
&= ||g|| |\mu|(Q) < \infty.
\end{aligned}$$

Therefore $\mu * (f * \tilde{f})$ is bounded. Now we have

$$\begin{aligned}
\int (\mu * f * \tilde{f})(q)(\tilde{g} * g)(q) dm(q) &= \int \int \Delta(f * \tilde{f})(t^\star, q)(\tilde{g} * g)(q) d\mu(t) dm(q) \\
&= \int \int \Delta(f * \tilde{f})^-(q^\star, t)(\tilde{g} * g)(q) d\mu(t) dm(q) \\
&= \int [(\tilde{g} * g) * (f * \tilde{f})^-](t) d\mu(t) \\
&= \int [(f^- * g) * (f^- * \tilde{g)}](t) d\mu(t).
\end{aligned}$$

Since $\mu \in M^p(Q)$, the last integral is nonnegative. So by [5, Lemma 8.3], $\mu * (f * \tilde{f})$ is positive definite which completes the proof. □

Definition 2.11. A function $\chi \in C_b(Q)$ is called a character of the KPC-hypergroup Q if $(\Delta\chi)(p, q) = \chi(p)\chi(q)$ and $\chi(p^\star) = \overline{\chi(p)}$, for all $p, q \in Q$.

Definition 2.12. For any $f \in L^1(Q)$ and $\mu \in M(Q)$, the Fourier-Stieltjes transform $\hat{\mu}$ of μ and the Fourier transform $\hat{f}$ of f are defined by

$$\hat{\mu}(\xi) = \int_Q \overline{\xi(t)} d\mu(t) \text{ and } \hat{f}(\xi) = \int_Q \overline{\xi(t)} f(t) dm(t),$$

respectively, where $\xi \in \hat{Q}$, [5].

Definition 2.13. Let $f \in L^1(\hat{Q})$ and $\mu \in M(\hat{Q})$. The inverse Fourier transform $\check{f}$ and $\check{\mu}$ of f and μ are defined by

$$\check{f}(p) = \int_{\hat{Q}} \xi(p) f(\xi) d\rho(\xi) \text{ and } \check{\mu}(p) = \int_{\hat{Q}} \xi(p) d\mu(\xi),$$

respectively, where $p \in Q$.
Note that ρ is the Plancherel measure and $\hat{Q} = \text{supp } \rho$.

In fact $\hat{Q}$ is not a KPC-hypergroup in general. But under some conditions it is a KPC-hypergroup (see Theorem 3.7)

Lemma 2.14. *For any* $f \in C_c(Q)$, $(f^\star\hat{)} = \overline{\hat{f}}$.

Proof. Let $\xi \in \hat{Q}$. Then

$$\begin{aligned}
(f^\star\hat{)}(\xi) &= \int_Q \overline{\xi}(p) f^\star(p) dm(p) \\
&= \int_Q \overline{\xi}(p) \overline{f}(p^\star)\delta(p^\star) dm(p) \quad \text{(Lemma 2.8}i\text{)} \\
&= \int_Q \overline{\xi}(p) \overline{f}(p^\star) dm(p) \quad (p := p^\star) \\
&= \int_Q \overline{\xi(p^\star) f(p)} dm(p^\star) \quad \text{(Lemma 2.8}i\text{)} \\
&= \overline{\int_Q \xi(p^\star) f(p) dm(p)} = \overline{\hat{f}}(\xi).
\end{aligned}$$

□

Lemma 2.15. *For any* $f, g \in C_c(Q)$, $(f * g\hat{)} = \hat{f}\hat{g}$.

Proof. Let $f, g \in C_c(Q)$. For any $\xi \in \hat{Q}$

$$\begin{aligned}
(f * g\hat{)}(\xi) &= \int_Q \overline{\xi}(p)(f * g)(p) dm(p) \\
&= \int_Q \int_Q \overline{\xi}(p) f(q) \Delta g(q^\star, p) dm(q) dm(p) \\
&= \int_Q f(q) (\int_Q \overline{\xi}(p) \Delta g(q^\star, p) dm(p)) dm(q) \quad (H_4) \\
&= \int_Q f(q) (\int_Q \Delta\overline{\xi}(q, p) g(p) dm(p)) dm(q) \quad (H_3) \\
&= \int_Q f(q) (\int_Q \Delta\xi(p^\star, q^\star) g(p) dm(p)) dm(q) \\
&= \int_Q f(q) (\int_Q \xi(p^\star)\xi(q^\star) g(p) dm(p)) dm(q) \\
&= \int_Q \xi(q^\star) f(q) dm(q) \int_Q \xi(p^\star) g(p) dm(p) \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

□

Lemma 2.16. *For any* $g \in C_c(Q)$ *and* $\sigma \in M(\hat{Q})$, *we have* $\check{\sigma} * (|\hat{g}|^2\check{)} = (|\hat{g}|^2\sigma\check{)}$.

Proof. By Lemma 2.15, for any $g \in C_c(Q)$, $(g * g^\star)\hat{} = |\hat{g}|^2$. We define $h := (g * g^\star)$. For any $p \in Q$ we have

$$\begin{aligned}
(|\hat{g}|^2\sigma)\check{}(p) &= (\hat{h}\sigma)\check{}(p) \\
&= \int_{\hat{Q}} \xi(p)\hat{h}(\xi)d\sigma(\xi) \\
&= \int_{\hat{Q}} \xi(p)(\int_Q \overline{\xi}(q)h(q)dm(q))d\sigma(\xi) \\
&= \int_{\hat{Q}} \int_Q \xi(p)\xi(q^\star)h(q)dm(q)d\sigma(\xi) \\
&= \int_{\hat{Q}} \int_Q \Delta\xi(p,q^\star)h(q)dm(q)d\sigma(\xi) \quad (Q \text{ is cocommutative}) \\
&= \int_{\hat{Q}} \int_Q \Delta\xi(q^\star,p)h(q)dm(q)d\sigma(\xi) \quad (H_3) \\
&= \int_{\hat{Q}} \int_Q \Delta\overline{\xi}(p,q^\star)h(q)dm(q)d\sigma(\xi) \quad (H_4) \\
&= \int_{\hat{Q}} \int_Q \Delta h(p,q)\overline{\xi}(q)dm(q)d\sigma(\xi) \quad (q := q^\star) \\
&= \int_{\hat{Q}} \int_Q \Delta h(p,q^\star)\xi(q)\delta(q^\star)dm(q)d\sigma(\xi) \quad (\text{Lemma } 2.8) \\
&= \int_{\hat{Q}} \int_Q \Delta h(q^\star,p)\xi(q)dm(q)d\sigma(\xi) \\
&= \int_Q \check{\sigma}(q)\Delta h(q^\star,p)dm(q)) \\
&= (\check{\sigma} * h)(p).
\end{aligned}$$

□

Theorem 2.17. *Every $\mu \in M^p(Q)$ corresponds to a unique $\sigma \in M_+(\hat{Q})$ such that for all $g, h \in C_c(Q)$ and $p \in Q$,*
i. $\int |\hat{g}|^2 d\sigma < \infty$,
ii. $(\mu * g * g^\star)(p) = \int_{\hat{Q}} \xi(p)|\hat{g}(\xi)|^2 d\sigma(\xi)$,
where $\xi \in \hat{Q}$.

The measure σ is called the associated measure of μ.

Proof. In Proposition 2.10 we proved that $\mu * (g * \tilde{g}) \in P_b(Q)$. Therefore by [5, Theorem 10.4], there is a $\sigma_g \in M_+^b(\hat{Q})$ satisfying $\mu * (g * \tilde{g}) = \check{\sigma_g}$. By

Lemma 2.16,

$$
\begin{aligned}
(|\hat{g}|^2\sigma_f)\check{} &= \check{\sigma_f} * (|\hat{g}|^2)\check{} \\
&= \check{\sigma_f} * ((g * g^\star)\hat{})\check{} \\
&= (\mu * f * f^\star) * (g * g^\star) \quad \text{(Lemma 2.8)} \\
&= (\mu * g * g^\star) * (f * f^\star) \\
&= \check{\sigma_g} * (|\hat{f}|^2)\check{} \\
&= (|\hat{f}|^2\sigma_g)\check{}
\end{aligned}
$$

By the uniqueness of the inverse Fourier transform, we have

$$
|\hat{g}|^2\sigma_f = |\hat{f}|^2\sigma_g. \tag{2.8}
$$

For any $\sigma \in M_+(\hat{Q})$ satisfying *(i)* and *(ii)*, we should have $|\hat{g}|^2\sigma = \sigma_g$ where $g \in C_c(Q)$ and define σ accordingly. We will show that σ is well defined. Choose $g \in C_c(Q)$ such that $\hat{g} \neq 0$ on $\mathrm{supp}(h)$ [2], so that

$$
\int_{\hat{Q}} h d\sigma = \int_{\hat{Q}} \frac{h}{|\hat{g}|^2} d\sigma_g
$$

where $\frac{h}{|\hat{g}|^2}$ is defined to be zero where $\hat{g}(\xi) = 0$. By (2.8), $\int_{\hat{Q}} h d\sigma$ is independent from the choice of g. Clearly $h \mapsto \int_{\hat{Q}} h d\sigma$ is positive and linear on $C_c(\hat{Q})$, and $\sigma \in M_+(\hat{Q})$. □

Definition 2.18. Let $\mu \in M(Q)$. If there is $\sigma \in M_+(\hat{Q})$ such that for all $f \in C_c(Q)$

$$
\int_Q |\hat{f}|^2 d\sigma < \infty \qquad \text{and} \qquad \int_Q f * f^\star d\mu = \int_{\hat{Q}} |\hat{f}|^2 d\sigma,
$$

then $F\mu := \sigma$ is called the generalized Fourier transform of μ.

Corollary 2.19. *Let Q be a cocommutative KPC-hypergroup. If $\mu \in M_b^p(Q)$, and $\sigma \in M_+(\hat{Q})$, then following statements are equivalent*

i. $\sigma = F\mu$;

ii. $\int_Q g * g^\star d\mu = \int_{\hat{Q}} |(g^-)\hat{}|^2 d\sigma \qquad (g \in C_c(Q))$.

Proof. The proof is similar to the case of DJS-hypergroups. Let $\sigma = F\mu$. By using polarization, we have

$$
\int_Q f * g^\star d\mu = \int_Q \hat{f}\bar{\hat{g}}, \quad (f, g \in C_c(Q)).
$$

If we replace f by $\delta_p * \bar{f}$, and g by $\bar{f}$, we get

$$
(\mu * f * f^\star)(p) = \int_{\hat{Q}} \xi(p)|\hat{f}(\xi)|^2 d\sigma(\xi).
$$

Thus σ is the associated measure of μ. Now if we replace p by e,

$$\begin{aligned}\int_Q \xi(e)|(f^-\hat{)}|^2(\xi)d\sigma(\xi) &= \int_Q |(f^-\hat{)}|^2(\xi)d\sigma(\xi) \quad \text{(Theorem 2.17 } ii)\\ &= (\mu * f^- * (f^-)^\star)(e) \quad \text{(by 2.7)}\\ &= \int_Q \Delta(f^- * (f^-)^\star)(p^\star, e)d\mu(p) \quad \text{(by } H_2)\\ &= \int_Q f^- * (f^-)^\star(p^\star)d\mu(p) \quad \text{(by } H_3)\\ &= \int_Q (f^- * (f^-)^\star)^-(p)d\mu(p) \quad \text{(by Lemma 3.1)}\\ &= \int_Q (f^\star * f)(p)d\mu(p) \quad \text{(by Lemma 2.8 } ii)\\ &= \int_Q (f * f^\star)(p)d\mu(p).\end{aligned}$$

Therefore $\int_Q |(f^-\hat{)}|^2 d\sigma = \int_Q f * f^\star d\mu$. The converse is proved similarly by using the polarization. □

Corollary 2.20. *The following statements are equivalent*

i. $\mu \in M^p(Q)$.

ii. There exists $\sigma \in M_+(\hat{Q})$ *such that*

$$\int_Q g * g^\star d\mu = \int_{\hat{Q}} |(g^-\hat{)}|^2 d\sigma \qquad (g \in C_c(Q)).$$

3. Main results

In this section we present the main theorem of this paper.

Lemma 3.1. *Let* $f, g \in C_c(Q)$. *Then* $(f * g)^- = g^- * f^-$.

Proof. Let $f, g \in C_c(Q)$. We have

$$\begin{aligned}(f * g)^-(p) &= (f * g)(p^\star) \quad \text{(by 2.6)}\\ &= \int f(q)\Delta g(q^\star, p^\star)dm(q) \quad \text{(by 2.3)}\\ &= \int f(q)\Delta g^-(p, q)dm(q) \quad \text{(by 2.4)}\\ &= \int g^-(q)\Delta f(p^\star, q)dm(q) \quad \text{(by 2.3)}\\ &= \int g^-(q)\Delta f^-(q^\star, p)dm(q) \quad \text{(by 2.6)}\\ &= (g^- * f^-)(p).\end{aligned}$$

Therefore $(f * g)^{-} = g^{-} * f^{-}$. □

Lemma 3.2. *Let $f, g, h \in C_c(Q)$. Then $(f * g) * h = f * (g * h)$.*

Proof. For $f, g, h \in C_c(Q)$ by [5, Proposition 3.4] we have $(fm * gm) * hm = fm * (gm * hm)$. On the other hand by Definition 2.3, we have

$$(fm * gm) * hm = (f * g)m * hm = [(f * g) * h]m,$$

$$fm * (gm * hm) = fm * (g * h)m = [f * (g * h)]m.$$

Thus by uniqueness in the Riesz representation theorem, $(f * g) * h = f * (g * h)$. □

Theorem 3.3. *Let $f, g, h \in C_c(Q)$. Then*

$$\int [(f * g)h]dm = \int f(h * g^{-})dm.$$

Proof. By Definition 2.1, we have $e^{\star} = e$. Thus by H_2,

$$\begin{aligned}
\int_Q [(f * g)h](p)dm(p) &= \int_Q (f * g)(p)\Delta h(e, p)dm(p) \quad (\text{by 2.3 and } e^{\star} = e) \\
&= \int_Q (f * g)(p)\Delta h^{-}(p^{\star}, e)dm(p) \quad (\text{by 2.6}) \\
&= [(f * g) * h^{-}](e) \quad (\text{Lemma 3.2}) \\
&= [f * (g * h^{-})](e) \quad (\text{Lemma 3.1}) \\
&= [f * (h * g^{-})^{-}](e) \quad (\text{by 2.6}) \\
&= \int_Q f(p)\Delta(h * g^{-})^{-}(p^{\star}, e)dm(p) \quad (\text{by 2.3}) \\
&= \int_Q f(p)\Delta(h * g^{-})(e^{\star}, p)dm(p) \quad (\text{by H}_2 \text{ and } e^{\star} = e) \\
&= \int_Q f(p)(h * g^{-})(p)dm(p),
\end{aligned}$$

where $f, g, h \in C_c(Q)$. □

Lemma 3.4. *Let η be a character of Q. Then for any $f \in C_b(Q)$ and $q \in Q$, we have*

$$(\eta * \overline{f})(q) = \int_Q \eta(q)\overline{(\eta f)}(p^{\star})dm(p). \tag{3.1}$$

Proof. By (2.6) for any $q \in Q$

$$\begin{aligned}
(\eta * \overline{f})(q) &= \int_Q \eta(p)\Delta\bar{f}(p^\star, q)dm(p) \quad \text{(by 2.3)} \\
&= \int_Q \eta(p)\Delta(\bar{f})^-(q^\star, p)dm(p) \quad \text{(by 2.4)} \\
&= \int_Q \Delta\eta(q,p)(\bar{f})^-(p)dm(p) \quad (\text{by } H_3) \\
&= \int_Q \Delta\eta(q,p)\bar{f}(p^\star)dm(p) \\
&= \int_Q \eta(q)\eta(p)\bar{f}(p^\star)dm(p) \\
&= \int_Q \eta(q)\bar{\eta}(p^\star)\bar{f}(p^\star)dm(p) \\
&= \int_Q \eta(q)\overline{(\eta f)}(p^\star)dm(p).
\end{aligned}$$

□

Remark 3.5. In the following theorem, we will assume that the approximate identity (e_n) (which is introduced in [5, Theorem 5.9]) satisfies the following properties: $e_n \in C_c^+(Q)$ and $\text{supp}(e_n) \subseteq V_n$, where $(V_n)_{n\in\mathbb{N}}$ is a fundamental system of open relatively compact neighborhoods of e such that $\cap_{n\in\mathbb{N}} V_n = \{e\}$ and $V_n \supseteq V_{n+1}$.

We recall the following lemma and theorem from [5] without proof. In the following lemma, we denote the space of bounded characters on Q by X_h.

Lemma 3.6. *Let $\chi_1.\chi_2$ be a positive definite function on Q for all $\chi_1, \chi_2 \in \hat{Q}$. Then there exists a nonnegative finite regular Borel measure ρ_{χ_1,χ_2} on X_h such that*

$$\chi_1(p)\chi_2(p) = \int_{X_h} \chi(p)d\rho_{\chi_1,\chi_2}(\chi). \tag{3.2}$$

Theorem 3.7. *Let Q be a cocommutative hypergroup satisfying the following properties:*

(1) *the character ϵ defined in (H_2) belongs to $\hat{Q}$;*

(2) *the product of two characters $\chi_1, \chi_2 \in \hat{Q}$ is a positive defnite function, and the support of the measure ρ_{χ_1,χ_2} defined by (3.2) is contained in $\hat{Q}$;*

(3) *the comultiplication* $\hat{\Delta} : C_b(\hat{Q}) \longrightarrow C_b(\hat{Q} \times \hat{Q})$ *defined by*

$$\hat{\Delta}(F)(\chi_1, \chi_2) = \int_{\hat{Q}} F(\chi) d\rho_{\chi_1,\chi_2}(\chi), \quad F \in C_b(\hat{Q}),$$

satisfies axiom $(H_1)(iv)$*. Then* $\hat{Q}$ *is also a locally compact cocommutative KPC-hypergroup, called dual hypergroup, that satisfies the conditions of this theorem, and the hypergroup* $\hat{\hat{Q}}$ *coincides with* Q*. The dual of a compact hypergroup is a discrete hypergroup, and the dual of a discrete hypergroup is a compact hypergroup.*

Theorem 3.8. *Let* Q *be as above and* μ *be a shift-bounded positive definite measure on* Q *with associated measure* σ*. For every* γ *in* $\hat{Q}$*, the measure* $\gamma\mu$ *is also a positive definite measure with associated measure* $\delta_\gamma * \sigma$*.*

Proof. Let $g \in C_c^+(Q)$ and put $h^- := g * \tilde{g}$. For each $f \in C_c(Q)$, we have

$$\begin{aligned}
&\int_Q [\gamma(f * \tilde{f}) * h](p) d\mu(p) \\
&= \int_Q \int_Q \gamma(q)(f * \tilde{f})(q) \Delta h(q^\star, p) dm(q) d\mu(p) \\
&= \int_Q \gamma(q)(f * \tilde{f})(q) \int_Q \Delta h(q^\star, p) d\mu(p) dm(q) \quad \text{(by 2.3)} \\
&= \int_Q \gamma(q)(f * \tilde{f})(q) \int_Q \Delta h^-(p^\star, q) d\mu(p) dm(q) \quad \text{(by 2.7)} \\
&= \int_Q \gamma(q)(f * \tilde{f})(q)(\mu * h^-)(q) dm(q) \\
&= \int_Q \gamma(q)(f * \tilde{f})(q)(\mu * g * \tilde{g})(q) dm(q) \quad \text{(Theorem 2.17)} \\
&= \int_Q \int_{\hat{Q}} \gamma(q)(f * \tilde{f})(q) \xi(q) |\hat{g}(\xi)|^2 d\sigma(\xi) dm(q) \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 \int_Q (f * \tilde{f})(q) \eta(q) dm(q) d\rho_{\gamma,\xi}(\eta) d\sigma(\xi) \quad \text{(Theorem 3.3)} \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 \int_Q f(q)(\eta * (\tilde{f})^-)(q) dm(q) d\rho_{\gamma,\xi}(\eta) d\sigma(\xi) \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 \int_Q f(q)(\eta * \bar{f})(q) dm(q) d\rho_{\gamma,\xi}(\eta) d\sigma(\xi) \quad \text{(by 3.1)} \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 \int_Q \int_Q (\eta f)(q) \overline{(\eta f)}(p^\star) dm(p) dm(q) d\rho_{\gamma,\xi}(\eta) d\sigma(\xi) \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 (\eta f)\hat{}(1) \overline{(\eta f)\hat{}}(1) d\rho_{\gamma,\xi}(\eta) d\sigma(\xi) \\
&= \int_{\hat{Q}} \int_{\hat{Q}} |\hat{g}(\xi)|^2 \mid (\eta f)\hat{}(1) \mid^2 d\rho_{\gamma,\xi}(\eta) d\sigma(\xi)
\end{aligned}$$

If we put $j(\eta) =| (\eta f\hat{)}(1) |^2$, then by Theorem 3.7 (iii), we have

$$\int_{\hat{Q}}\int_{\hat{Q}} |\hat{g}(\xi)|^2 \mid (\eta f\hat{)}(1) \mid^2 d\rho_{\gamma,\xi}(\eta)d\sigma(\xi) = \int_{\hat{Q}} |\hat{g}(\xi)|^2 \hat{\Delta} j(\gamma,\xi)d\sigma(\xi).$$

Now we replace g by e_n in the above relations (the net (e_n) has been introduced in [5, Theorem 5.9]). By Urysohn's lemma, there is an $h_0 \in C_c(Q)$ such that $h_0 \equiv \| f \|_\infty$ on the compact set $\text{supp}(f) * U$, where U is a compact neighborhood of the identity $e \in Q$ that contains a fundamental system of neighborhoods $\{V_n\}$ as in the above remark. By [5, Lemma 4.5] we have $\| \mid f \mid * e_n \|_\infty \leq \| f \|_\infty \| e_n \|_1 = \| f \|_\infty$. Since (e_n) is an approximate identity, by [5, Theorem 5.9] $e_n^\star = e_n$ and $\mid \overline{e}_n \mid = e_n$. Now since Q is a cocommutative KPC-hypergroup, by Lemma 2.8, we have

$$\begin{aligned} \mid \gamma(f * \tilde{f}) * (e_n * \overline{e}_n) \mid &\leq \mid \gamma \mid (|f| * |\tilde{f}|) * (e_n * \overline{e}_n) \\ &\leq (|f| * e_n) * (|f| * e_n)^- \\ &\leq h_0 * h_0^- \in L^1(Q, m). \end{aligned}$$

For any $\xi \in \hat{Q}$,

$$\mid \hat{e_n}(\xi) \mid \leq \int_Q \mid \overline{\xi(p)} \mid e_n(p)dm(p) \leq \| e_n \|_1 = 1.$$

Thus for any $\xi \in \hat{Q}$, we have $\mid \hat{e_n}(\xi) \mid \Delta j(\gamma,\xi) \leq \Delta j(\gamma,\xi)$. Also

$$\begin{aligned} \int_{\hat{Q}} \hat{\Delta} j(\gamma,\xi)d\sigma(\xi) &= \int_Q \int_{\hat{Q}} \gamma(q)(f * \tilde{f})(q)\xi(q)d\sigma(\xi)dm(q) \\ &= \int_Q \gamma(q)(f * \tilde{f})(q)\check{\sigma}(q)dm(q) < \infty, \end{aligned}$$

since $\check{\sigma} \in C(Q)$, and so that $\gamma(f * \tilde{f})\check{\sigma} \in C_c(Q)$. Thus $\hat{\Delta} j(\gamma,\xi) \in L^1(\hat{Q},\sigma)$. Therefore, applying the dominated convergence theorem in both sides of the equality

$$\int_Q [\gamma(f * \tilde{f}) * (e_n * \tilde{e_n})](p)d\mu(p) = \int_{\hat{Q}} \mid \hat{e}_n(\xi) \mid^2 \hat{\Delta} j(\gamma,\xi)d\sigma(\xi),$$

we have

$$\begin{aligned} \int_Q [\gamma(f * \tilde{f})](p)d\mu(p) &= \int_{\hat{Q}} \hat{\Delta} j(\gamma,\xi)d\sigma(\xi) \quad \text{(by 2.5)} \\ &= \int_{\hat{Q}} j(q)d(\delta_\gamma * \sigma)(q). \end{aligned}$$

Thus

$$(\eta f\hat{)}(1) = \int_Q f(p)\eta(p)dm(p) = \int_Q f(p^*)\eta(p^*)dm(p^*)$$

$$= \int_Q f(p^\star)\eta(p^\star)dm(p) = (f^-\hat{)}(\eta).$$

Therefore $j(\eta) = \mid (\eta f\hat{)}(1) \mid^2 = \mid (f^-\hat{)}(\eta) \mid^2$. Thus

$$\int_Q (f * \tilde{f})d(\gamma\mu) = \int_{\hat{Q}} \mid (f^-\hat{)} \mid^2 d(\delta_\gamma * \sigma).$$

This completes the proof. □

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