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φ-CONNES AMENABILITY OF DUAL BANACH ALGEBRAS

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Abstract. Generalizing the notion of character amenability for Banach algebras, we study the concept of φ-Connes amenability of a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$, where φ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ that lies in $\mathcal{A}_*$. Several characterizations of φ-Connes amenability are given. We also prove that the following are equivalent for a unital weakly cancellative semigroup algebra $l^1(S)$:
(i) S is χ-amenable.
(ii) $l^1(S)$ is $\hat{\chi}$-Connes amenable.
(iii) $l^1(S)$ has a $\hat{\chi}$-normal, virtual diagonal.
Keywords: Banach algebras, Connes amenability, derivation, dual Banach algebra, virtual diagonal, weak*-weak* continuous.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. Introduction

In 1972, Barry Johnson introduced the cohomological notion of an amenable Banach algebra [14]. Amenable Banach algebras have since proved themselves to be widely applicable in modern analysis. In many instances, the classical concept of amenability is, however, too strong. For this reason, by relaxing some of the constrains in the definition of amenability, new concepts have been introduced. In [15], B. E. Johnson, R. V. Kadison and J. Ringrose introduced a notion of amenability for Von Neumann algebras which modifies Johnson's original definition for general Banach algebras. This notion of amenability was later dubbed Connes amenability by A. Ya. Helemskii [12].

In [21], Runde extended the notion of Connes amenability to dual Banach algebras. More recently Kaniuth, Lau and Pym have introduced and studied the concept of φ-amenability for Banach algebras [16]. Let $\mathcal{A}$ be a Banach algebra and φ be a character of $\mathcal{A}$, that is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. $\mathcal{A}$ is called φ-amenable if there exists a bounded linear functional m on A^*

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satisfying $m(\varphi) = 1$ and $m(a.f) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Several authors have studied various notions of φ-amenability (see [1, 13, 17, 18]).

In this paper we introduce the concept of φ-Connes amenability for dual Banach algebras. A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. It is easy to show that a Banach algebra $\mathcal{A}$ is dual if and only if it is a dual Banach space such that multiplication in $\mathcal{A}$ is separately weak*-continuous. We define φ-Connes amenability of $\mathcal{A}$ through the vanishing of some cohomology groups. The Banach $\mathcal{A}$-bimodules E that are relevant to us are those where the left action is of the form $a.x = \varphi(a)x$. For the sake of brevity, such E will occasionally be called a Banach φ-bimodule. Throughout the paper, $\Delta(\mathcal{A})$ will denote the set of all homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$. We characterize φ-Connes amenability of $\mathcal{A}$ in terms of the existence of a φ-invariant mean on a weak*-dense subspace of $\mathcal{A}^*$.

Connes amenability of certain Banach algebras is characterized in terms of normal virtual diagonals (see [3,8,9]). Like Connes amenability, the notion of a normal virtual diagonal adapts naturally to the context of general dual Banach algebras. In [21], Runde introduced a stronger variant of Connes amenability, called strong Connes amenability, and showed that the existence of a normal, virtual diagonal for a dual Banach algebra is equivalent to it being strongly Connes amenable.

In [23], Runde proved that the following statements are equivalent for a locally compact group G:

(i) G is amenable.
(ii) $M(G)$ is Connes amenable.
(iii) $M(G)$ has a normal virtual diagonal.

We investigate the above statements for a χ-amenability of semigroup S, $\hat{\chi}$-Connes amenability ($\hat{\chi}$-strong Connes amenability) and the existence of $\hat{\chi}$-normal virtual diagonal for a weakly cancellative semigroup algebra $l^1(S)$. We also discuss some results for general dual Banach algebras.

2. φ-Connes amenability

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. A dual φ-bimodule E is called normal φ-bimodule if for each $x \in E$ the map $a \mapsto x.a$ is weak*-continuous. Note that φ is taken to be in a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$. For an element φ in $\mathcal{A}^*$, the map $a \mapsto a \cdot x = \varphi(a)x$ is not weak* continuous unless in general, $\varphi \in \mathcal{A}_*$ and E is not normal.

Definition 2.1. A dual Banach algebra $\mathcal{A}$ is φ-Connes amenable if, for every normal φ-bimodule E, every bounded weak*-continuous derivation $D : \mathcal{A} \to E$ is inner.

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Let E be a Banach φ-bimodule. Then an element $x \in E$ is called weak*-weakly continuous if the module action $a \mapsto x.a$ is weak*-weak continuous. The collection of all weak*-weakly continuous elements of E is denoted by $\sigma wc(E)$.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. A linear functional $m \in \mathcal{A}^{**}$ is called a mean if $m(\varphi) = 1$. A m is φ-invariant mean if $m(a \cdot f) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}_*$.

Clearly every φ-amenable Banach algebra is φ-Connes amenable. We will see in Example 4.5 that the converse is not true. Our first result is a generalization of the main result of [16].

Theorem 2.3. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Then $\mathcal{A}$ is φ-Connes amenable if and only if $\mathcal{A}^{**}$ has a φ-invariant mean.*

Proof. Let $\mathcal{A}$ be φ-Connes amenable. Let $\mathcal{A}$ be the normal φ-bimodule whose underling space is $\mathcal{A}$, but on which $\mathcal{A}$ acts by

$$a \cdot x = \varphi(a)x \quad \text{and} \quad x \cdot a = xa \quad (a \in \mathcal{A},\ x \in \mathcal{A}).$$

We know that $\mathcal{A}$ is canonically mapped into $\sigma wc(\mathcal{A}^*)^*$. Choose $u \in \mathcal{A}$ with $\varphi(u) = 1$ and define

$$D : \mathcal{A} \to \sigma wc(\mathcal{A}^*)^*, \quad a \mapsto u \cdot a - \varphi(a)u.$$

It is easy to verify that D is a bounded derivation of $\mathcal{A}$ into $\sigma wc(\mathcal{A}^*)^*$. Since $\varphi \in \mathcal{A}_*$, D is a weak*-continuous derivation on $\mathcal{A}$. By [24, Proposition 4.4], $\sigma wc(\mathcal{A}^*)^*$ is a normal φ-bimodule. Clearly, D attains its values in the weak*-closed φ-bimodule $\ker\varphi$. Since $\mathcal{A}$ is φ-Connes amenable, there exists some $n \in \ker\varphi$ such that $D(a) = n \cdot a - \varphi(a)n$ for all $a \in \mathcal{A}$. Letting $m = n - u$, we obtain a φ-invariant mean, see Corollary 4.6 in [24].

Conversely, suppose there exists $m \in \mathcal{A}^{**}$ such that $m(\varphi) = 1$ and $m(a.f) = \varphi(a)m(f)$ for all $f \in \mathcal{A}_*$ and $a \in \mathcal{A}$. Let E be a normal φ-bimodule and let $D : \mathcal{A} \to E$ be a bounded weak*-continuous derivation. Let $D^* : E^* \to \mathcal{A}^*$ denote the adjoint of D. $D' := D^*|_{E_*}$ maps the predual module E_* of E^* into $\mathcal{A}_*$, see section 3.14 in [19]. Let $\pi : \mathcal{A}^{**} \to \mathcal{A}$ be the Dixmier projection. It is well known that the Dixmier projection from $\mathcal{A}^{**}$ onto $\mathcal{A}$ is a module homomorphism. Let $g = D \circ \pi(m)$. Then for all $a \in A$ and $x \in E_*$,

$$\begin{aligned}\langle x, a \cdot g\rangle &= \langle g, x \cdot a\rangle \\ &= \langle D \circ \pi(m), x \cdot a\rangle \\ &= \langle \pi(m), D'(x \cdot a)\rangle \\ &= \varphi(a)\langle \pi(m), D'(x)\rangle \\ &= \varphi(a)\langle D \circ \pi(m), x\rangle \\ &= \varphi(a)\langle g, x\rangle\end{aligned}$$

and hence $a.g = \varphi(a)g$. Since D is a derivation, for the left action of $\mathcal{A}$ on E we get

$$\begin{aligned}\langle D'(a \cdot x), b\rangle &= \langle a \cdot x, D(b)\rangle \\ &= \langle x, D(b) \cdot a\rangle \\ &= \langle x, D(ba)\rangle - \langle x, b \cdot D(a)\rangle \\ &= \langle ba, D'(x)\rangle - \varphi(b)\langle x, D(a)\rangle\end{aligned}$$

for all $b \in \mathcal{A}$. This implies that $D'(a \cdot x) = a \cdot D'(x) - \langle x, D(a)\rangle\varphi$. It follows that

$$\begin{aligned}\langle g \cdot a, x\rangle &= \langle g, a \cdot x\rangle \\ &= \langle Do\pi(m), a \cdot x\rangle \\ &= \langle \pi(m), D'(a \cdot x)\rangle \\ &= \langle \pi(m), a \cdot D'(x)\rangle - \langle \pi(m), \varphi\rangle\langle x, D(a)\rangle \\ &= \varphi(a)\langle \pi(m), D'(x)\rangle - \langle x, D(a)\rangle \\ &= \varphi(a)\langle Do\pi(m), x\rangle - \langle x, D(a)\rangle.\end{aligned}$$

This shows that $D(a) = \varphi(a)g - g \cdot a = a \cdot g - g \cdot a$. Thus D is inner. $\square$

We write $\hat{\otimes}$ for the projective tensor product of Banach spaces. For any dual Banach algebra $\mathcal{A}$, let $\mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})$ denote the set of separately weak*-continuous elements of $\mathcal{L}^2(\mathcal{A}, \mathbb{C}) \cong (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Note that the dual Banach $\mathcal{A}$-bimodule $\mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})$ need not be normal. Let $\triangle$: $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be the multiplication operator, i.e. $\triangle(a \otimes b) = ab$. Since $\triangle^*$ maps $\mathcal{A}_*$ into $\mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})$, it follows that $\triangle^{**}$ drops to an $\mathcal{A}$-bimodule homomorphism $\triangle^{**}_{\mathcal{A}} : \mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})^* \longrightarrow \mathcal{A}$.

Given $F \in \mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})$ and $M \in \mathcal{L}^2_{w^*}(\mathcal{A}, \mathbb{C})^*$, we put

$$\langle M, F\rangle = \int F dM =: \int_{\mathcal{A}\otimes\mathcal{A}} F(a, b) dM(a, b).$$

More generally, let X^* be a dual Banach space and let $F : \mathcal{A} \times \mathcal{A} \to X^*$ be a bilinear map such that $a \to F(a, b)$ and $b \to F(a, b)$ are weak*-weak* continuous. We define $\int F dM \in X^*$ by

$$\langle \int F dM, x\rangle = \int \langle F(a, b), x\rangle dM(a, b).$$

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra. Then $M \in \mathcal{L}^2_{w^*}(\mathcal{A},\mathbb{C})^*$ is called a φ-normal virtual diagonal for $\mathcal{A}$ if $\langle \triangle^{**}_{\mathcal{A}}(M), \varphi\rangle = 1$ and $M.c = \varphi(c)M$ for all $c \in \mathcal{A}$.

The next two theorems are analogs of [20, Theorem 4.4.15] and [21, Theorem 4.7]. It is easy to check that in Definition 2.4, $M.c = \varphi(c)M$ is equivalent to

$$\varphi(c)\int F(a,b)dM(a,b) = \int F(a,bc)dM(a,b)$$

for all $F \in \mathcal{L}^2_{w^*}(\mathcal{A},\mathbb{C})$ and $c \in \mathcal{A}$.

Theorem 2.5. *Let $\mathcal{A}$ be a dual Banach algebra and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. If $\mathcal{A}$ has a φ-normal virtual diagonal then $\mathcal{A}$ is φ-Connes amenable.*

Proof. Let X be a normal dual Banach φ-bimodule with predual X_*. Let $D : \mathcal{A} \to X$ be a weak*-continuous derivation. As in the proof of [20, Theorem 4.4.15], define $F : \mathcal{A} \times \mathcal{A} \to X$ by $F(a,b) = \varphi(a)D(b)$. Put $g = \int F(a,b)dM(a,b) = \int \varphi(a)D(b)dM(a,b)$. For every $x \in X_*$ and $c \in \mathcal{A}$,

$$\begin{aligned}
\langle c\cdot g, x\rangle &= \langle g, x\cdot c\rangle = \varphi(c)\langle g, x\rangle = \varphi(c)\langle \int \varphi(a)D(b)dM(a,b), x\rangle \\
&= \varphi(c)\langle \int F(a,b)dM(a,b), x\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle g.c, x\rangle &= \langle g, c.x\rangle \\
&= \langle int\varphi(a)D(b)dM(a,b), c\cdot x\rangle \\
&= \langle \int \varphi(a)D(b)\cdot c dM(a,b), x\rangle \\
&= \langle \int \varphi(a)D(bc)dM(a,b), x\rangle - \langle \int \varphi(a)b\cdot D(c)dM(a,b), x\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle c\cdot g - g\cdot c, x\rangle &= \langle \int \varphi(ac)D(b) - \varphi(a)D(bc) + \varphi(a)b\cdot D(c)dM(a,b), x\rangle \\
&= \langle \int \varphi(c)F(a,b) - F(a,bc) + D(c)\varphi(ab)dM(a,b), x\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\int\langle\triangle(a\otimes b),\varphi\rangle dM(a,b) &= \int\langle a\otimes b,\triangle^*(\varphi)\rangle dM(a,b)\\ &= \langle\int a\otimes b dM(a,b),\triangle^*(\varphi)\rangle\\ &= \langle M,\triangle^*(\varphi)\rangle\\ &= \langle\triangle^{**}(M),\varphi\rangle = 1.\end{aligned}$$

Consequently $D(c) = g.c - c.g$, and so D is inner. □

Let $\mathcal{A}$ be a dual Banach algebra and let E be a Banach φ-bimodule. Then we call an element $x \in E^*$ a weak*-element if the map $a \mapsto a.x$ is weak*-continuous.

Definition 2.6. Let $\mathcal{A}$ be a dual Banach algebra. $\mathcal{A}$ is called φ-strongly Connes amenable if for each Banach φ-bimodule X^*, every weak*-continuous derivation $D : \mathcal{A} \to X^*$ whose range consists of weak*-elements is inner.

We do not know whether the converse of Theorem 2.5 is true, but for φ-strong Connes amenability, the corresponding question is easy to answer. We follow the standard argument of [21, Theorem 4.7].

Theorem 2.7. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Then $\mathcal{A}$ is φ-strongly Connes amenable if and only if $\mathcal{A}$ has a φ-normal virtual diagonal.*

Proof. Consider the Banach $\mathcal{A}$-bimodule $\mathcal{A}\hat{\otimes}\mathcal{A}$ with the module actions given by

$$(a\otimes b).c = a\otimes bc, \quad c.(a\otimes b) = \varphi(c)a\otimes b \quad (a, b \in \mathcal{A}, c \in \mathcal{A}).$$

Then $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is a φ-bimodule. Let $u \in \mathcal{A}$ such that $\varphi(u) = 1$. As in the proof of [21, Theorem 4.7], define the derivation

$$D : \mathcal{A} \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}, \quad a \mapsto (u\otimes u)a - \varphi(a)(u\otimes u).$$

It is easy to see that D is weak*-continuous and attains its values in the weak*-closed submodule $\ker\varphi\otimes\varphi$. Hence, there is $N \in \ker\varphi\otimes\varphi$ such that $D(a) = a\cdot N - N\cdot a$ for all $a \in \mathcal{A}$. Let $M = N - u\otimes u$, and let $a \in \mathcal{A}$. We have $\varphi(a)M = M\cdot a$ and $\langle\triangle_{\mathcal{A}}^{**}(M),\varphi\rangle = 1$.

The converse follows directly from the above theorem. □

The purpose of the present note is to investigate the relation between the φ-amenability and φ-Connes amenability of dual Banach algebras.

Proposition 2.8. *Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Then $\mathcal{A}$ is φ-Connes amenable if and only if* $\ker\varphi$ *has a left identity.*

Proof. Let $e_{\mathcal{A}}$ be the unit of $\mathcal{A}$. We make $\mathcal{A}$ into a normal φ-bimodule by taking the right multiplication and the left module multiplication to be defined by $a.x = \varphi(a)x$. Let D be the mapping of $\mathcal{A}$ into $\mathcal{A}$ given by $D(a) = a - \varphi(a)e_{\mathcal{A}}$. Then a simple calculation shows that D is a bounded weak*-continuous derivation. Moreover, D maps $\mathcal{A}$ into the weak*-closed submodule $\ker\varphi$. The algebra $\mathcal{A}$ is assumed to be φ-Connes amenable. So there exists $m \in \ker\varphi$ such that $D(a) = a.m - m.a$ for all $a \in \mathcal{A}$. Given $a \in \ker\varphi$, we have $D(a) = a.m - m.a = a - \varphi(a)e_{\mathcal{A}}$, and so $a = -m.a$. This proves that $\ker\varphi$ has a left identity.

The converse statement is an immediate consequence of [16, Proposition 2.1]. □

Corollary 2.9. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Suppose that $\mathcal{A}$ has a bounded approximate identity. Then $\mathcal{A}$ is φ-Connes amenable if and only if $\mathcal{A}$ is φ-amenable.*

Proof. This statement follows from Proposition 2.8 and using the fact that a dual Banach algebra has an identity if and only if it has a bounded approximate identity [21]. □

Our next goal is to characterize the φ-Connes amenability of $\mathcal{A}$ in terms of $\varphi - \sigma wc$- diagonal (See [24]). Let $\triangle$ be the multiplication map. From [24, Corollary 4.6], $\triangle^*$ maps $\mathcal{A}_*$ into $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$. It follows that $\triangle^{**}$ descends to a homomorphism $\triangle_{\sigma wc}$: $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$.

Definition 2.10. A $\varphi - \sigma wc$-diagonal for a Banach algebra $\mathcal{A}$ is an element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$M \cdot a = \varphi(a)M$$

for all $a \in \mathcal{A}$ and $\langle \triangle_{\sigma wc}(M), \varphi\rangle = 1$.

Theorem 2.11. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Then $\mathcal{A}$ is φ-Connes amenable if and only if it has a $\varphi - \sigma wc$-diagonal.*

Proof. Choose $u \in \mathcal{A}$ such that $\varphi(u) = 1$. The projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a φ-bimodule with module multiplications determined by

$$(a \otimes b) \cdot c = a \otimes bc, \quad c \cdot (a \otimes b) = \varphi(c)a \otimes b \qquad (a, b, c \in \mathcal{A}).$$

Note that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is canonically mapped into $\sigma((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Define

$$D : \mathcal{A} \to \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*, \; a \mapsto (u \otimes u).a - \varphi(a)(u \otimes u).$$

Since the dual module $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is normal, it follows that D is a bounded weak*-continuous derivation. Moreover, D attains its values in the weak*-closed submodule $\ker(\varphi \otimes \varphi)$. Since $\mathcal{A}$ is φ-Connes amenable, there is $N \in \ker(\varphi \otimes \varphi)$ such that $D(a) = N \cdot a - \varphi(a)N$ for all $a \in \mathcal{A}$. Let $M = u \otimes u - N$. We obtain an element as required by Definition 2.10.

Conversely, let E be a normal φ-bimodule, and let $D : \mathcal{A} \to E$ be a bounded weak*-continuous derivation. Let $\mu_D : \mathcal{A} \otimes \mathcal{A} \to E$ be the continuous linear map determined by $\mu_D(a \otimes b) = \varphi(a)D(b)$. μ_D^* is a module homomorphism, and also μ_D^* maps the predual module E_* of E^* to $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$ [24]. Suppose that $\mathcal{A}$ has a $\varphi - \sigma wc$-diagonal M. Then there exists a bounded net $\{m_\alpha := \sum_{k=1}^{n_\alpha} a_k^\alpha \otimes b_k^\alpha\}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that the canonical image of $\{m_\alpha\}$ converges to M in the weak*-topology on $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Let $x := (\mu_D^*|_{E_*})^*(M)$. Let $\tilde{\varphi}$ denote the extension of φ to $\mathcal{A}^{**}$. For $c \in \mathcal{A}$,

$$\begin{aligned} x \cdot c &= w^* - \lim_\alpha \sum_{k=1}^{n_\alpha} \varphi(a_k^\alpha) D(b_k^\alpha).c \\ &= w^* - \lim_\alpha \sum_{k=1}^{n_\alpha} \varphi(a_k^\alpha) D(b_k^\alpha \cdot c) - w^* - \lim_\alpha \sum_{k=1}^{n_\alpha} \varphi(a_k^\alpha) b_k^\alpha \cdot D(c) \\ &= (\mu_D^*|_{E_*})^*(M \cdot c) - w^* - \lim_\alpha \tilde{\varphi}(\triangle(m_\alpha)) \cdot D(c) \end{aligned}$$

and $c.x = \varphi(c)x = \varphi(c)(\mu_D^*|_{E_*})^*(M)$. We conclude from this that $D(c) = c \cdot x - x \cdot c$, and so D is inner. □

3. Some hereditary properties

In this section we discuss some hereditary properties of φ-Connes amenability. The main result concerns the projective tensor product of two dual Banach algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and let $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$. Let $f \otimes g$ denote the element of $(\mathcal{A}\hat{\otimes}\mathcal{B})^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Note that with this notation

$$\Delta(\mathcal{A}\hat{\otimes}\mathcal{B}) = \{\varphi \otimes \psi;\ \varphi \in \Delta(\mathcal{A}),\ \psi \in \Delta(\mathcal{B})\}.$$

We use $\otimes_w$ to denote the injective tensor product of two Banach spaces. The following Theorem is an analog of [16, Theorem 3.3].

Theorem 3.1. *Let $\mathcal{A}_*$ and $\mathcal{B}_*$ be the preduals of dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ and $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. If $\mathcal{A}\hat{\otimes}\mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \otimes_w \mathcal{B}_*$, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $(\varphi \otimes \psi)$-Connes amenable if and only if $\mathcal{A}$ is φ-Connes amenable and $\mathcal{B}$ is ψ-Connes amenable.*

Proof. Let $\mathcal{A}$ be φ-Connes amenable and let $\mathcal{B}$ be ψ-Connes amenable. For each $f \in \mathcal{A}_*$ and $g \in \mathcal{B}_*$, the mapping $T(a, b) = \langle a, f\rangle\langle b, g\rangle$ is a bilinear map from $\mathcal{A} \times \mathcal{B}$ into $\mathbb{C}$. By [2, Theorem 6], there exists a unique linear mapping $\Lambda_T : \mathcal{A}\hat{\otimes}\mathcal{B} \to \mathbb{C}$ such that $\Lambda_T(a \otimes b) = T(a, b)$. It is known that T has a continuous extension $\overline{T} : \mathcal{A}^{**} \times \mathcal{B}^{**} \to \mathbb{C}$ such that for $F \in \mathcal{A}^{**}$, $G \in \mathcal{B}^{**}$ and nets $\{a_\alpha\} \subseteq \mathcal{A}$, $\{b_\beta\} \subseteq \mathcal{B}$ with $w^* - \lim_\alpha a_\alpha = F$ and $w^* - \lim_\beta b_\beta = G$, we have

$$\overline{T}(F, G) = \lim_\alpha \lim_\beta T(a_\alpha, b_\beta).$$

By [11, Lemma 1.7], there exists a continuous linear mapping $\Psi : \mathcal{A}^{**}\hat{\otimes}\mathcal{B}^{**} \to (\mathcal{A}\hat{\otimes}\mathcal{B})^{**}$ such that $\langle \Psi(F\otimes G), \Lambda_T\rangle = \overline{T}(F,G)$. Let m_1 be a φ-invariant mean on $\mathcal{A}_*$ and m_2 be a ψ-invariant mean on $\mathcal{B}_*$. Choose nets $\{a_\alpha\}$ in $\mathcal{A}$ and $\{b_\beta\}$ in $\mathcal{B}$ such that $a_\alpha \to m_1$ in the weak*-topology on $\mathcal{A}^{**}$ and $b_\beta \to m_2$ in the weak*-topology on $\mathcal{B}^{**}$. We have

$$\begin{aligned}
\langle \Psi(m_1\otimes m_2), (a\otimes b)\cdot(f\otimes g)\rangle &= \langle \Psi(m_1\otimes m_2), a.f\otimes b\cdot g)\rangle \\
&= \lim_\alpha \lim_\beta a\cdot f\otimes b.g\langle a_\alpha, b_\beta\rangle \\
&= \lim_\alpha \lim_\beta \langle a\cdot f, a_\alpha\rangle\langle b\cdot g, b_\beta\rangle \\
&= \langle a\cdot f, m_1\rangle\langle b\cdot g, m_2\rangle \\
&= \varphi(a)\psi(b)\langle f, m_1\rangle\langle g, m_2\rangle \\
&= (\varphi\otimes\psi)(a\otimes b)\langle f, m_1\rangle\langle g, m_2\rangle \\
&= (\varphi\otimes\psi)(a\otimes b) f\otimes g\langle m_1, m_2\rangle \\
&= (\varphi\otimes\psi)(a\otimes b)\langle \Psi(m_1\otimes m_2), f\otimes g\rangle,
\end{aligned}$$

for all $a\otimes b\in\mathcal{A}\hat{\otimes}\mathcal{B}$. Therefore $\Psi(m_1\otimes m_2)$ is a $\varphi\otimes\psi$-invariant mean. By Theorem 2.3, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\varphi\otimes\psi$-Connes amenable.

The converse follows as in the proof of [16, Theorem 3.3]. □

The notions of dual tensor norms are given in [7]. These properties are studied for arbitrary tensor norms. In particular, injective and projective tensor norms will be described. With the aid of the Radon-Nikodym property and the approximation property, the duality of tensor norms will be investigated. The Radon-Nikodym property and the approximation property are discussed in [7]. Various equivalent formulations of the Radon-Nikodym property and the approximation property are given, along with a list of spaces having or lacking these properties. For example l^1 and c_0 have the approximation property, l^1 has the Radon-Nikodym property and c_0 lacks the Radon-Nikodym property. Theorem 3.2 is [7, Theorem 1.6.16].

Theorem 3.2. *Let E and F be Banach spaces such that*
(i) E^* or F^* has the approximation property, and
(ii) E^* or F^* has the Radon-Nikodym property.
Then $\kappa_{E,F} : E^*\hat{\otimes}F^* \to (E\otimes_w F)^*$, $\kappa_{E,F}(g,h)(x,y) = \langle x,g\rangle\langle y,h\rangle$ is an isometric isomorphism.

An example of duality of tensor norms appears in Corollary 4.2

Lemma 3.3. *Let $\mathcal{A}$ be a dual Banach algebra with a bounded approximate identity. Then the following conditions are equivalent:*

(1) *For each normal φ-bimodule E every bounded weak*-continuous derivation $D : \mathcal{A}\to E$ is inner.*

(2) *If E is a normal φ-bimodule such that its predual is pseudo-unital, then every bounded weak*-continuous derivation $D : \mathcal{A} \to E$ is inner.*

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): Let E be a normal φ-bimodule with predual E_*, and let $D : \mathcal{A} \to E$ be a bounded weak*-continuous derivation. Let

$$E_0 = \{\varphi(a)x \cdot b;\ a, b \in \mathcal{A},\ x \in E_*\}.$$

By Cohen's Factorization theorem, E_0 is a closed submodule of E_*. Let $\Lambda : E \to E_0^*$ be the restriction map. It is easy to see that Λ is a module homomorphism. So $\Lambda \circ D : \mathcal{A} \to E_0^*$ is a weak*-continuous derivation. By assumption, there is some $x_0 \in E_0^*$ such that $\Lambda \circ D(a) = \varphi(a)x_0 - x_0.a$ for all $a \in \mathcal{A}$. Choose $x \in E$ such that $\Lambda(x) = x_0$. Thus $\tilde{D} = D - ad_x$ is a weak*-continuous derivation from $\mathcal{A}$ into $E_0^{\perp}$. On the other hand, $E_0^{\perp} \cong (E/E_0)^*$ and $\mathcal{A}.(E/E_0) = \{0\}$. It follows that $H^1(\mathcal{A}, E_0^{\perp}) = \{0\}$, see [20, Proposition 2.1.3]. This implies that D is inner. □

Theorem 3.4. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Let I be a closed two-sided ideal of $\mathcal{A}$ with a bounded approximate identity such that $\varphi|_I \neq 0$. Let I be a dual Banach algebra. If $\mathcal{A}$ is φ-Connes amenable, then I is $\varphi|_I$-Connes amenable.*

Proof. The proof uses the standard argument of [14, Proposition 5.1]. Let E be a normal $\varphi|_I$-bimodule. By Lemma 3.3, there is no loss of generality if we suppose that the predual of E is pseudo-unital. Let $D : I \to E$ be a bounded weak*-continuous derivation. Clearly E is a normal φ-bimodule, see [20, Proposition 2.1.6] and its proof. Indeed, let $\{a_\alpha\}$ be a net in $\mathcal{A}$ such that $a_\alpha \to a$ in the weak*-topology of $\mathcal{A}$ and let $x \in E$. Pick $x_* \in E_*$. Since E_* is pseudo-unital, there are $i \in I$ and $y_* \in E_*$ such that $x_* = i \cdot y_*$. It follows that

$$\langle x.a_\alpha, x_*\rangle = \langle x \cdot a_\alpha, i \cdot y_*\rangle = \langle x \cdot a_\alpha i, y_*\rangle \to \langle x \cdot ai, y_*\rangle,$$

because E is a normal $\varphi|_I$-bimodule. To extend D, let

$$\tilde{D} : \mathcal{A} \to E, \quad a \mapsto w^* - \lim_\alpha (D(ae_\alpha) - a.D(e_\alpha)).$$

It is well known that $\tilde{D}$ is a continuous derivation. To see that $\tilde{D}$ is weak*-continuous, again let $\{a_\alpha\}$ be a net in $\mathcal{A}$ such that $a_\alpha \to a$ in the weak*-topology of $\mathcal{A}$. For $x_* \in E_*$, let $i \in I$ and $y_* \in E_*$ be such that $x_* = i.y_*$. Then

$$\begin{aligned}
\langle \tilde{D}(a_\alpha), x_*\rangle &= \langle \tilde{D}(a_\alpha), i \cdot y_*\rangle = \langle \tilde{D}(a_\alpha i) - a_\alpha.\tilde{D}(i), y_*\rangle \\
&\to \langle \tilde{D}(ai) - a \cdot \tilde{D}(i), y_*\rangle = \langle \tilde{D}(a) \cdot i, y_*\rangle \\
&= \langle \tilde{D}(a), i \cdot y_*\rangle = \langle \tilde{D}(a), x_*\rangle
\end{aligned}$$

because D is weak*-continuous and E is a $\varphi|_I$-bimodule. From the φ-Connes amenability of E we conclude that $\tilde{D}$ (and hence D) is inner. □

Theorem 3.5. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Let I be a closed two-sided ideal of $\mathcal{A}$ such that $\varphi|_I \neq 0$. Let I be a dual Banach algebra. If I is $\varphi|_I$-Connes amenable, then $\mathcal{A}$ is φ-Connes amenable.*

Proof. Let E be a normal $\mathcal{A}$-bimodule such that $a \cdot x = \varphi(a)x$ for all $a \in \mathcal{A}$ and $x \in E$ and let $D : \mathcal{A} \to E$ be a bounded weak*-continuous derivation. Clearly, $D|_I$ is a weak*-continuous derivation. Since I is $\varphi|_I$-Connes amenable, there exists $x_0 \in E$ such that $D(i) = i \cdot x_0 - x_0 \cdot i$ for all $i \in I$. Choose $i_0 \in I$ such that $\varphi(i_0) = 1$, and put $x = x_0 \cdot i_0$. For $a \in \mathcal{A}$,

$$\begin{aligned}
\varphi(a)x - x \cdot a &= \varphi(a)x_0 \cdot i_0 - \varphi(a)\varphi(i_0)x_0 + \varphi(a)\varphi(i_0)x_0 - x_0 \cdot i_0 a \\
&= D|_I(i_0 a) - \varphi(a)D|_I(i_0) \\
&= D(a) + D|_I(i_0) \cdot a - \varphi(a)D|_I(i_0) \\
&= D(a) + (x_0 - x_0 \cdot i_0) \cdot a - \varphi(a)(x_0 - x_0 \cdot i_0).
\end{aligned}$$

This shows that $D(a) = a.x_0 - x_0.a$, and hence D is inner. □

Let $\mathcal{A}$ be a Banach algebra with predual $\mathcal{A}_*$. Let $\mathcal{A}^\#$ denote the Banach algebra obtained by adjoining an identity e. Then $\mathcal{A}^\#$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus \mathbb{C}$. We have the following analogue of [14] (see p.16).

Proposition 3.6. *Let $\mathcal{A}$ be a Banach algebra with predual $\mathcal{A}_*$. Let $\mathcal{A}^\#$ denote the Banach algebra obtained by adjoining an identity e. Let $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ and let $\varphi^\#$ be the unique extension of φ to an element of $\Delta(A^\#)$. Then $\mathcal{A}$ is φ-Connes amenable if and only if $\mathcal{A}^\#$ is $\varphi^\#$-Connes amenable.*

Proof. Suppose that m is a φ-invariant mean on $\mathcal{A}_*$ and define $n \in {\mathcal{A}^\#}^{**}$ by $\langle n, f\rangle = \langle m, f|_{\mathcal{A}}\rangle$. Then $\langle n, \varphi^\#\rangle = 1$. On the other hand

$$\begin{aligned}
\langle n, f.(a, \alpha)\rangle &= \langle n, f.(a, 0)\rangle + \alpha\langle n, f\rangle = \langle m, f.(a, 0)|_{\mathcal{A}}\rangle + \alpha\langle n, f\rangle \\
&= \varphi(a)\langle m, f|_{\mathcal{A}}\rangle + \alpha\langle n, f\rangle = \varphi^\#(a, \alpha)\langle n, f\rangle.
\end{aligned}$$

for all $f \in {\mathcal{A}^\#}_*$, $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. By Theorem 2.3, $\mathcal{A}^\#$ is $\varphi^\#$-Connes amenable.

Conversely, let E be a normal φ-bimodule and let $D : \mathcal{A} \to E$ be a bounded weak*-continuous derivation. For $x \in E$, define $e.x = x.e = x$. Clearly, E is a normal $\varphi^\#$-bimodule. Define $\tilde{D} : \mathcal{A}^\# \to E$ by $\tilde{D}(a + \alpha e) = D(a)$ $(a \in \mathcal{A},\ \alpha \in \mathbb{C})$. It is easy to see that $\tilde{D}$ is a weak*-continuous derivation. The algebra $\mathcal{A}^\#$ is assumed to be $\varphi^\#$-Connes amenable, thus $\tilde{D}$ is inner. It follows that D is inner. □

Proposition 3.7. *Let $\mathcal{A}_*$ and $\mathcal{B}_*$ be the preduals of dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose that $\theta : \mathcal{A} \to \mathcal{B}$ is a bounded weak*-continuous homomorphism with weak*-dense range. If $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$ and $\mathcal{A}$ is $\psi \circ \theta$-Connes amenable, then $\mathcal{B}$ is ψ-Connes amenable.*

Proof. We follow the standard argument of [14, Proposition 5.3]. Let E be a normal ψ-bimodule, and let D be a bounded weak*-continuous derivation. Suppose that $\mathcal{A}$ acts on E by $a \cdot x = (\psi \circ \theta)(a)x$ and $x \cdot a = x \cdot \theta(a)$ for $x \in E$ and $a \in \mathcal{A}$. Now set $\tilde{D}(a) = D \circ \theta(a)$ $(a \in \mathcal{A})$. Then $\tilde{D} : \mathcal{A} \to E$ is a weak*-continuous derivation. The algebra $\mathcal{A}$ is assumed to be $\psi \circ \theta$-Connes amenable. Thus there exists $x \in E$ with $\tilde{D}(a) = a \cdot x - x \cdot a = \psi \circ \theta(a)x - x \cdot \theta(a)$ for $a \in \mathcal{A}$. Therefore $D(b) = \psi(b)x - x \cdot b$, and hence D is inner. $\square$

4. $\hat{\chi}$-Connes amenability of semigroup algebras

In this section, we apply these results to semigroup algebra $l^1(S)$. Before stating our results, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [6]. We denote by $\hat{S}$ the set of all nonzero bounded continuous characters of S. More precisely $\hat{S} = \{\chi \in c_b(S);\ \chi \neq 0 \text{ and } \chi(xy) = \chi(x)\chi(y) \text{ for all } x, y \in S\}$. For $\chi \in \hat{S}$ and $\mu \in l^1(S)$, define $\hat{\chi}(\mu) = \int \overline{\chi(x)} d\mu$. A semigroup S is said to be χ-amenable if there exists a bounded linear functional M on $l^1(S)^*$ satisfying $M(\hat{\chi}) = 1$ and $M(\delta_x f) = \hat{\chi}(\delta_x)M(f)$ for all $x \in S$ and $f \in l^1(S)^*$(see [10]).

We write $\mathcal{B}(l^1(S), l^\infty(S))$ for the Banach space of bounded linear maps from $l^1(S)$ to $l^\infty(S)$. We write T^* for adjoint of $T \in \mathcal{B}(l^1(S), l^\infty(S))$.

It is standard that $\mathcal{B}(l^1(S), l^\infty(S)) = l^\infty(S \times S)$ where $T(s, s') = \langle T(\delta_s), \delta_{s'} \rangle$. For a Banach space E we have the canonical map $k_E : E \to E^{**}$ defined by $\langle k_E(x), f \rangle = \langle f, x \rangle$ for $f \in E^*$ and $x \in E$.

Proposition 4.1. *Let S be a weakly cancellative semigroup and let $\mathcal{A} = l^1(S)$. Then $\mathcal{A}$ is $\hat{\chi}$-Connes amenable if and only if there exists $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that:*

- (i) $\langle M, (\bar{\chi}(ss'))_{(s,s') \in S \times S} \rangle = 1$;
- (ii) $\langle M, (f(s'k, s) - \hat{\chi}(\delta_t) f(s, s'))_{(s,s') \in S \times S} \rangle = 0$ *for each $k \in S$ and every bounded function $f : S \times S \to \mathbb{C}$ so that the map $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}^*)$ defined by $\langle T(\delta_s), \delta_{s'} \rangle = f(s, s')$ satisfies the following. If $T(\mathcal{A}) \subseteq k_{c_0(S)}(c_0(S))$ and $T^*(k_{\mathcal{A}}(\mathcal{A})) \subseteq k_{c_0(S)}(c_0(S))$, then for each sequence $\{k_n\}$ of distinct elements of S and each sequence $\{(s_m, s'_m)\}$ of distinct elements of $S \times S$ such that the repeated limits*
$$\lim_n \lim_m \langle T(\delta_{s'_m}), \delta_{k_n s_m} \rangle, \quad \lim_n \lim_m \langle T(\delta_{s'_m k_n}), \delta_{s_m} \rangle$$
exist, we have
$$\lim_n \lim_m \langle T(\delta_{s'_m}), \delta_{k_n s_m} \rangle = \lim_n \lim_m \langle T(\delta_{s'_m k_n}), \delta_{s_m} \rangle = 0.$$

Proof. By Theorem 2.11, $\mathcal{A}$ is $\hat{\chi}$-Connes amenable if and only if there exists $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $\langle \triangle_{\mathcal{A}}^{**}(M), \hat{\chi} \rangle = 1$ and $\langle M, \delta_t.T - \hat{\chi}(\delta_t)T \rangle = 0$ for every $T \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$ and $t \in S$. By [6, Corollary 3.5 and Proposition 5.5] this holds if and only if for each $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}^*)$ which satisfies the above conditions, we have $\langle M, (\hat{\chi}(\delta_{ss'}))_{(s,s')\in S\times S}\rangle = 1$ and $\langle M, \delta_t.T - \hat{\chi}(\delta_t)T\rangle = 0$ for every $t \in S$. On the other hand,

$$\begin{aligned}\langle \delta_t.T - \hat{\chi}(\delta_t)T, \delta_s \otimes \delta_{s'}\rangle &= \langle T, \delta_s \otimes \delta_{s't}\rangle - \hat{\chi}(\delta_t)\langle T(\delta_{s'}), \delta_s\rangle \\ &= \langle T(\delta_{s't}), \delta_s\rangle - \bar{\chi}(t)\langle T(\delta_{s'}), \delta_s\rangle.\end{aligned}$$

This shows that $\langle M, (f(s't, s) - \bar{\chi}(t)f(s', s))_{(s',s)\in S\times S}\rangle = 0$ for all $t \in S$. This completes the proof. □

Corollary 4.2. *Let S be a weakly cancellative semigroup. Then $l^1(S)$ is $\hat{\chi}$-Connes amenable if and only if $l^1(S)\hat{\otimes}l^1(S)$ is $\hat{\chi}\otimes\hat{\chi}$-Connes amenable.*

Proof. By [2, Example 14] and [6, proposition 5.1], we have $l^1(S)\hat{\otimes}l^1(S) = (c_0(S) \otimes_w c_0(S))^*$. The result is now an immediate consequence of Theorem 3.1. □

In [6], it is proved that if $l^1(S)$ is Connes amenable with respect to the predual $c_0(S)$ and S is cancellative or unital, then S is a group. It would be interesting to know whether the result extends to $\hat{\chi}$-Connes amenability.

Theorem 4.3. *Let S be a weakly cancellative semigroup, and let $\mathcal{A} = l^1(S)$ be unital with unit $e_{\mathcal{A}}$. Let $\mathcal{A}$ be $\hat{\chi}$-Connes amenable with respect to the predual $c_0(S)$. If S is cancellative or unital, then S is a group.*

Proof. Let $\mathcal{A}$ be $\hat{\chi}$-Connes amenable and let $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ be as in Proposition 4.1. By [6, Theorem 5.13] and its proof, if S is cancellative then S is unital with unit u_s. Now suppose that S is a unital weakly cancellative semigroup. Suppose that $s_0 \in S$ has no right inverse. Define $f : S \times S \to \mathbb{C}$ by $f(s', s_0s) = 1$ if $ss' = u_s$, and $f(s', s_0s) = 0$ if $ss' \neq u_s$. Moreover, we put $f(s's_0, s) = 0$. It is routinely checked that f is a bounded function. Let $T : \mathcal{A} \to \mathcal{A}^*$ be the operator associated with f. It is easy to see that T satisfies all the conditions of Proposition 4.1. On the other hand, $f(s's_0, s) - \bar{\chi}(s_0)f(s', s) = -\bar{\chi}(ss')$ if $ss' = s_0$ and $f(s's_0, s) - \bar{\chi}(s_0)f(s', s) = 0$ if $ss' \neq s_0$. By Proposition 4.1, $\langle M, (\bar{\chi}(ss'))_{(s,s')\in S\times S}\rangle = 0$, which is a contradiction. Hence every element of S has a right inverse. A similar proof shows that every element of S has a left inverse. Thus S is a group. □

Parallel to [6, Theorem 5.14], we state the following result.

Proposition 4.4. *Let S be a weakly cancellative semigroup, and let $l^1(S)$ be unital with unit $e_{l^1(S)}$. Then $l^1(S)$ is $\hat{\chi}$-Connes amenable if and only if $l^1(S)$ is $\hat{\chi}$-amenable.*

Proof. This is an immediate consequence of Corollary 2.9. □

The following example shows that there are $\hat{\chi}$-Connes amenable Banach algebras which are not $\hat{\chi}$-amenable.

Example 4.5. Let S be an infinite left zero semigroup and βS be the Stone-Cech compactifcation of S. If $st = s$ $(s, t \in \beta S)$, then $\mu * \nu = \hat{\chi}(\nu)\mu$ for all $\mu, \nu \in l^1(S)$. Thus $l^1(S)$ is not $\hat{\chi}$-amenable. But $l^1(\beta S)$ is $\hat{\chi}$-Connes amenable whence $l^1(S)$ is $\hat{\chi}$-Connes amenable. (See [5])

Theorem 4.6. *The following statements are equivalent for a weakly cancellative semigroup S:*

(i) *S is χ-amenable.*
(ii) *$l^1(S)$ is $\hat{\chi}$-strongly Connes amenable.*
(iii) *$l^1(S)$ has a $\hat{\chi}$-normal virtual diagonal.*

Proof. Assume that S is χ-amenable. Then $l^1(S)$ is $\hat{\chi}$-amenable [10]. It follows that $l^1(S)$ is $\hat{\chi}$-strongly Connes amenable.

To prove $(ii) \Longrightarrow (i)$, we first let $l^1(S)$ be the normal $\hat{\chi}$-bimodule whose underling space is $l^1(S)$. Then $l^\infty(S)^*$ is a $\hat{\chi}$-bimodule. Choose $u \in S$ such that $\hat{\chi}(\delta_u) = 1$. Define the derivation

$$D : l^1(S) \to l^\infty(S)^*, \quad D(\delta_x) = \delta_{ux} - \overline{\chi(x)}\delta_u.$$

It is easy to see that D is weak*-continuous and attains its values in $\ker \hat{\chi}$. Since $l^1(S)$ is $\hat{\chi}$-strongly Connes amenable, there is some $n \in \ker \hat{\chi}$ such that $D(\delta_x) = \delta_x.n - n.\delta_x$ for all $x \in S$. Then the element $m := \delta_u - n$ has the desired properties.

Since $(ii) \Longleftrightarrow (iii)$ for any dual Banach algebra, the proof of the Theorem is complete. □

In view of [23], it would be of considerable interest to know if the result holds for unital weakly cancellative semigroup algebras.

Theorem 4.7. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be unital. The following statements are equivalent:*

(i) *S is χ-amenable.*
(ii) *$l^1(S)$ is $\hat{\chi}$-Connes amenable.*
(iii) *$l^1(S)$ has a $\hat{\chi}$-normal virtual diagonal.*

Proof. This follows from Proposition 4.4 and [10, Theorem 1] □

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