ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 1, pp. 17–24

Title:

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Author(s):

N. Jing and C. Wang

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 1, pp. 17–24
Online ISSN: 1735-8515

MODULES OF THE TOROIDAL LIE ALGEBRA $\widehat{\widehat{\mathfrak{sl}}}_2$

N. JING AND C. WANG*

(Communicated by Fariborz Azarpanah)

ABSTRACT. Highest weight modules of the double affine Lie algebra $\widehat{\widehat{\mathfrak{sl}}}_2$ are studied under a new triangular decomposition. Singular vectors of Verma modules are determined using a similar condition with horizontal affine Lie subalgebras, and highest weight modules are described under the condition $c_1 > 0$ and $c_2 = 0$.
Keywords: Double affine Lie algebras, Verma module, integrability, irreducibility.
MSC(2010): Primary: 17B67; Secondary: 17B10, 17B65.

1. Introduction

Toroidal Lie algebras are multiloop generalization of the affine Lie algebras. Their representations can be studied with similar but distinctive methods as their affine counterparts. Classification of irreducible integrable modules with finite dimensional weight spaces has been carried out and properties of the integrable modules have been investigated in [5–9]. Berman and Billig [1] constructed general modules by the standard induction procedure and studied their irreducible quotients using vertex operator techniques. As a special subalgebra of the toroidal Lie algebra, the double affine algebras also have similar representation theory. However, the integrable modules are no longer completely reducible [4].

In this paper, we use a different triangular decomposition to study representations of the double affine Lie algebra $\mathfrak{T}$ [11]. We determine all singular vectors of the Verma modules and give relatively easier description of their submodule structures. When one canonical center is positive and the other center is trivial, we are able to determine integrability and irreducibility of the Verma modules, which then enable us to describe general highest weight modules.

Article electronically published on February 22, 2017.
Received: 22 May 2015, Accepted: 7 October 2015.
*Corresponding author.

The paper is organized as follows. In Section 2, we describe a new triangular decomposition of the double affine algebra of $\mathfrak{sl}_2$. In Section 3, we study the Verma modules based on the triangular decomposition. Using the affine Weyl group, we fix the singular vectors of $M(\lambda)$ and study integrability of the quotient $W(\lambda)$. In Section 4, we give a necessary and sufficient condition for the irreducibility of $M(\lambda)$ and $W(\lambda)$.

Throughout the paper, we will denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ the sets of integers, nonnegative integers and positive integers, respectively.

2. The toroidal Lie algebra $\widehat{\widehat{\mathfrak{sl}}}_2$

Let $\mathfrak{sl}_2(\mathbb{C})$ be the three dimensional simple Lie algebra generated by $e, f, \alpha^\vee$ with the canonical bilinear form given by $(e|f) = \frac{1}{2}(\alpha^\vee|\alpha^\vee) = 1$, where α is the simple root. The toroidal Lie algebra $\widehat{\widehat{\mathfrak{sl}}}_2 = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ is a central extension of the 2-loop algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ with the following Lie bracket:

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + (x|y)\delta_{r_1,-s_1}\delta_{r_2,-s_2}(r_1c_1 + r_2c_2), \tag{2.1a}$$

$$[x \otimes t^r, c_1] = [x \otimes t^r, c_2] = 0, \tag{2.1b}$$

$$[d_i, x \otimes t^r] = r_i x \otimes t^r, \quad [d_i, c_j] = 0 \tag{2.1c}$$

where $x, y \in \mathfrak{sl}_2(\mathbb{C})$, $r = (r_1, r_2) \in \mathbb{Z}^2$, $s = (s_1, s_2) \in \mathbb{Z}^2$, and $t^r = t_1^{r_1} t_2^{r_2}$. In the following we also denote $x(m, n) = x \otimes t_1^m t_2^n$.

The Cartan subalgebra is $\mathfrak{h} = \mathbb{C}\alpha^\vee \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, where c_1 and c_2 are central elements. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. For a functional $\beta \in \mathfrak{h}^*$, let $\mathfrak{T}_\beta = \{x \in \mathfrak{T} | [h, x] = \beta(h)x, \forall\, h \in \mathfrak{h}\}$ be the root subspace. The root system Δ of $\mathfrak{T}$ consists of all nonzero $\beta \in \mathfrak{h}^*$ such that $\mathfrak{T}_\beta \neq 0$.

Let $\delta_1, \delta_2, \omega_1, \omega_2 \in \mathfrak{h}^*$ be the linear functionals defined by $\delta_i(d_j) = \delta_{ij}$, $\delta_i(c_j) = \delta_i(\alpha^\vee) = 0$ and $\omega_i(c_j) = \delta_{ij}$, $\omega_i(d_j) = \omega_i(\alpha^\vee) = 0$ for $i, j = 1, 2$. It is easy to check that the root system $\Delta = \{\pm\alpha + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2\} \cup (\{\mathbb{Z}\delta_1 + \mathbb{Z}\delta_2\} \backslash \{0\})$. Let $(\ |\ )$ be the invariant form on $\mathfrak{h}$ defined by

$$(\alpha^\vee|\alpha^\vee) = 2,\ (\alpha^\vee|c_i) = (\alpha^\vee|d_j) = 0,$$
$$(c_i|d_j) = \delta_{ij},\ (c_i|c_j) = (d_i|d_j) = 0.$$

Then the associated invariant form on $\mathfrak{h}^*$ is given by

$$(\alpha|\alpha) = 2,\ (\alpha|\delta_i) = (\alpha|\omega_j) = 0,$$
$$(\delta_i|\omega_j) = \delta_{ij},\ (\delta_i|\delta_j) = (\omega_i|\omega_j) = 0,$$

where $i, j = 1, 2$. The real and imaginary roots are given respectively by

$$\Delta^{re} = \{\pm\alpha + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2\}, \tag{2.2a}$$

$$\Delta^{im} = \{\mathbb{Z}\delta_1 + \mathbb{Z}\delta_2\} \backslash \{0\}. \tag{2.2b}$$

Clearly $\widehat{\mathfrak{g}_i} = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t_i, t_i^{-1}] \oplus \mathbb{C}c_i \oplus \mathbb{C}d_i$ $(i = 1, 2)$ are two subalgebras of $\mathfrak{T}$ that are isomorphic to the affine Lie algebra $A_1^{(1)}$. Denote the root system of $\widehat{\mathfrak{g}_i}$ by $\Delta_{\widehat{\mathfrak{g}_i}}$. It is well-known that $\Delta_{\widehat{\mathfrak{g}_1}}$ is decomposed into positive and negative parts [12]: $\Delta_{\widehat{\mathfrak{g}_1}} = \Delta_{\widehat{\mathfrak{g}_1}+} \cup \Delta_{\widehat{\mathfrak{g}_1}-}$, where

$$\begin{aligned}\Delta_{\widehat{\mathfrak{g}_1}+} =&\{\alpha + \mathbb{Z}_+\delta_1\} \cup \{-\alpha + \mathbb{N}\delta_1\} \cup \mathbb{N}\delta_1,\\ \Delta_{\widehat{\mathfrak{g}_1}-} =&\{-\alpha - \mathbb{Z}_+\delta_1\} \cup \{\alpha - \mathbb{N}\delta_1\} \cup \{-\mathbb{N}\delta_1\}.\end{aligned}$$

Let $\alpha_1 = \alpha$, $\alpha_0 = \delta_1 - \alpha$ and $\alpha_{-1} = \delta_2 - \alpha$. Then all roots are integral linear combination of $\alpha_i, i = -1, 0, 1$, and they are called the "simple" roots [15] of $\mathfrak{T}$. However some roots can not be represented as negative or positive linear combinations of the "simple" roots. In this paper, we view $\mathfrak{T}$ as an affinization of the Lie algebra $\widehat{\mathfrak{g}_1}$ and define the following partition of Δ:

$$\begin{aligned}\Delta_+ =&\{\alpha + \mathbb{Z}_+\delta_1 + \mathbb{Z}_+\delta_2\} \cup \{-\alpha + \mathbb{N}\delta_1 + \mathbb{Z}_+\delta_2\} \cup \{\mathbb{N}\delta_1 + \mathbb{Z}_+\delta_2\}\\ &\cup \{-\alpha - \mathbb{Z}_+\delta_1 + \mathbb{N}\delta_2\} \cup \{\alpha - \mathbb{N}\delta_1 + \mathbb{N}\delta_2\} \cup \{-\mathbb{N}\delta_1 + \mathbb{N}\delta_2\} \cup \mathbb{N}\delta_2\end{aligned}$$

and

$$\begin{aligned}\Delta_- =&\{-\alpha - \mathbb{Z}_+\delta_1 - \mathbb{Z}_+\delta_2\} \cup \{\alpha - \mathbb{N}\delta_1 - \mathbb{Z}_+\delta_2\} \cup \{-\mathbb{N}\delta_1 - \mathbb{Z}_+\delta_2\}\\ &\cup \{\alpha + \mathbb{Z}_+\delta_1 - \mathbb{N}\delta_2\} \cup \{-\alpha + \mathbb{N}\delta_1 - \mathbb{N}\delta_2\} \cup \{\mathbb{N}\delta_1 - \mathbb{N}\delta_2\} \cup \{-\mathbb{N}\delta_2\}.\end{aligned}$$

The corresponding positive (resp. negative) root space is denoted by $\mathfrak{T}_+ = \bigoplus_{\beta\in\Delta_+} \mathfrak{T}_\beta$ (resp. $\mathfrak{T}_- = \bigoplus_{\beta\in\Delta_-} \mathfrak{T}_\beta$). Then $\mathfrak{T} = \mathfrak{T}_+ \oplus \mathfrak{h} \oplus \mathfrak{T}_-$ is the associated triangular decomposition of $\mathfrak{T}$.

Let $\mathcal{Q}_+ = \mathbb{Z}_+$-span of Δ_+. Similarly, $\mathcal{Q}_{1+} = \mathbb{Z}_+$-span of $\Delta_{\widehat{\mathfrak{g}_1}+} = \mathbb{Z}_+\alpha_0 + \mathbb{Z}_+\alpha_1$ is the positive root lattice of $\widehat{\mathfrak{g}_1}$. Let $\lambda, \mu \in \mathfrak{h}^*$. We say that $\lambda \geq \mu$ if $\lambda - \mu$ is a nonnegative linear combination of roots in Δ_+.

The Weyl group of $\mathfrak{T}$ is defined as usual [15].

Definition 2.1. For a real root $\beta = \pm\alpha + n_1\delta_1 + n_2\delta_2$, we define the reflection r_β on $\mathfrak{h}^*$ by

$$r_\beta(\lambda) = \lambda - \lambda(\beta^\vee)\beta,$$

where $\lambda \in \mathfrak{h}^*$ and $\beta^\vee = \pm\alpha^\vee + n_1c_1 + n_2c_2$. The Weyl group $W_{\mathfrak{T}}$ is generated by r_β $(\beta \in \Delta^{re})$.

3. The Verma module $M(\lambda)$

In this section, we study highest weight modules of $\mathfrak{T}$. Integrable modules are constructed by Chari [4] for double affine Lie algebras, and a classification has been given by Rao [5–8] and Jiang [9] for irreducible integrable modules of the toroidal Lie algebras. We will take the new triangular decomposition to study the Verma modules $M(\lambda)$.

Definition 3.1. A module M of $\mathfrak{T}$ is called a highest weight module if there exists some $0 \neq v \in M$ such that

(1) the vector v is a weight vector, that is $h.v = \lambda(h)v$ for some $\lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$,
(2) $\mathfrak{T}_+.v = 0$,
(3) $U(\mathfrak{T}).v = M$,

where $U(\mathfrak{T})$ is the universal enveloping algebra of $\mathfrak{T}$.

Definition 3.2. A module M of $\mathfrak{T}$ is integrable if M is a weight module and all $x_\alpha(m, n)$'s are locally nilpotent, i.e. for any nonzero $v \in M$ there exists $N = N(\alpha, m, n, v)$ such that $x_\alpha(m, n)^N.v = 0$.

Definition 3.3. A nonzero element $v \in M$ is called a singular vector if it is a weight vector and $\mathfrak{T}_+.v = 0$.

Let $\lambda \in \mathfrak{h}^*$. The one-dimensional vector space $\mathbb{C}1_\lambda$ can be viewed as a $\mathfrak{T}_+ \oplus \mathfrak{h}-$module with $\mathfrak{T}_+.1_\lambda = 0$ and $h.1_\lambda = \lambda(h) \cdot 1_\lambda$ for all $h \in \mathfrak{h}$. Assume that the central element c_1 acts as a scalar $k_1 \geq 0$, and the other center c_2 acts trivially. Then we have the induced Verma module:

$$M(\lambda) = U(\mathfrak{T}) \otimes_{U(\mathfrak{T}_+ \oplus \mathfrak{h})} \mathbb{C}1_\lambda.$$

Proposition 3.4. (1) *$M(\lambda)$ is a $U(\mathfrak{T}_-)-$free module generated by the highest weight vector: $1 \otimes 1_\lambda = v_\lambda$.*
(2) *$\dim M(\lambda)_\lambda = 1$; $0 < \dim M(\lambda)_{\lambda-\beta} < +\infty$ for every $\beta \in \mathcal{Q}_{1+}$; otherwise, $\dim M(\lambda)_{\lambda-\gamma} = \infty$ for any $\gamma \in \mathcal{Q}_+$.*
(3) *The module $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$.*

We now determine all singular vectors of $M(\lambda)$. In the following, we will consider the properties of $M(\lambda)$ and $L(\lambda)$ under the assumptions that the highest weight λ is dominant on $\widehat{\mathfrak{g}_1}$.

Proposition 3.5. *If $\lambda(\alpha_i^\vee) = n_i$ $(i = 0, 1)$ are nonnegative integers, then $\sum_{i=0}^{1} U(\mathfrak{T}_-).y_i^{n_i+1}v_\lambda$ is a proper submodule of $M(\lambda)$, where $y_1 = y, y_0 = x \otimes t_1^{-1}$.*

Proof. We need to show that $y_i^{n_i+1}v_\lambda$ $(i = 0, 1)$ are singular vectors of $M(\lambda)$, i.e, $\mathfrak{T}_+.y_i^{n_i+1}v_\lambda = 0$ for $i = 0, 1$. It suffice to show that $e(m, n) \in \mathfrak{T}_+$ and $f(m', n') \in \mathfrak{T}_+$ act trivially on $y_i^{n_i+1}.v_\lambda$ $(i = 0, 1)$. Because the weight of $e(m, n)y_i^{n_i+1}.v_\lambda$ $(m \in \mathbb{Z}, n \in \mathbb{N})$ or $f(m', n')y_i^{n_i+1}.v_\lambda$ $(m' \in \mathbb{Z}, n' \in \mathbb{N})$ is higher than λ, we get $e(m, n)y_i^{n_i+1}.v_\lambda = f(m', n')y_i^{n_i+1}.v_\lambda = 0$ $(m, m' \in \mathbb{Z}, n, n' \in \mathbb{N})$. Therefore, we only need to consider $z.y_i^{n_i+1}.v_\lambda$ with $z \in \widehat{\mathfrak{g}}_{1+}$. But this is zero as it is the case of affine Lie algebras. □

Denote the quotient $W(\lambda) = M(\lambda)/\sum_{i=0}^{1} U(\mathfrak{T}_-).y_i^{n_i+1}v_\lambda$.

Let $W_{\widehat{\mathfrak{g}_1}}$ be the Weyl group of the affine Lie algebra $\widehat{\mathfrak{g}_1}$ defined as above. It has two generators given by the simple reflections r_{α_0} and r_{α_1}. For arbitrary $w \in W_{\widehat{\mathfrak{g}_1}}$, we define $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ satisfies $\rho(\alpha_i^\vee) = 1$ $(i = 0, 1)$.

Corollary 3.6. *For arbitrary $w \in W_{\widehat{\mathfrak{g}_1}}$, the module $M(w \cdot \lambda)$ is a submodule of $M(\lambda)$.*

Proof. This can be proved by induction on the length of w as in Proposition 3.5. □

We now look at the integrability of $W(\lambda)$, where $\lambda(\alpha_i^\vee) = n_i \geq 0$, $i = 0, 1$.

Proposition 3.7. *Suppose that c_1 acts on $W(\lambda)$ as a scalar $k_1 > 0$ and c_2 is trivial. Then $W(\lambda)$ is not integrable.*

Proof. Let $W(\lambda) = U(\mathfrak{T}_-).w_\lambda$, where w_λ is the image of v_λ in $W(\lambda)$. We claim that $e(0,-1)$ is not locally nilpotent. Suppose that $e(0,-1)$ is nilpotent on w_λ and assume that N is the minimum positive integer such that $e(0,-1)^N.w_\lambda = 0$. Then we have

$$\begin{aligned} 0 =& f(0,1)e(0,-1)^N.w_\lambda \\ =& [f(0,1), e(0,-1)^N].w_\lambda \\ =& -Ne(0,-1)^{N-1}((N-1)\cdot 1 - (c_2 - \alpha^\vee)).w_\lambda \\ =& -N(N-1+n_1)e(0,-1)^{N-1}.w_\lambda, \end{aligned}$$

where we have used $\lambda(\alpha_{-1}^\vee) = \lambda(c_2 - \alpha^\vee) = -n_1$.

By the minimality of N we obtain that $(N-1) + n_1 = 0$, then $N = 1$ and $n_1 = 0$. Then $e(0,-1).w_\lambda = 0$ and

$$f(1,0)e(0,-1).w_\lambda = -\alpha^\vee(1,-1).w_\lambda = 0. \tag{3.1}$$

Applying $\alpha^\vee(-1,1)$ to Eq. (3.1), we obtain

$$\alpha^\vee(-1,1)\alpha^\vee(1,-1).w_\lambda = (\alpha^\vee|\alpha^\vee)(-c_1 + c_2).w_\lambda = -2k_1 w_\lambda = 0.$$

This is a contradiction as we have assumed $k_1 > 0$. Therefore, the quotient $W(\lambda)$ is not integrable. □

Proposition 3.8. *Suppose that c_1 acts as a positive constant k_1 and c_2 is trivial. Then some weight spaces of $W(\lambda)$ are infinite dimensional.*

Proof. Observe that $\alpha^\vee(-m,-1).w_\lambda \neq 0$, $m \in \mathbb{N}$. Otherwise, we have

$$\alpha^\vee(m,1)\alpha^\vee(-m,-1).w_\lambda = [\alpha^\vee(m,1), \alpha^\vee(-m,-1)].w_\lambda = 2mk_1 w_\lambda = 0.$$

Similarly we also get that $\alpha^\vee(m,-1).w_\lambda \neq 0$ for $m \in \mathbb{N}$.

We claim that $\{\alpha^\vee(-m,-1)\alpha^\vee(m,-1).w_\lambda, m \in \mathbb{N}\}$ is linearly independent. Suppose there exist $a_m \neq 0$ such that

$$\sum_m a_m \alpha^\vee(-m,-1)\alpha^\vee(m,-1).w_\lambda = 0. \tag{3.2}$$

Let $s \in \{m | a_m \neq 0\}$. Applying $\alpha^\vee(s,1)$ to Eq. (3.2), we obtain

$$
\begin{aligned}
0 =& \sum_m a_m([\alpha^\vee(s,1), \alpha^\vee(-m,-1)]\alpha^\vee(m,-1) \\
&+ \alpha^\vee(-m,-1)[\alpha^\vee(s,1), \alpha^\vee(m,-1)]).w_\lambda \\
=& \sum_m a_m \delta_{s,m}(\alpha^\vee|\alpha^\vee)\alpha^\vee(m,-1)sk_1.w_\lambda \\
=& 2a_s sk_1 \alpha^\vee(1,-m).w_\lambda.
\end{aligned}
$$

This contradiction proves our claim. □

4. Highest weight modules of $\widehat{\widehat{\mathfrak{sl}}}_2$

Futorny [10] studied the imaginary Verma modules (IVM) for affine Lie algebras and proved that an IVM is irreducible if and only if $\lambda(c) \neq 0$. In this section, we prove an irreducibility criterion for Verma modules $M(\lambda)$ when $c_1 \neq 0$ and $c_2 = 0$.

Lemma 4.1. *Let $0 \neq v \in M(\lambda)$ and $M = \bigoplus_{\eta \in \mathcal{Q}_{1+}} M(\lambda)_{\lambda-\eta}$. Then $U(\mathfrak{T})v \cap M \neq 0$.*

Proof. Suppose $v \in M(\lambda)_{\lambda-\mu}$, where $\mu \in \mathcal{Q}_+$. Let $\mu = \sum_{i=0}^{1} n_i \alpha_i + k\delta_2$ for $n_i \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$. Define the height of μ by $ht(\mu) = k$. If $k = 0$, then $\lambda - \mu = \lambda - \sum_{i=0}^{1} n_i \alpha_i$ for some nonnegative integers n_i, so the result holds. Suppose $k > 0$, since $M(\lambda)$ is a free $U(\mathfrak{T}_-)-$module, there exists a homogenous element $u \in U(\mathfrak{T}_-)$ such that $v = uv_\lambda$. By the PBW theorem

$$
u = \sum_p X^{l_{1p}}_{\phi_{1p}-n_{1p}\delta_2} X^{l_{2p}}_{\phi_{2p}-n_{2p}\delta_2} \cdots X^{l_{s(p)p}}_{\phi_{s(p)p}-n_{s(p)p}\delta_2} u_p,
$$

where $u_p \in U(\widehat{\mathfrak{g}_1}-)$, $X_{\phi_{ip}-n_{ip}\delta_2} \in \mathfrak{T}_-$, $\phi_{ip} \in \Delta_{\widehat{\mathfrak{g}_1}}$, $l_{ip}, n_{ip} \in \mathbb{N}$, and $k = \sum_i n_{ip} l_{ip}$ $(i = 1, 2, \ldots, s(p))$ for all p. If $i \neq j$, $\phi_{ip} - n_{ip}\delta_2 \neq \phi_{jp} - n_{jp}\delta_2$ for all p. We will also assume $n_{1p} \geq n_{2p} \geq \cdots \geq n_{s(p)p}$ for all p. Next we consider the set $\Omega \subset \{n_{ip}\delta_2\}$ consisting of φ_{ip} such that $ht(-\varphi_{ip}) = \min_p\{n_{s(p)p}\}$. In Ω, we consider the subset $\Omega^{'}$ consisting of $\varphi_{s(p)p}$ such that $l_{s(j)j} \geq l_{s(p)p}$ for all $\varphi_{ij} \in \Omega$. We then take a subset Ω_0 in $\Omega^{'}$ consisting of all $\varphi_{s(p)p}$ such that $\phi_{ij} \geq \phi_{s(p)p}$. Without loss of generality, we can assume $\phi_{s(1)1} \in \Omega_0$. Since $\phi_{s(1)1} - n_{s(1)1}\delta_2 = \phi_{ip} - n_{ip}\delta_2$ and $l_{s(1)1} = l_{ip}$, we have $i = s(p)$. Choose sufficiently large $\gamma \in \Delta_{\widehat{\mathfrak{g}_1}+}$ such that $\gamma > \phi_{ij}$ for all i, j and $ht_{\widehat{\mathfrak{g}_1}+}(\gamma - \phi_{s(1)1}) < ht_{\widehat{\mathfrak{g}_1}+}(-u_p)$ for all p. Let $0 \neq z \in \mathfrak{T}_{-\gamma+n_{s(1)1}\delta_2}$. We have $zu_pv_\lambda = 0$ for all p since the weight of zu_pv_λ is larger than λ. The choice of $n_{s(1)1}$ and γ ensures that $zv \neq 0$. Since $ht(\mu + \gamma - n_{s(1)1}\delta_2) < ht(\mu)$, by induction hypothesis we get $U(\mathfrak{T})(zv) \cap M \neq 0$. Then $U(\mathfrak{T})v \cap M \neq 0$ because $U(\mathfrak{T})(zv) \subset U(\mathfrak{T})v$. □

Let $M(\lambda)^+ = \{v \in M(\lambda) | \mathfrak{T}_+.v = 0\}$. Clearly it is $\mathfrak{h}$-invariant. For an arbitrary nonzero element $v \in M(\lambda)^+$, we see that $U(\mathfrak{T}_-).v$ is a submodule of $M(\lambda)$. As for the form of elements in $M(\lambda)^+$, we have the following result.

Corollary 4.2. *$M(\lambda)^+ \subset M$.*

Proof. Suppose there exists a nonzero $v \in M(\lambda)^+$, and the weight of v is not of the form $\lambda - \beta$ $(\beta \in \mathcal{Q}_{1+})$. Since $U(\mathfrak{T})v = U(\mathfrak{T}_-)v$, the weight of every element in $U(\mathfrak{T})v$ can not be of the form $\lambda - \gamma$ for any $\gamma \in \mathcal{Q}_{1+}$. Hence $U(\mathfrak{T})v \cap M = 0$, which contradicts Lemma 4.1. □

The subspace M can be viewed as a Verma $\widehat{\mathfrak{g}_1}-$module. Kac and Kazhdan [13] gave a necessary and sufficient condition for the reducibility of the Verma modules for affine Lie algebras.

Theorem 4.3. ([13]) *The Verma module $V(\lambda)$ of $\widehat{\mathfrak{g}_1}$ is reducible if and only if for some positive root β of the algebra $\widehat{\mathfrak{g}_1}$ and some positive integer l, one has $(\lambda+\rho)(\beta^\vee) = l$, where $\rho(\alpha_i^\vee) = 1$ $(i = 0, 1)$. Then $V(\lambda - l\beta)$ are submodules of $V(\lambda)$.*

Theorem 4.4. *The module $M(\lambda)$ is reducible if and only if for some positive root β of the algebra $\widehat{\mathfrak{g}_1}$ and some positive integer l, one has $(\lambda+\rho)(\beta^\vee) = l$, where $\rho(\alpha_i^\vee) = 1$ $(i = 0, 1)$.*

Proof. Suppose that $M(\lambda)$ is reducible. Assume on the contrary that there does not exist a root β of the algebra $\widehat{\mathfrak{g}_1}$ and a positive integer l such that $(\lambda+\rho)(\beta^\vee) = l$. By Theorem 4.3, M is irreducible as a $\widehat{\mathfrak{g}_1}$-module. By Lemma 4.1, for arbitrary $0 \neq v \in M(\lambda)$, we have $U(\mathfrak{T})v \cap M \neq 0$. Since $U(\mathfrak{T})v \cap M$ is a $\widehat{\mathfrak{g}_1}$-submodule of M, $U(\mathfrak{T})v \cap M = M$ by the $\widehat{\mathfrak{g}_1}$-irreducibility of M. Then $M \subset U(\mathfrak{T})v$ and $v_\lambda \in U(\mathfrak{T})v$. Subsequently $U(\mathfrak{T})v = M(\lambda)$. Therefore $M(\lambda)$ is irreducible, which contradicts our assumption and we have proved the necessity.

On the other hand, if for some positive root β of the algebra $\widehat{\mathfrak{g}_1}$, $(\lambda+\rho)(\beta^\vee) = l$ holds for a positive integer l, then M is reducible as a Verma module for $\widehat{\mathfrak{g}_1}$ by Theorem 4.3, i.e., there exists some singular vector $v \notin \mathbb{C}v_\lambda$ of M. Meanwhile, it is also a singular vector of $M(\lambda)$ since the weight of $z(m,n).v$ $(z \in \mathfrak{sl}_2, m \in \mathbb{Z}, n \in \mathbb{N})$ is higher than λ. Thus $M(\lambda)$ is reducible. □

Corollary 4.5. *Let $\lambda \in \mathfrak{h}^*$ such that $\lambda(\alpha_i^\vee)$ $(i = 0, 1)$ are nonnegative. Then $W(\lambda) \cong L(\lambda)$.*

Proof. Since $\lambda(\alpha_i^\vee)$ $(i = 0, 1)$ are nonnegative, $\sum_{i=0}^{1} U(\mathfrak{T}_-).y_i^{n_i+1}v_\lambda$ is a maximal submodule of M as $\widehat{\mathfrak{g}_1}-$module by [12]. According to Theorem 4.3 and Theorem 4.4, we know that the reducibility of the $U(\mathfrak{T})-$module $M(\lambda)$ is equivalent to that of M as $\widehat{\mathfrak{g}_1}-$module. Hence, $W(\lambda)$ is irreducible. □

Corollary 4.6. *Let* $\Delta^+(\lambda) = \{(\beta, l)|(\lambda+\rho)(\beta^\vee) = l, \beta \in \widehat{\mathfrak{g}_1}, l \in \mathbb{N}\}$. *Then* $J(\lambda) = \sum_{(\beta,l)\in\Delta^+(\lambda)} M(\lambda - l\beta)$ *is the maximal submodule of* $M(\lambda)$. *If* V *is the highest weight module of weight* λ, *then* $V \cong M(\lambda)/N(\lambda)$, *where* $N(\lambda) \subset J(\lambda)$.

Acknowledgements

This paper was supported in part by NSFC grants(11271138, 11531004) and Simons Foundation (No. 198129).

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(Naihuan Jing) Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail address: jing@math.ncsu.edu

(Chunhua Wang) School of Information and Mathematics, Yangtze University, Jingzhou 434023, China.
E-mail address: tiankon_g1987@163.com