ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 1, pp. 53–67

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 1, pp. 53–67
Online ISSN: 1735-8515

CONSTRUCTION OF MEASURES OF NONCOMPACTNESS OF $C^k(\Omega)$ AND C_0^k AND THEIR APPLICATION TO FUNCTIONAL INTEGRAL-DIFFERENTIAL EQUATIONS

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In this paper, first, we investigate the construction of compact sets of C^k and C_0^k by proving "$C^k, C_0^k - version$" of Arzelà-Ascoli theorem, and then introduce new measures of noncompactness on these spaces. Finally, as an application, we study the existence of entire solutions for a class of the functional integral-differential equations by using Darbo's fixed point theorem associated with these new measures of noncompactness. Further, some examples are presented to show the efficiency of our results.

Keywords: Measure of noncompactness, Darbo's fixed point theorem, Arzelà-Ascoli theorem, integral-differential equations.
MSC(2010): Primary: 47H09; Secondary: 47H10.

1. Introduction

Compactness results in the spaces $L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ and $C(K)$ (the space of continuous functions over a compact metric space K with values in $\mathbb{R}$) are often vital for proving existence results of differential, integral and functional integral equations ([1,4,15,16,20], for example). A necessary and sufficient condition for a subset of $L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ and $C(K)$ to be compact are given in what are often called the Kolmogorov compactness theorem and the Arzelà-Ascoli theorem, respectively. On the other hand, measures of noncompactness are very useful tools in functional analysis. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integral-differential equations, optimal control theory, and in the characterizations of compact operators between Banach spaces [2, 3, 5–14, 18, 19, 21, 22]. In particular, in recent years, a lot of authors used the concept of a measure of noncompactness in conjunction with

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Received:17 December 2014, Accepted: 8 October 2015.
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the Darbo's fixed point theorem in order to prove the existence of solutions for a wide variety of functional integral equations (cf. [3, 5, 6, 8, 9, 11, 13, 14]).

The paper is organized as follows. In Section 2, we prove a "$C^k - version$" of Arzelà-Ascoli theorem, and then introduce a new measure of noncompactness in the space $C^k(\Omega)$. In Section 3, again we give a "$C_0^k - version$" of Arzelà-Ascoli theorem, and present a new measure of noncompactness in C_0^k. Section 4 is devoted to the application of the results obtained to the functional integral-differential equations. Finally, two examples are provided to illustrate the efficiency and usefulness of our results.

Here, we recall some basic facts concerning measures of noncompactness, which is defined axiomatically in terms of some natural conditions. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and ConvX the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ the subfamily consisting of all relatively compact subsets of E.

Definition 1.1. ([8]) A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.
2° $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(\mathrm{ConvX}) = \mu(\mathrm{X})$.
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

The following Darbo's fixed point theorem will be needed in Section 4.

Theorem 1.2. ([11, Theorem 1]) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property*

$$\mu(FX) \le k\mu(X) \tag{1.1}$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

2. Measure of noncompactness on $C^k(\Omega)$

In this section, we characterize the compact subsets of $C^k(\Omega)$. Next we introduce the new measure of noncompactness on $C^k(\Omega)$. Let Ω is a compact subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C^k(\Omega)$ the space of functions f which are k times continuously differentiable on Ω with the standard norm

$$\|f\|_{C^k(\Omega)} = \max_{0\leq|\alpha|\leq k} \|D^\alpha f\|_u,$$

where $\|D^\alpha f\|_u = \sup\{|D^\alpha f(x)| : x \in \Omega\}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$. Now we present "$C^k - version$" of Arzelà-Ascoli theorem.

Theorem 2.1. *Let Ω be a compact subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$. Then $\mathcal{F} \subset C^k(\Omega)$ is totally bounded if and only if $\mathcal{F}^\alpha = \{D^\alpha f : f \in \mathcal{F}\}$ is bounded and equicontinuous for all $|\alpha| \leq k$.*

The proof relies on the following useful observation.

Lemma 2.2. ([17, Lemma 1]) *Let X be a metric space. Assume that for every $\varepsilon > 0$, there exists some $\delta > 0$, a metric space W, and a mapping $\Phi : X \longrightarrow W$ with $\Phi[X]$ totally bounded, and such that $d(x, y) < \varepsilon$ whenever $x, y \in X$ and $d(\Phi(x), \Phi(y)) < \delta$. Then X is totally bounded.*

Proof. Proof of Theorem 2.1. Assume $\mathcal{F}^\alpha$ are bounded and equicontinuous for all $|\alpha| \leq k$. Let $\varepsilon > 0$. Combining the equicontinuity of $\mathcal{F}^\alpha$ and compactness of Ω, we can find a finite set of points $y_1, \ldots, y_m \in \Omega$ with neighborhoods $U_1, \ldots, U_m$ covering all of Ω so that

$$|D^\alpha f(x) - D^\alpha f(y_j)| < \varepsilon$$

whenever $f \in \mathcal{F}$, $x \in U_j$ and $|\alpha| \leq k$. Define $\Phi : \mathcal{F} \longrightarrow \mathbb{R}^{m(l+1)}$ ($\{\alpha : |\alpha| \leq k\} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_l\}$) by

$$\Phi(f) = (f(y_1), \ldots, f(y_m), D^{\beta_1} f(y_1), \ldots, D^{\beta_1} f(y_m), \ldots, D^{\beta_l} f(y_m)).$$

By boundedness of $\mathcal{F}^\alpha$, the image $\Phi[\mathcal{F}]$ is bounded, and hence totally bounded, in $\mathbb{R}^{m(l+1)}$. Furthermore, if $f, g \in \mathcal{F}$ with $\|\Phi(f) - \Phi(g)\|_{C^k(\Omega)} < \varepsilon$, then as any $x \in \Omega$, belongs to some U_j,

$$\begin{aligned}|D^\alpha f(x) - D^\alpha g(x)| \leq& |D^\alpha f(x) - D^\alpha f(x_j)| + |D^\alpha f(x_j) - D^\alpha g(x_j)| \\ &+ |D^\alpha g(x_j) - D^\alpha g(x)| < 3\varepsilon,\end{aligned}$$

and so $\|f - g\|_{C^k(\Omega)} \leq 3\varepsilon$. By Lemma 2.2, $\mathcal{F}$ is totally bounded.

For the converse, assume that $\mathcal{F}$ is a totally bounded subset of $C^k(\Omega)$. Let us fix α arbitrarily such that $0 \leq |\alpha| \leq k$. The existence of a finite ε-cover for $\mathcal{F}$, for any ε, clearly implies the boundedness of $\mathcal{F}^\alpha$.

To prove the equicontinuity of $\mathcal{F}^\alpha$, let $x \in \Omega$ and $\varepsilon > 0$ be given. Pick an ε-cover $\{U_1, \ldots, U_m\}$ of $\mathcal{F}$, and choose $g_j \in U_j$ for $j = 1, \ldots, m$. Pick a neighborhood V_j of x so that $|D^\alpha g_j(y) - D^\alpha g_j(x)| < \varepsilon$ whenever $y \in V_j$, for $j = 1, \ldots, m$. Let $V = V_1 \cap \cdots \cap V_m$. If $f \in U_j$, then $\|f - g_j\|_{C^k(\Omega)} < \varepsilon$, and so when $y \in V$,

$$\begin{aligned}|D^\alpha f(y) - D^\alpha f(x)| \leq& |D^\alpha f(y) - D^\alpha g_j(y)| + |D^\alpha g_j(y) - D^\alpha g_j(x)| \\ &+ |D^\alpha g_j(x) - D^\alpha f(x)| < 3\varepsilon.\end{aligned}$$

Now, since Ω is compact, we have the equicontinuity of $\mathcal{F}^\alpha$. □

Now, we are ready to define a new measure of noncompactness on $C^k(\Omega)$.

Theorem 2.3. *Suppose* $1 \leq k < \infty$ *and* $\mathcal{F}$ *is a bounded subset of* $C^k(\Omega)$*. For* $f \in \mathcal{F}$, $\varepsilon > 0$ *and* $\alpha \in \mathbb{R}^N$ *such that* $0 \leq |\alpha| \leq k$*, let*

$$\omega(f, \varepsilon) = \sup\{|D^\alpha f(x) - D^\alpha f(y)| : x, y \in \Omega, \|x - y\| < \varepsilon, 0 \leq |\alpha| \leq k\},$$
$$\omega(\mathcal{F}, \varepsilon) = \sup\{\omega(f, \varepsilon) : f \in \mathcal{F}\}.$$

Then $\omega_0 : \mathfrak{M}_{C^k(\Omega)} \longrightarrow \mathbb{R}$ *given by*

$$\omega_0(\mathcal{F}) = \lim_{\varepsilon \to 0} \omega(\mathcal{F}, \varepsilon) \tag{2.1}$$

defines a measure of noncompactness on $C^k(\Omega)$ *and moreover,* $\ker(\omega_0) = \mathfrak{N}_{C^k(\Omega)}$.

Proof. First we show 1° holds. Take $\mathcal{F} \in \mathfrak{M}_{C^k(\Omega)}$ such that $\omega_0(\mathcal{F}) = 0$. Let us fix α arbitrarily such that $0 \leq |\alpha| \leq k$. Let $\eta > 0$ be arbitrary. Since $\omega_0(\mathcal{F}) = 0$, $\lim_{\varepsilon \to 0} \omega(\mathcal{F}, \varepsilon) = 0$ and thus, there exists $\delta > 0$ such that $\omega(\mathcal{F}, \delta) < \eta$. This implies that

$$|D^\alpha f(x) - D^\alpha f(y)| < \eta$$

for all $f \in \mathcal{F}$ and $x, y \in \Omega$ with $\|x - y\| < \delta$. Then $\mathcal{F}^\alpha$ is bounded and equicontinuous. Thus 1° is satisfied.

2° follows directly from definition of ω. We continue by showing that 3° holds. Suppose that $\mathcal{F} \in \mathfrak{M}_{C^k(\Omega)}$ and $(f_m) \subset \mathcal{F}$ such that $f_m \to f \in \overline{\mathcal{F}}$ in $C^k(\Omega)$. By the definition of $\omega(\mathcal{F}, \varepsilon)$ we have

$$|D^\alpha f_m(x) - D^\alpha f_m(y)| \leq \omega(\mathcal{F}, \varepsilon)$$

for all $m \in \mathbb{N}$, $0 \leq |\alpha| \leq k$ and $x, y \in \Omega$ with $\|x - y\| < \varepsilon$. Letting $m \to \infty$ we get

$$|D^\alpha f(x) - D^\alpha f(y)| \leq \omega(\mathcal{F}, \varepsilon)$$

for any $0 \leq |\alpha| \leq k$ and $x, y \in \Omega$ with $\|x - y\| < \varepsilon$, and hence

$$\lim_{\varepsilon \to 0} \omega(\overline{\mathcal{F}}, \varepsilon) \leq \lim_{\epsilon \to 0} \omega(\mathcal{F}, \varepsilon).$$

This implies that

$$\omega_0(\overline{\mathcal{F}}) \leq \omega_0(\mathcal{F}). \tag{2.2}$$

From (2.2) and 2° we get $\omega_0(\overline{\mathcal{F}}) = \omega_0(\mathcal{F})$. Therefore ω_0 satisfies the condition 3° of Definition 1.1.

4° follows directly from $[\mathrm{Conv}(\mathcal{F})]^\alpha = \mathrm{Conv}(\mathcal{F}^\alpha)$ and is therefore omitted. The proof of condition 5° can be carried out by using the equality

$$D^\alpha(\lambda f + (1 - \lambda)g) = \lambda D^\alpha f + (1 - \lambda) D^\alpha g$$

for all $\lambda \in [0, 1]$.

It remains to prove 6°. Suppose that $\{\mathcal{F}_n\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_E$ such that $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for $n = 1, 2, \cdots$, and $\lim_{n \to \infty} \omega_0(\mathcal{F}_n) =$

0. Now for any $n \in \mathbb{N}$, take $f_n \in \mathcal{F}_n$ and set $\mathcal{G} = \overline{\{f_n\}}$.

claim: $\mathcal{G}$ is a compact set in $C^k(\Omega)$.

To prove the claim, we need to verify $\mathcal{G}^\alpha$ is bounded and equicontinuous for all $|\alpha| \leq k$. Let $\varepsilon > 0$ be fixed. Since $\lim\limits_{n\to\infty} \omega_0(\mathcal{F}_n) = 0$, then there exists $N \in \mathbb{N}$ such that $\omega_0(\mathcal{F}_N) < \varepsilon$. Hence, we can find $\delta_1 > 0$ such that

$$\omega(\mathcal{F}_N, \delta_1) < \varepsilon.$$

Thus, for all $n \geq N$, $0 \leq |\alpha| \leq k$ and $\|x - y\| < \delta_1$ we have

$$|D^\alpha f_n(x) - D^\alpha f_n(y)| \leq \omega(\mathcal{F}_N, \delta_1) < \varepsilon.$$

Also we know that the set $\{x_1, x_2, \ldots, x_N\}$ is compact. Hence there exists $\delta_2 > 0$ such that

$$|D^\alpha f_n(x) - D^\alpha f_n(y)| < \varepsilon \tag{2.3}$$

for all $n = 1, 2, \ldots, N$, $0 \leq |\alpha| \leq k$ and $\|x - y\| < \delta_2$, which implies that

$$|D^\alpha f_n(x) - D^\alpha f_n(y)| < \varepsilon$$

for all $n \in \mathbb{N}$ and $\|x - y\| < \min\{\delta_1, \delta_2\}$. Therefore all the hypotheses of Theorem 2.1 are satisfied. This completes the proof of the claim.

Applying the claim shows that there exists a subsequence $\{f_{n_j}\}$ and $f_0 \in C^k(\Omega)$ such that $f_{n_j} \to f_0$. Since $f_n \in \mathcal{F}_n$, $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ and $\mathcal{F}_n$ is closed for all $n \in \mathbb{N}$, we get

$$f_0 \in \bigcap_{n=1}^{\infty} \mathcal{F}_n = \mathcal{F}_\infty,$$

finishing the proof of 6°. Finally, to prove that $\ker(\omega_0) = \mathfrak{N}_{C^k(\Omega)}$. Suppose that $\mathcal{F} \in \mathfrak{N}_{C^k(\Omega)}$. Thus, the closure of $\mathcal{F}$ in $C^k(\Omega)$ is compact, and by Theorem 2.1, $\mathcal{F}^\alpha$ is bounded and equicountinious for all $|\alpha| \leq k$. Let us fix an arbitrary $\varepsilon > 0$. Since $\mathcal{F}^\alpha$ is bounded and equicountinious for all $|\alpha| \leq k$, so there exists $\delta > 0$ such that

$$|D^\alpha f(x) - D^\alpha f(y)| < \varepsilon$$

for all $0 \leq |\alpha| < k$, $f \in \mathcal{F}$ and $\|x - y\| \leq \delta$. Then for all $f \in \mathcal{F}$ we have

$$\omega(f, \delta) = \sup\{|D^\alpha f(x) - D^\alpha f(y)| : \|x - y\| < \delta\} \leq \varepsilon$$

and

$$\omega(\mathcal{F}, \delta) = \sup\{\omega(f, \delta) : f \in \mathcal{F}\} \leq \varepsilon.$$

This implies that

$$\lim_{\delta\to 0} \omega(\mathcal{F}, \delta) = 0, \tag{2.4}$$

and the condition $\ker(\omega_0) = \mathfrak{N}_{C^k(\Omega)}$ holds. □

3. Measure of noncompactness on C_0^k

In this section, we characterize the compact subsets of C_0^k, and then introduce the new measure of noncompactness on C_0^k. Let us recall a few auxiliary facts needed in the sequel of the paper.

$$C_0^k = \{f \in C^k(\mathbb{R}^n) : D^\alpha f \in C_0 \ for \ |\alpha| \leq k\},$$
$$C_0 = \{f \in BC(\mathbb{R}^n) : \lim_{\|x\| \to \infty} f(x) = 0\},$$

where $BC(\mathbb{R}^n)$ is the Banach space of all bounded and continuous functions on $\mathbb{R}^n$ and C_0^k is a Banach space with $\|f\|_{C_0^k} = \sum\limits_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_u$.

Now, we give and prove "$C_0^k - version$" of Arzelà-Ascoli theorem.

Theorem 3.1. *Let $k \in \mathbb{N}$ and $\mathcal{F}$ be a bounded set in C_0^k. Then the following two conditions are equivalent:*

(i) *$\mathcal{F}^\alpha_{|\bar{B}_T}$ are equicountinious on $\bar{B}_T$ for any $T > 0$ and*

$$\lim_{\|x\| \to \infty} \mathrm{diam}\mathcal{F}^\alpha(\mathrm{x}) = 0 \tag{3.1}$$

for all $|\alpha| \leq k$, where $\mathrm{diam}\mathcal{F}^\alpha(\mathrm{x}) = \sup\{|\mathrm{D}^\alpha \mathrm{f}(\mathrm{x}) - \mathrm{D}^\alpha \mathrm{g}(\mathrm{x})| : \mathrm{f}, \mathrm{g} \in \mathcal{F}\}$ and $\mathcal{F}^\alpha_{|\bar{B}_T}$ denotes the restrictions to $\bar{B}_T$ of the functions $\mathcal{F}^\alpha$.

(ii) *$\mathcal{F}$ is totally bounded in C_0^k.*

Proof. Assume that $\mathcal{F}^\alpha$ satisfies condition (i). Let $\varepsilon > 0$. From (3.1) for $\varepsilon > 0$ there exists a $T > 0$ such that

$$\mathrm{diam}\mathcal{F}^\alpha(\mathrm{x}) < \varepsilon \quad \text{for all } \mathrm{x} \in \mathbb{R}^\mathrm{n} \setminus \bar{\mathrm{B}}_\mathrm{T},$$

and by the equicontinuity of $\mathcal{F}^\alpha_{|\bar{B}_T}$ we can find a finite set of points $y_1, \ldots, y_m \in \bar{B}_T$ with neighborhoods $U_1, \ldots, U_m$ covering all of $\bar{B}_T$ so that

$$|D^\alpha f(x) - D^\alpha f(y_j)| < \varepsilon$$

whenever $f \in \mathcal{F}$, $x \in U_j$ and $|\alpha| \leq k$. Define $\Phi : \mathcal{F} \longrightarrow \mathbb{R}^{m(l+1)}$ ($\{\alpha : |\alpha| \leq k\} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_l\}$) by

$$\Phi(f) = (f(y_1), \ldots, f(y_m), D^{\beta_1} f(y_1), \ldots, D^{\beta_1} f(y_m), \ldots, D^{\beta_l} f(y_m)).$$

By boundedness of $\mathcal{F}^\alpha$, the image $\Phi[\mathcal{F}]$ is bounded, and hence totally bounded, in $\mathbb{R}^{m(l+1)}$. Furthermore, if $f, g \in \mathcal{F}$ with $\|\Phi(f) - \Phi(g)\|_{C_0^k} < \varepsilon$, then since any $x \in \bar{B}_T$, belongs to some U_j,

$$\begin{aligned} |D^\alpha f(x) - D^\alpha g(x)| \leq & |D^\alpha f(x) - D^\alpha f(x_j)| + |D^\alpha f(x_j) - D^\alpha g(x_j)| \\ & + |D^\alpha g(x_j) - D^\alpha g(x)| < 3\varepsilon. \end{aligned} \tag{3.2}$$

On the other hand, for any $x \in \mathbb{R}^n \setminus \bar{B}_T$ we have

$$|D^\alpha f(x) - D^\alpha g(x)| \leq \mathrm{diam}\mathcal{F}^\alpha(\mathrm{x}) \leq \varepsilon. \tag{3.3}$$

So from (3.2) and (3.3) we get $\|f-g\|_{C_0^k} \leq 3l\varepsilon$. By Lemma 2.2, $\mathcal{F}$ is totally bounded and therefore condition (ii) is satisfied.

Conversely, assume that $\mathcal{F}$ satisfies condition (ii). Since $\mathcal{F}$ is totally bounded in C_0^k, hence for any $T>0$, $\mathcal{F}$ is totally bounded in $C^k(\bar{B}_T)$, and as an application of Theorem 2.1, $\mathcal{F}^{\alpha}_{|\bar{B}_T}$ are equicontinuous on $\bar{B}_T$. On the other hand, take an arbitrary $\varepsilon>0$. Thus, there exist $f_1,\dots,f_m \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{i=1}^{m} \bar{B}(f_i,\varepsilon)$. Since $f_i \in C_0^k$, then there exists a $T>0$ such that $|D^\alpha f_i(x)|<\varepsilon$ for all $1 \leq i \leq m$, $|\alpha| \leq k$ and $\|x\|>T$. Hence for each $f \in \mathcal{F}$, there is an $1 \leq i \leq m$ such that f belongs to $\bar{B}(f_i,\varepsilon)$, and therefore we get

$$\begin{aligned} |D^\alpha f(x)| &\leq |D^\alpha f(x) - D^\alpha f_i(x)| + |D^\alpha f_i(x)| \\ &\leq 2\varepsilon \end{aligned}$$

for all $\|x\|>T$ and $|\alpha| \leq k$, and consequently condition (i) is satisfied. □

The following theorem presents a new measure of noncompactness on C_0^k.

Theorem 3.2. *Suppose* $1 \leq k < \infty$ *and* $\mathcal{F}$ *is a bounded subset of* C_0^k. *For* $f \in \mathcal{F}$, $\varepsilon>0$, $T>0$ *and* $\alpha \in \mathbb{R}^n$ *put*

$$\begin{aligned} &\omega^T(f,\varepsilon) = \sup\{|D^\alpha f(x) - D^\alpha f(y)| : x,y \in \bar{B}_T, \|x-y\|<\varepsilon, 0 \leq |\alpha| \leq k\}, \\ &\omega^T(\mathcal{F},\varepsilon) = \sup\{\omega^T(f,\varepsilon) : f \in \mathcal{F}\}, \\ &\omega^T(\mathcal{F}) = \lim_{\varepsilon \to 0} \omega^T(\mathcal{F},\varepsilon), \\ &\omega(\mathcal{F}) = \lim_{T \longrightarrow \infty} \omega^T(\mathcal{F}), \\ &d_\alpha(\mathcal{F}) = \lim_{\|x\| \to \infty} \mathrm{diam}\mathcal{F}^\alpha(\mathrm{x}). \end{aligned}$$

Then $\omega_0 : \mathfrak{M}_{C_0^k} \longrightarrow \mathbb{R}$ *given by*

$$\omega_0(\mathcal{F}) = \omega(\mathcal{F}) + \max_{0 \leq |\alpha| \leq k}(d_\alpha(\mathcal{F})) \tag{3.4}$$

defines a measure of noncompactness on C_0^k *and moreover,* $\ker(\omega_0) = \mathfrak{N}_{C_0^k}$.

Proof. The proof 2°, 4° and 5° are obvious. Now, we show that 1° holds. To do this, take $\mathcal{F} \in \mathfrak{M}_{C_0^k}$ such that $\omega_0(\mathcal{F})=0$. Let us arbitrarily fix an α with $0 \leq |\alpha| \leq k$. Let $\eta>0$ be arbitrary. Since $\omega_0(\mathcal{F})=0$, then

$$\lim_{T \to \infty} \lim_{\varepsilon \to 0} \omega^T(\mathcal{F},\varepsilon) = 0.$$

Thus, there exists $\delta>0$ and $T'>0$ such that $\omega^T(\mathcal{F},\delta)<\eta$ for all $T \geq T'$. This yields

$$|D^\alpha f(x) - D^\alpha f(y)| < \eta$$

for all $f \in \mathcal{F}$ and $x,y \in \bar{B}_T$ such that $\|x-y\|<\delta$. Then $\mathcal{F}^{\alpha}_{|\bar{B}_T}$ is bounded and equicontinuous for all $T \geq T'$. On the other hand, since $\mathcal{F}^{\alpha}_{|\bar{B}_{T'}}$ is bounded and equicontinuous, it follows that $\mathcal{F}^{\alpha}_{|\bar{B}_T}$ is bounded and equicontinuous for all

$T < T'$. Using again the fact that $\omega_0(\mathcal{F}) = 0$, we have $d_\alpha(\mathcal{F}) = 0$. Hence the condition (3.1) is valid. Now, by Theorem 3.1 we conclude that 1° holds.

Next, we check that 3° holds. Suppose that $\mathcal{F} \in \mathfrak{M}_{C_0^k}$. Similar to the proof of Theorem 2.3, we have

$$\omega(\overline{\mathcal{F}}) = \omega(\mathcal{F}). \tag{3.5}$$

Also, since $\operatorname{diam}\overline{\mathcal{F}}^\alpha = \operatorname{diam}\overline{\mathcal{F}^\alpha}$, we get $d_\alpha(\overline{\mathcal{F}}) = d_\alpha(\mathcal{F})$. This implies that $\omega_0(\overline{\mathcal{F}}) = \omega_0(\mathcal{F})$. Then condition 3° is satisfied.

To prove 6°, suppose that $\{\mathcal{F}_n\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_{C_0^k}$ such that $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \omega_0(\mathcal{F}_n) = 0$. For any $n \in \mathbb{N}$, take $f_n \in \mathcal{F}_n$ and set $\mathcal{G} = \overline{\{f_n\}}$. Similar to the proof of Theorem 2.3 we have $\omega^T(\mathcal{G}) = 0$, and therefore we deduce

$$\omega(\mathcal{G}) = \lim_{T\to\infty} \omega^T(\mathcal{G}) = 0. \tag{3.6}$$

Also, since $\lim_{n\to\infty} \omega_0(\mathcal{F}_n) = 0$, so we have $\lim_{n\to\infty} d_\alpha(\mathcal{F}_n) = 0$ for all $0 \le |\alpha| \le k$. Let $\varepsilon > 0$ and $0 \le |\alpha| \le k$ be fixed. There exists an $N \in \mathbb{N}$ such that $d_\alpha(\mathcal{F}_N) < \varepsilon$. Hence, we can find $T_1 > 0$ such that

$$\mathrm{diam}\mathcal{F}_{\mathrm{N}}^\alpha(\mathrm{x}) < \varepsilon,$$

for all $\|x\| > T_1$. Thus, for all $n, m \ge N$ and $\|x\| > T_1$ we can write

$$|D^\alpha f_n(x) - D^\alpha f_m(x)| < \varepsilon. \tag{3.7}$$

On the other hand, since $f_n \in C_0^k$, so we have $\lim_{\|x\|\to\infty} D^\alpha f_n(x) = 0$ for all $0 \le |\alpha| \le k$. Hence, there exists $T_2 > 0$ such that $|D^\alpha f_n(x)| < \varepsilon$ for all $\|x\| > T_2$ and $n = 1, 2, \ldots, N$. Moreover, we can write

$$|D^\alpha f_n(x) - D^\alpha f_m(x)| \le |D^\alpha f_n(x)| + |D^\alpha f_m(x)| < 2\varepsilon \tag{3.8}$$

for all $n, m \le N$ and $\|x\| > T_2$. This implies that

$$|D^\alpha f_n(x) - D^\alpha f_m(x)| \le \varepsilon \tag{3.9}$$

for all $n, m \in N$ and $\|x\| > T$ ($T = \max\{T_1, T_2\}$). Now, from (3.9) we conclude that

$$\max_{0\le|\alpha|\le k} d_\alpha(\mathcal{G}) = 0, \tag{3.10}$$

and therefore $\mathcal{G}$ is a compact set in C_0^k. Hence there exists a subsequence $\{f_{n_j}\}$ and $f_0 \in C_0^k$ such that $f_{n_j} \to f_0$. Since $f_n \in \mathcal{F}_n$, $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ and $\mathcal{F}_n$ is closed for all $n \in \mathbb{N}$ we get

$$f_0 \in \bigcap_{n=1}^{\infty} \mathcal{F}_n = \mathcal{F}_\infty,$$

which finishes the proof of 6°. Since the proof of $\ker(\omega_0) = \mathfrak{N}_{C_0^k}$ follows the main lines of the proof of $\ker(\omega_0) = \mathfrak{N}_{C^k(\Omega)}$, we omit the details. □

4. **Application**

In this section, we apply the results of the previous section to study the solvability of functional integral-differential equations on $C^1(\Omega)$.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(i) $p, q \in C^1(\Omega)$ *such that*

$$\lambda := \sup\{\|q\|_u + \|\frac{\partial q}{\partial x_i}\|_u : 1 \leq i \leq n\} < 1. \tag{4.1}$$

(ii) $g : \Omega \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ *is continuous and there exists a continuous function* $a : \Omega \longrightarrow \mathbb{R}_+$ *and a continuous and nondecreasing function* $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$|g(x, u_0, u_1, \dots, u_n)| \leq a(x)\zeta(\max_{0\leq i\leq n} |u_i|). \tag{4.2}$$

(iii) $k : \Omega \times \Omega \longrightarrow \mathbb{R}$ *is continuous and has a continuous derivative of order 1 with respect to the first argument.*

(iv) *There exists a positive solution* r_0 *of the inequality*

$$\|p\|_{C^1(\Omega)} + \lambda r + D\zeta(r) \leq r \tag{4.3}$$

where

$$D = \sup\Big\{\{\int_\Omega |\frac{\partial k}{\partial x_i}(x, y)|a(y)dy : x \in \Omega\} \cup \{\int_\Omega |k(x, y)|a(y)dy : x \in \Omega\}\Big\}.$$

Then the functional integral-differential equation

$$u(x) = p(x) + q(x)u(x) + \int_\Omega k\big(x, y\big)g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \dots, \frac{\partial u}{\partial x_n}(y)\big)dy \tag{4.4}$$

has at least one solution in the space $C^1(\Omega)$.

Proof. We define the operator $F : C^1(\Omega) \longrightarrow C^1(\Omega)$ by

$$Fu(x) = p(x)+q(x)u(x)+\int_\Omega k\big(x, y\big)g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \dots, \frac{\partial u}{\partial x_n}(y)\big)dy. \tag{4.5}$$

First notice that the continuity of $Fu(x)$ for any $u \in C^1(\Omega)$ is obvious. Also, for any $x \in \Omega$ we have

$$\begin{aligned}\frac{\partial(Fu)}{\partial x_i}(x) =& \frac{\partial p}{\partial x_i}(x) + \frac{\partial q}{\partial x_i}(x)u(x) + q(x)\frac{\partial u}{\partial x_i}(x)\\ &+ \int_\Omega \frac{\partial k}{\partial x_i}\big(x, y\big)g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \dots, \frac{\partial u}{\partial x_n}(y)\big)dy,\end{aligned}$$

and Fu has a continuous derivative. Thus, $Fu \in C^1(\Omega)$. Using conditions (i)-(iv), for arbitrarily fixed $x \in \Omega$, we have

$$\begin{aligned}|Fu(x)| \leq & |p(x)| + |q(x)||u(x)| \\ & + \Big| \int_\Omega k\big(x,y\big) g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) dy \Big| \\ \leq & \|p\|_u + \|q\|_u \|u\|_u + D\zeta(\|u\|_{C^1(\Omega)})\end{aligned}$$

and

$$\begin{aligned}|\frac{\partial Fu}{\partial x_i}(x)| &\leq |\frac{\partial p}{\partial x_i}(x)| + |\frac{\partial q}{\partial x_i}(x)||u(x)| + |q(x)||\frac{\partial u}{\partial x_i}(x)| \\ &\quad + \Big| \int_\Omega \frac{\partial k}{\partial x_i}\big(x,y\big) g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) dy \Big| \\ &\leq \|\frac{\partial p}{\partial x_i}\|_u + \|\frac{\partial q}{\partial x_i}\|_u \|u\|_u + \|q\|_u \|\frac{\partial u}{\partial x_i}\|_u + D\zeta(\|u\|_{C^1(\Omega)}).\end{aligned}$$

Thus, we obtain

$$\|Fu\|_{C^1(\Omega)} \quad \leq \quad \|p\|_{C^1(\Omega)} + \lambda \|u\|_{C^1(\Omega)} + D\zeta(\|u\|_{C^1(\Omega)}).$$

By considering condition (iv), we infer that F is a mapping from $\bar{B}_{r_0}$ into $\bar{B}_{r_0}$. Now, we show that the map F is continuous. For this, take $u \in C^1(\Omega)$ and $\varepsilon > 0$ arbitrarily, and consider $v \in C^1(\Omega)$ with $\|u - v\|_{C^1(\Omega)} < \varepsilon$. Then we have

$$\begin{aligned}|Fu(x) - Fv(x)| &\leq |q(x)||u(x) - v(x)| \\ &\quad + \int_\Omega |k(x,y)| \Big| g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) \\ &\quad - g\big(y, v(y), \frac{\partial v}{\partial x_1}(y), \ldots, \frac{\partial v}{\partial x_n}(y)\big) \Big| dy \\ &\leq \|q\|_u \|u - v\|_u + D\vartheta(\varepsilon),\end{aligned}$$

and by similar argument, we have

$$\begin{aligned}\left| \frac{\partial (Fu)}{\partial x_i}(x) - \frac{\partial (Fv)}{\partial x_i}(x) \right| &\leq \left| \frac{\partial q}{\partial x_i}(x) \right| |u(x) - v(x)| + |q(x)| \left| \frac{\partial u}{\partial x_i}(x) - \frac{\partial v}{\partial x_i}(x) \right| \\ &\quad + \int_\Omega \left| \frac{\partial k}{\partial x_i}(x,y) \right| \Big| g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) \\ &\quad - g\big(y, v(y), \frac{\partial v}{\partial x_1}(y), \ldots, \frac{\partial v}{\partial x_n}(y)\big) \Big| \, dy \\ &\leq \lambda \, |u - v|_{C^1(\Omega)} + D\vartheta(\varepsilon)\end{aligned}$$

where

$$\begin{aligned}\vartheta(\varepsilon) = \sup\{ \big| g(y, u_0, u_1 \ldots, u_n) - g(y, v_0, v_1, \ldots, v_n) \big| : y \in \Omega, u_i, v_i \in [-r_0, r_0], \\ |u_i - v_i| \leq \varepsilon \}.\end{aligned}$$

Since g is continuous on $\Omega \times [-r_0, r_0]^{n+1}$, then we have $\vartheta(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus F is a continuous operator from $C^1(\Omega)$ into $C^1(\Omega)$. In order to finish the proof, we only need to verify that condition (1.1) is satisfied. Let U be a nonempty and bounded subset of $C^1(\Omega)$, and assume that $\varepsilon > 0$. Let us choose $u \in U$ and $x_1, x_2 \in \Omega$ with $\|x_1 - x_2\| \leq \varepsilon$, thus we have

$$\begin{aligned}
&|Fu(x_1) - Fu(x_2)| = \\
=&\Big|p(x_1) + q(x_1)u(x_1) \\
&+ \int_\Omega k\big(x_1, y\big) g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) dy \\
&- \Big(p(x_2) + q(x_2)u(x_2) \\
&+ \int_\Omega k\big(x_2, y\big) g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big) dy\Big)\Big| \\
\leq& |p(x_1) - p(x_2)| + |q(x_1) - q(x_2)||u(x_1)| + |q(x_2)||u(x_1) - u(x_2)| \\
&+ \int_\Omega |k(x_1, s) - k(x_2, s)| \Big| g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big)\Big| dy \\
\leq& \omega(p, \varepsilon) + r_0 \omega(q, \varepsilon) + \lambda \omega(u, \varepsilon) + U_{r_0} m(\Omega) \omega(k, \varepsilon),
\end{aligned} \tag{4.6}$$

and by a similar argument, we deduce that

$$\begin{aligned}
&\left|\frac{\partial(Fu)}{\partial x_i}(x_1) - \frac{\partial(Fu)}{\partial x_i}(x_2)\right| \\
\leq& \left|\frac{\partial p}{\partial x_i}(x_1) - \frac{\partial p}{\partial x_i}(x_2)\right| + \left|\frac{\partial q}{\partial x_i}(x_1) - \frac{\partial q}{\partial x_i}(x_2)||u(x_1)\right| \\
&+ |q(x_2)| \left|\frac{\partial u}{\partial x_i}(x_1) - \frac{\partial u}{\partial x_i}(x_2)\right| \\
&+ \int_\Omega \left|\frac{\partial k}{\partial x_i}(x_1, y) - \frac{\partial k}{\partial x_i}(x_2, y)\right| \Big| g\big(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y)\big)\Big| dy \\
\leq& \omega(p, \varepsilon) + r_0 \omega(q, \varepsilon) + \lambda \omega(u, \varepsilon) + U_{r_0} m(\Omega) \omega(\frac{\partial k}{\partial x_i}, \varepsilon),
\end{aligned} \tag{4.7}$$

where m is the Lebesgue measure on Ω and

$$\begin{aligned}
U_{r_0} =& \sup\{|g(y, u_0, u_1 \ldots, u_n)| : y \in \Omega, |u_i| \leq r_0\}, \\
\omega(k, \varepsilon) =& \sup\{|k(x_1, y) - k(x_2, y)| : y, x_1, x_2 \in \Omega, \|x_1 - x_2\| \leq \varepsilon\} \\
\omega(\frac{\partial k}{\partial x_i}, \varepsilon) =& \sup\{|\frac{\partial k}{\partial x_i}(x_1, y) - \frac{\partial k}{\partial x_i}(x_2, y)| : y, x_1, x_2 \in \Omega, \\
& \|x_1 - x_2\| \leq \varepsilon, 1 \leq i \leq n\}.
\end{aligned}$$

Since u was an arbitrary element of U in (4.6) and (4.7), so we obtain

$$\begin{aligned}\omega(F(U),\varepsilon) \leq &\omega(p,\varepsilon)+r_0\omega(q,\varepsilon)+\lambda\omega(U,\varepsilon)\\ &+U_{r_0}m(\Omega)\max\{\omega(k,\varepsilon),\omega(\frac{\partial k}{\partial x_1},\varepsilon),\dots,\omega(\frac{\partial k}{\partial x_n},\varepsilon)\}.\end{aligned}$$

Now, by the uniform continuity of p, q, $\frac{\partial p}{\partial x_i}$ and $\frac{\partial q}{\partial x_i}$ on the compact set Ω for all $1 \leq i \leq n$, k and $\frac{\partial k}{\partial x_i}$ on the compact set $\Omega \times \Omega$ for all $1 \leq i \leq n$, we derive that $\omega(p,\varepsilon) \longrightarrow 0$, $\omega(q,\varepsilon) \longrightarrow 0$, $\omega(k,\varepsilon) \longrightarrow 0$ and $\omega(\frac{\partial k}{\partial x_i},\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, we get

$$\omega_0(F(U)) \leq \lambda\omega_0(U), \tag{4.8}$$

where $\lambda \in [0,1)$. Finally, from (4.8) and applying Theorem 1.2, we conclude that the functional integral-differential equation (4.4) has at least one solution in the space $C^1(\Omega)$. ☐

Now, we provide two examples illustrating the main result contained in Theorem 4.1 and showing its applicability.

Example 4.2. Consider the following the functional integral-differential equation

$$\begin{aligned}x(t,s) =&\sqrt{t^5}+\frac{e^{-t-2}x(t,s)}{s+4}\\ &+\int_0^1\int_0^1 \frac{t^2 su\cos(v)}{t^2+e^s}\tanh\Big(ux(u,v)\frac{\partial x}{\partial t}(u,v)+v^2\frac{\partial x}{\partial s}(u,v)\Big)dudv.\end{aligned} \tag{4.9}$$

Eq. (4.9) is a special case of Eq. (4.4) with

$$\begin{aligned}&p(t,s)=\sqrt{t^5}, \qquad q(t,s)=\frac{e^{-t-2}}{s+4}, \qquad k(t,s,u,v)=\frac{t^2su\cos(v)}{t^2+e^s},\\ &\Omega=[0,1]\times[0,1], g(u,v,x_0,x_1,x_2)=\tanh(ux_0x_1+v^2x_2).\end{aligned}$$

It is easy to see that $p,q \in C^1(\Omega)$ and $\lambda = \frac{9}{16e^2}$. Also, g is continous, and if we define $a(t,s)=\zeta(t)=1$, then condition (ii) holds. Moreover, k is continuous and has a continuous derivative of order 1 with respect to the first argument, and we deduce

$$\sup\left\{\int_0^1\int_0^1 |k(t,s,u,v)|a(u,v)dudv : t,s\in[0,1]\right\}$$

$$= \sup\left\{\int_0^1\int_0^1 \left|\frac{t^2su\cos(v)}{t^2+e^s}\right| ds : t,s\in[0,1]\right\} < \frac{1}{2},$$

$$\sup\left\{\int_0^1\int_0^1 \left|\frac{\partial k}{\partial t}(t,s,u,v)\right| a(u,v)dudv : t,s\in[0,1]\right\}$$

$$= \sup\left\{\int_0^1\int_0^1 \left|\frac{2tsu\cos(v)(t^2+e^s)-2t(t^2su\cos(v))}{(t^2+e^s)^2}\right| dudv : t,s\in[0,1]\right\}$$

$$< 2,$$

$$\sup\left\{\int_0^1\int_0^1 \left|\frac{\partial k}{\partial s}(t,s,u,v)\right| a(u,v)dudv : t,s\in[0,1]\right\} =$$

$$= \sup\left\{\int_0^1\int_0^1 \left|\frac{t^2u\cos(v)(t^2+e^s)-e^s(t^2su\cos(v))}{(t^2+e^s)^2}\right| dudv : t,s\in[0,1]\right\}$$

$$< \frac{1}{2}(1+e).$$

Thus, by choosing $D < 2$, it is easy to see that each number $r \geq 5$ satisfies the inequality in condition (iv), i.e.,

$$\|p\|_{C^1(\Omega)} + \lambda r + D\zeta(r) \leq \frac{5}{2} + \frac{9}{16e^2}r + 2 \leq r.$$

Hence, as the number r_0 we can take $r_0 = 5$. Consequently, all the conditions of Theorem 4.1 are satisfied. This implies that the functional integral-differential equation (4.9) has at least one solution which belongs to the space $C^1(\Omega)$.

Example 4.3. Consider the following the functional integral-differential equation

$$x(t) = \frac{x(t)}{t+4} + \int_0^2 e^{t^2-s}\frac{\sqrt[3]{s^2x'(s)+3x^{(3)}(s)}}{1+x^2(s)e^{s\sin(x'(s))}}ds. \tag{4.10}$$

Eq. (4.10) is a special case of Eq. (4.4) with

$$p(t)=0, \qquad q(t)=\frac{1}{t+4}, \qquad k(t,s)=e^{t^2-s}, \qquad \Omega=[0,2],$$

$$g(s,x_0,x_1,x_2,x_3)=\frac{\sqrt[3]{s^2x_1+3x_3}}{1+x_0^2e^{s\sin(x_1)}}.$$

It is easy to see that $p,q\in C^1(\Omega)$ and $\lambda = \frac{5}{16}$. Also, g is continuous, and if we define $a(t)=\sqrt[3]{7}$ and $\zeta(t)=\sqrt[3]{t}$, then condition (ii) holds. Moreover, k is continuous and has a continuous derivative of order 1 with respect to the first argument, and we have

$$\begin{aligned}\sup\left\{\int_0^2 |k(t,s)|a(s)ds : t\in[0,2]\right\} &= \sqrt[3]{7}\sup\left\{\int_0^2 |e^{t^2-s}|ds : t\in[0,2]\right\}\\ &< \sqrt[3]{7}e^4,\\ \sup\left\{\int_0^2 |\frac{\partial k}{\partial t}(t,s)|a(s)ds : t\in[0,2]\right\} &= \sqrt[3]{7}\sup\left\{\int_0^2 |2te^{t^2-s}|ds : t\in[0,2]\right\}\\ &< 4\sqrt[3]{7}e^4.\end{aligned}$$

Thus, by choosing $D < 4\sqrt[3]{7}e^4$, it is easy to see that each number $r \geq e^{10}$ satisfies the inequality in condition (iv), i.e.,

$$\|p\|_{C^1(\Omega)} + \lambda r + D\zeta(r) \leq \frac{5}{16}r + 4\sqrt[3]{7}e^4\sqrt[3]{r} \leq r.$$

Hence, as the number r_0 we can take $r_0 = e^{10}$. Consequently, all the conditions of Theorem 4.1 are satisfied. This show that the functional integral-differential equation (4.10) has at least one solution which belongs to the space $C^1(\Omega)$.

REFERENCES

[1] R. P. Agarwal, S. Arshad, D. O'Regan and V. Lupulescu, A Schauder fixed point theorem in semilinear spaces and applications, *Fixed Point Theory Appl.* **2013** (2013), no. 306, 13 pages.

[2] R. P. Agarwal, M. Benchohra and D. Seba, On the application of measure of noncompactness to the existence of solutions for fractional differential equations, *Results Math.* **55** (2009), no. 3-4, 221–230.

[3] R. P. Agarwal, M. Meehan and D. O'Regan, Fixed Point Theory and Applications, Cambridge Univ. Press, Cambridge, 2004.

[4] R. P. Agarwal and D. O'Regan, Fredholm and Volterra integral equations with integrable singularities, *Hokkaido Math. J.* **33** (2004), no. 2, 443–456.

[5] A. Aghajani, J. Banaś and Y. Jalilian, Existence of solutions for a class of nonlinear Volterra singular integralm equations, *Comput. Math. Appl.* **62** (2011), no. 3, 1215–1227.

[6] A. Aghajani and Y. Jalilian, Existence and global attractivity of solutions of a nonlinear functional integral equation, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), no. 11, 3306–3312.

[7] J. Banaś, Measures of noncompactness in the study of solutions of nonlinear differential and integral equations, *Cent. Eur. J. Math.* **10** (2012), no. 6, 2003–2011.

[8] J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York, 1980.

[9] J. Banaś and R. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, *Appl. Math. Lett.* **16** (2003), no. 1, 1–6.

[10] F. Bloom, Asymptotic bounds for solutions to a system of damped integrodifferential equations of electromagnetic theory, *J. Math. Anal. Appl.* **73** (1980), no. 2, 524–542.

[11] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Semin. Mat. Univ. Padova* **24** (1955) 84–92.

[12] M. A. Darwish, On monotonic solutions of a quadratic integral equation with supremum, *Dynam. Systems Appl.* **17** (2008), no. 3-4, 539–549.

[13] M. A. Darwish, J. Henderson and D. O'Regan, Existence and asymptotic stability of solutions of a perturbed fracttional functional-integral equation with linear modification of the arrgument, *Bull. Korean Math. Soc.* **48** (2011), no. 3, 539–553.

[14] B. C. Dhage and S. S. Bellale, Local asymptotic stability for nonlinear Quadratic functional integral equations, *Electron. J. Qual. Theory Differ. Equ.* **2008** (2008), no. 10, 13 pages.

[15] C. González and A. Jiménez-Melado, Existence of monotonic asymptotically constant solutions for second order differential equations, *Glasg. Math. J.* **49** (2007), no. 3, 515–523.

[16] D. Guo, Existence of solutions for n^{th}-order integro-differential equations in Banach spaces, *Comput. Math. Appl.* **41** (2001) 597–606.

[17] H. Hanche-Olsen and H. Holden, The Kolmogorov-Riesz compactness theorem, *Expo. Math.* **28** (2010), no. 4, 385–394.

[18] R. P. Kanwal, Linear Integral Differential Equations: Theory and Technique, Academic Press, New York, 1971.

[19] L. Liu, F. Guo, C. Wu and Y. Wu, Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces, *J. Math. Anal. Appl.* **309** (2005), no. 2, 638–649.

[20] Z. Liu and S. M. Kang, Applications of Schauder's fixed point theorem with respect to iterated functional equations, *Appl. Math. Lett.* **14** (2001), no. 8, 955–962.

[21] M. Mursaleen and S. A. Mohiuddine, Applications of noncompactness to the infinite system of differential equations in l_p spaces, *Nonlinear Anal.* **75** (2012), no. 4, 2111–2115.

[22] L. Olszowy, Solvability of infinite systems of singular integral equations in Fréchet space of coninuous functions, *Comput. Math. Appl.* **59** (2010), no. 8, 2794–2801.

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