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ON THE DIMENSION OF A SPECIAL SUBALGEBRA OF DERIVATIONS OF NILPOTENT LIE ALGEBRAS

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Abstract. Let L be a Lie algebra, $\mathrm{Der}(L)$ be the set of all derivations of L and $\mathrm{Der}_c(L)$ denote the set of all derivations $\alpha \in \mathrm{Der}(L)$ for which $\alpha(x) \in [x, L] := \{[x, y] | y \in L\}$ for all $x \in L$. We obtain an upper bound for dimension of $\mathrm{Der}_c(L)$ of the finite dimensional nilpotent Lie algebra L over algebraically closed fields. Also, we classify all finite dimensional nilpotent Lie algebras L over algebraically closed fields for which $\dim\mathrm{Der}_c(L)$ attains its maximum value.
Keywords: Lie algebra, derivation, nilpotent Lie algebra.
MSC(2010): Primary: 17B30; Secondary: 17B40.

1. Introduction and preliminaries

In 2007, Yadav [6] classified all finite p-groups G for which $|\mathrm{Aut}_c(G)|$ attains its maximum value, where $\mathrm{Aut}_c(G)$ is the group of all class preserving automorphisms of G. An automorphism α of G is called class preserving if $\alpha(x) \in x^G$ for all $x \in G$, where x^G denotes the conjugay class of x in G. The aim of this paper is to study an anologous version of Yadav's result for Lie algebras.

Let L be a Lie algebra over an arbitrary field F. A *derivation* of L is an F-linear transformation $\alpha : L \to L$ such that $\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)]$ for all $x, y \in L$. We denote by $\mathrm{Der}(L)$ the set of all derivations of L, which itself forms a Lie algebra with respect to the commutator of linear transformations, called the *derivation algebra* of L. The map $\mathrm{ad}_x : L \to L$ given by $y \to [x, y]$ is a derivation called the *inner derivation* corresponding to x for all $x \in L$. Clearly, the set $\mathrm{IDer}(L) = \{\mathrm{ad}_x | x \in L\}$ of all inner derivations of L is an ideal of $\mathrm{Der}(L)$.

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Let $\mathrm{Der}_c(L)$ denote the set of all derivations $\alpha \in \mathrm{Der}(L)$ such that $\alpha(x) \in \mathrm{Im}\ \mathrm{ad}_x$ for all $x \in L$. If $[x, L] := \{[x, y] | y \in L\}$, then

$$\mathrm{Der}_c(L) = \{\alpha \in \mathrm{Der}(L) : \alpha(x) \in [x, L], \forall x \in L\}.$$

Notice that $[x, L]$ is a vector space. Using the Jacobi identity, one can prove that $\mathrm{Der}_c(L)$ is a subalgebra of $\mathrm{Der}(L)$. Clearly, $\mathrm{IDer}(L)$ is contained in $\mathrm{Der}_c(L)$.

Cicalò, Graaf and Schneider [1] classified all nilpotent Lie algebras of dimension 6. Notice that, in this classification, the nilpotent Lie algebras are denoted by $L_{d,k}, L_{d,k}(\varepsilon), L^{(2)}_{d,k}$, or $L^{(2)}_{d,k}(\varepsilon)$ where d is the dimension of the corresponding Lie algebra, k is its nilindex among the nilpotent Lie algebras with dimension d, ε is a possible parameter, and the superscript "(2)" refers to the fact that the algebra is defined over a field of characteristic 2. The list of all nilpotent Lie algebras with dimension at most 5, described with the same notations, can be find in [2].

We use the following notations all over this paper. The center and derived algebra of L will be denoted by $Z(L)$ and L^2, respectively. For $x \in L, C_L(x)$ denotes the centralizer of x in L. Clearly, $C_L(x) = \{y \in L \mid [x, y] = 0\}$ is a subspace of L. Also, for each ideal J of L, $C_L(J) = \{y \in L \mid [x, y] = 0, \forall x \in J\}$ is the centralizer of J in L. It is easily checked that $C_L(J)$ is an ideal of L. We write $\langle x_1, \ldots, x_m\rangle$ for the subspace of L spanned by a given set $\{x_1, \ldots, x_m\}$, where $x_i \in L$.

In this paper, only finite dimensional Lie algebras over the algebraically closed fields are considered.

Our main results are as follows. Note that in Theorem A, we give an upper bound for dimension of $\mathrm{Der}_c(L)$ and in Theorem B, we classify all finite dimensional nilpotent Lie algebras L for which $\mathrm{dimDer}_c(L)$ attains its maximum value.

Theorem 1.1 (Theorem A)**.** *Let L be an n-dimensional nilpotent Lie algebra. Then*

$$\dim \mathrm{Der}_c(L) \leq \begin{cases} \frac{n^2-4}{4}, & n \text{ is even}, \\ \frac{n^2-1}{4}, & n \text{ is odd}. \end{cases}$$

Theorem 1.2 (Theorem B)**.** *Let L be a non-abelian nilpotent Lie algebra. Then, the equality holds in Theorem A if and only if one of the followings is satisfied.*

(i) *L is a 3-dimensional Heisenberg Lie algebra.*
(ii) $L = L_{4,3} = \langle x_1, \ldots, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle$.
(iii) $L = L_{6,22}(\varepsilon) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \varepsilon x_6, [x_3, x_4] = x_5\rangle$, *where* $\varepsilon \neq 0$.
(iv) $L = L_{6,24}(\varepsilon) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \varepsilon x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5\rangle$, *where* $\varepsilon \neq 0$.

(v) $L = L^{(2)}_{6,8}(\eta) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \eta x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5 + x_6 \rangle$, *where* $\eta \neq 0$.

Notice that over a field F of characteristic $\neq 2$, we have $L_{6,22}(\varepsilon_1) \cong L_{6,22}(\varepsilon_2)$, $L_{6,24}(\varepsilon_1) \cong L_{6,24}(\varepsilon_2)$ if and only if $\varepsilon_1 = \gamma^2 \varepsilon_2$ for some $0 \neq \gamma \in F$, and over a field F of characteristic 2, we have $L_{6,22}(\varepsilon_1) \cong L_{6,22}(\varepsilon_2)$, $L_{6,24}(\varepsilon_1) \cong L_{6,24}(\varepsilon_2)$ if and only if $\varepsilon_1 = \gamma^2 \varepsilon_2 + \delta^2$ for some $0 \neq \gamma \in F$ and $\delta \in F$. Also, η denotes a fixed element from $F \setminus \{x^2 + x \mid x \in F\}$, where F is a field of characteristic 2. For details, we refer the reader to [1].

We recall that a Lie algebra L is called Heisenberg provided that $L^2 = Z(L)$ and $\dim L^2 = 1$. Such algebras are odd dimensional with basis $x_1, \ldots, x_{2m}, x$ and the only non-zero multiplication between basis elements are $[x_{2i-1}, x_{2i}] = x$ for $i = 1, \ldots, m$.

2. Camina and almost Camina Lie algebras

Let L be a Lie algebra and $I \neq 0$ be a proper ideal of L. Then (L, I) is called a *Camina pair* if $I \subseteq [x, L]$ for all $x \in L \setminus I$. Also, L is called a *Camina Lie algebra* if (L, L^2) is a Camina pair. In [4] we have introduced and studied some properties of Camina Lie algebras. In this section, we present some new results on Camina Lie algebras. Also, we introduce almost Camina Lie algebras and give some properties of such Lie algebras. Our results here will be use to proof main theorems.

The next two results come from [4].

Lemma 2.1. *If (L, I) is a Camina pair and J is an ideal of L contained in I, then $(L/J, I/J)$ is a Camina pair.*

Theorem 2.2. *Let L be a nilpotent Camina Lie algebra with nilindex 3, where $dim(L/L^2) = m$, $dim(L^2/L^3) = n$ and $dim L^3 = r$. Then*

(i) *(L, L^3) is Camina pair, $m = 2n$ and n is even,*
(ii) *$L^3 = Z(L)$,*
(iv) *if $r = 1$ then $dim(L/C_L(L^2)) = n$.*

Lemma 2.3. *Let L be a Lie algebra and i, j be two positive integers. Then $[L^i, L^j] \subseteq L^{i+j}$.*

Proof. We proceed by induction on j. For $j = 1$ the result follows by definition. Now assume the result holds for j, i.e, $[L^i, L^j] \subseteq L^{i+j}$. By Jacobi identity,

$$[L^i, L^{j+1}] = [[L^j, L], L^i] \subseteq [[L, L^i], L^j] + [[L^i, L^j], L] \subseteq L^{i+j+1},$$

hence the result holds for $j + 1$. □

Lemma 2.4. *Let L be a Lie algebra and H be a subalgebra of L such that $L^2 = H^2$. Then $L^i = H^i$ for all $i \geq 2$.*

Proof. Since $L^2 = H^2 \subseteq H$, H is an ideal of L. By Jacobi identity, one can prove that the subalgebras $H^{i+1} = [H^i, H]$ are also ideals of L for all $i \geq 1$. We use induction on i to prove the result. Assume that the result holds by induction for $i \geq 2$. Then

$$[[L, H^{i-1}], H] \subseteq [L^i, H] = [H^i, H] = H^{i+1}.$$

Since $[H, L] \subseteq L^2 = H^2$, by Lemma 2.3, one gets

$$[[H, L], H^{i-1}] \subseteq [H^2, H^{i-1}] \subseteq H^{i+1}.$$

Thus, by using the Jacobi identity, we obtain

$$\begin{aligned} L^{i+1} = [L^i, L] = [H^i, L] &= [[H^{i-1}, H], L] \\ &\subseteq [[H, L], H^{i-1}] + [[L, H^{i-1}], H] \\ &\subseteq H^{i+1}, \end{aligned}$$

from which the result follows. □

Let L be a Lie algebra and $\Phi(L)$ be the Frattini subalgebra of L. Stagg [5] proved that for any Lie algebra L, $\Phi(L)$ is the set of all non-generators of L. Also, Marshall [3] proves that a maximal subalgebra of a nilpotent Lie algebra L is an ideal of L. Using this result he proves that the Frattini subalgebra of a nilpotent Lie algebra equals the derived subalgebra of L. Therefore, a nilpotent Lie algebra L such that $\dim(L/L^2) = m$ must have a minimal generating set $\{x_1, x_2, \ldots, x_m\}$ with m elements.

Lemma 2.5. *Let L be a Camina Lie algebra with nilindex 2 and $\{x_1, \ldots, x_m\}$ be a minimal generating set of L, where $m > 2$. Then, a subalgebra H with minimal generating set $\{x_1, \ldots, x_{m-1}\}$ satisfies $L^2 = H^2$. In particular, H is a maximal ideal of L.*

Proof. Since L is a Camina Lie algebra with nilindex 2, it follows that $Z(L) = L^2$. First suppose that $\dim L^2 = 1$. Then L is a Heisenberg Lie algebra and so m is even. If $L^2 \neq H^2$ then $H^2 = 0$ and hence $H + Z(L)$ is an abelian subalgebra of the Heisenberg Lie algebra L. Thus $\dim(H + Z(L)) = m \leq \frac{m}{2} + 1$. This implies that $m \leq 2$, which contradicts to our assumption.

Now suppose that $\dim L^2 > 1$. If $L^2 \neq H^2$, then H^2 is contained in some maximal subalgebra M of $L^2 = Z(L)$. Thus, by Lemma 2.1, L/M is a Camina Lie algebra. Clearly, L/M has nilindex 2 and $\{x_1+M, \ldots, x_m+M\}$ is a minimal generating set of L/M. But then $(H + M)/M$ has a minimal generating set $\{x_1 + M, \ldots, x_{m-1} + M\}$ with $(\frac{H+M}{M})^2 = 0$, which is a contradiction by the previous paragraph. Clearly, H is a maximal subalgebra of L and hence it is an ideal by [3, Corollary 1]. □

Proposition 2.6. *Let $\{x_1, \ldots, x_m\}$ be a minimal generating set for a Camina Lie algebra L with nilindex 3. Then a subalgebra H with minimal generating set $\{x_1, \ldots, x_{m-1}\}$ satisfies $H^2 = L^2 \nsubseteq Z(H)$. In particular, H is a maximal ideal of L.*

Proof. Set $\dim(L/L^2) = m$, $\dim(L^2/L^3) = n$ and $\dim L^3 = r$. From Theorem 2.2 it follows that $L^3 = Z(L)$, $m = 2n$ and n is even. So $m > 2$. Also, by Lemma 2.1, L/L^3 is a Camina Lie algebra. Clearly L/L^3 has nilindex 2, and $\{x_1 + L^3, \ldots, x_m + L^3\}$ and $\{x_1 + L^3, \ldots, x_{m-1} + L^3\}$ are minimal generating sets for L/L^3 and $(H + L^3)/L^3$, respectively. Thus, Lemma 2.5 gives us $(L/L^3)^2 = ((H+L^3)/L^3)^2$, which implies that $L^2/L^3 = (H^2+L^3)/L^3$ and hence $H^2 + L^3 = L^2$. First suppose that $r = 1$ and $H^2 \subset L^2$. Since $\dim L^3 = 1$, L^2 is the direct sum of two abelian Lie algebras $Z(L) = L^3$ and $H^2 \cong L^2/L^3$ both of which are centralized by H. So $[L^2, H] = 0$. Since L^2 is abelian, $H + L^2 \subseteq C_L(L^2)$. On the other hand, $H \subset H + L^2 \subset L$ from which we obtain $\dim(L/(H + L^2)) = 1$. Therefore, by Theorem 2.2,

$$n = \dim\left(\frac{L}{C_L(L^2)}\right) \leq \dim\left(\frac{L}{H + L^2}\right) = 1,$$

which is impossible as $n > 0$ is even.

Now assume that $r > 1$ and $H^2 \subset L^2$. Since $L^2 = H^2 + L^3$, we have $H^2 \cap L^3 \subset L^3$. So, there is a maximal subalgebra M of $L^3 = Z(L)$ containing $H^2 \cap L^3$. We know that L/M is a Camina Lie algebra with nilindex 3, and $\{x_1 + M, \ldots, x_m + M\}$ and $\{x_1 + M, \ldots, x_{m-1} + M\}$ are minimal generating sets for L/M and $(H + M)/M$, respectively. Since $\dim(L^3/M) = 1$ and the result holds for $r = 1$, $(L/M)^2 = ((H+M)/M)^2$. Hence $L^2/M = (H^2+M)/M$ and so $H^2 + M = L^2$. Therefore,

$$L^3 = (H^2 + M) \cap L^3 \subseteq (H^2 \cap L^3) + M = M \subset L^3,$$

which is impossible. This contradiction shows that $H^2 = L^2$.

Finally, by Lemma 2.4, $H^3 = L^3 \neq 0$. So, $H^2 \subseteq Z(H)$ when $L^2 \subseteq Z(H)$ and hence $H^3 = 0$, which is a contradiction. Therefore $L^2 \nsubseteq Z(H)$. □

Lemma 2.7. *Let L be a Camina Lie algebra with nilindex 3 such that $\dim L^3 \geq 2$. Then $Z(M) = Z(L)$ for every maximal ideal M of L.*

Proof. Let M be a maximal ideal of L and $x \in L$ be such that $L = M \oplus \langle x \rangle$. Since L is a Camina Lie algebra, $Z(L) \subset L^2 = \Phi(L) \subseteq M$. Therefore, $Z(L) \subseteq Z(M)$.

Suppose on the contrary that $Z(L) \subset Z(M)$ and $z \in Z(M) \setminus Z(L)$. If $z \in L^2 \setminus L^3$, then $L^3 = [z, L]$ for, by Theorem 2.2, (L, L^3) is a Camina pair. If $z \in L \setminus L^2$, then $L^2 = [z, L]$ for L is a Camina Lie algebra. In both cases, we have $2 \leq \dim L^3 \leq \dim[z, L]$. Since $z \in Z(M)$ and $L = M \oplus \langle x \rangle$, we have $\dim[z, L] = 1$, from which it follows that $2 \leq 1$. This contradiction shows that $Z(L) = Z(M)$. □

Definition 2.8. A finite dimensional nilpotent Lie algebra L is called an almost Camina Lie algebra if for any minimal generating set $\{x_1, \ldots, x_m\}$ of L, we have $[x_i, L] = L^2$ for all but at most one i.

Notice that every nilpotent Camina Lie algebra is almost Camina Lie algebra. But the converse is not true in general. For instance $L_{4,3} = \langle x_1, \ldots, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$ (see [2]) is an almost Camina Lie algebra, which it is not a Camina Lie algebra, because $[x_1, L] = L^2, [x_2, L] \subset L^2$.

Lemma 2.9. *Let L be an almost Camina Lie algebra and $\{x = x_1, \ldots, x_m\}$ be a minimal generating set of L such that $[x, L] \subset L^2$. Then*

(i) *$[y, L] = L^2$ for all $y \in L \setminus (\langle x \rangle + L^2)$*
(ii) *$[y, L] = L^2$ for all $y \in H \setminus L^2$, where $H = \langle x_2, \ldots, x_m \rangle + L^2$.*

Proof. (i) It is obvious that L/L^2 is an m-dimensional abelian Lie algebra with basis $\{x_1 + L^2, \ldots, x_m + L^2\}$. Let K/L^2 be an ideal of L/L^2 spanned by $x_1 + L^2$. Suppose on the contrary that there is an element y in $L \setminus (\langle x \rangle + L^2)$ such that $[y, L] \subset L^2$. Therefore, $y + L^2 \in L/L^2 \setminus K/L^2$ and the set $\{x_1 + L^2, y + L^2\}$ is linearly independent. Hence, we can extend this set to a basis $\{x_1 + L^2, y + L^2, y_3 + L^2, \ldots, y_m + L^2\}$ of L/L^2, which contradicts the fact that L is an almost Camina Lie algebra, since $[x, L]$ and $[y, L]$ are proper subspace of L^2.

(ii) Let $y \in H \setminus L^2$. We claim that $y \in L \setminus (\langle x \rangle + L^2)$. If $y \in \langle x \rangle + L^2$, then $y = \alpha x + u$ for some $u \in L^2$ and a non-zero scalar $\alpha \in F$. This implies that $\alpha x = y - u \in H$ and hence $x \in H$. This contradiction shows that our claim holds. Now (ii) follows from (i). $\square$

Proposition 2.10. *Let L be an almost Camina Lie algebra such that $dimL = 2m$ and $dimL^2 = m$, where $m \geq 3$. Then there exists a maximal ideal H of L such that $L^2 = H^2$. Moreover, if the nilindex of L is at least 3, then $L^2 \nsubseteq Z(H)$ for any such maximal ideal H.*

Proof. Since every nilpotent Camina Lie algebra has nilindex at most 3 (see [4]), the result holds from Lemma 2.5 and Proposition 2.6 when L is a Camina Lie algebra. Now, we proceed in three steps.

Step 1. Let $x' \in L \setminus L^2$ such that $C_L(x') + L^2 \neq \langle x' \rangle + L^2$. Then there exists an element $y' \in L \setminus (\langle x' \rangle + L^2)$ such that $[x', y'] = 0$. Hence, $M = \langle x', y' \rangle$ is an abelian subalgebra of L and $\dim(M + L^2)/L^2 = 2$. Choose $x \in M \setminus L^2$ such that $\dim[x, L] \leq \dim[z, L]$ for all $z \in M \setminus L^2$, and let $\{x = x_1, x_2, \ldots, x_m\}$ be a minimal generating set of L and $H = \langle x_2, \ldots, x_m \rangle + L^2$. Clearly, $\dim(\langle x \rangle + L^2)/L^2 = 1$ and so $\langle x \rangle + L^2 \subset M + L^2$. Let $y \in M \setminus (\langle x \rangle + L^2)$. Then $[x, y] = 0$. If $[x, L] = L^2$ then, by minimality of $\dim[x, L]$, it follows that $[y, L] = L^2$ and if $[x, L] \subset L^2$, then it follows from Lemma 2.9 that $[y, L] = L^2$. Thus, in both cases, we have $[y, L] = L^2$. Let $u \in L^2$ be an arbitrary element of L^2. Then $u = [y, l]$ for some $l \in L$. Now l can be written as $l = \alpha x + h$ for some $h \in H$

and $\alpha \in F$. Therefore,

$$u = [y, l] = [y, \alpha x + h] = \alpha[y, x] + [y, h] = [y, h] \in H^2.$$

This implies that $L^2 \subseteq H^2$. Since $H^2 \subseteq L^2$, we have $L^2 = H^2$.

Step 2. Let $x \in L \setminus L^2$ such that $\dim[x, L] \leq m - 2$. Since $\dim[x, L] = \dim(L/C_L(x))$, we have $\dim C_L(x) \geq m+2$. But $\dim(\langle x\rangle + L^2) = m+1$, which implies that $C_L(x) + L^2 \neq \langle x\rangle + L^2$. By step 1, we have $L^2 = H^2$ for some maximal ideal H of L.

Step 3. Let $C_L(u) + L^2 = \langle u\rangle + L^2$ for all $u \in L \setminus L^2$. Then, it follows from step 2 that $\dim[u, L] \geq m - 1$ for all $u \in L \setminus L^2$. Let $x \in L \setminus L^2$ such that $\dim[x, L] = m - 1$, $\{x = x_1, \ldots, x_m\}$ be a minimal generating set for L and $H = \langle x_2, \ldots, x_m\rangle + L^2$.

We know that $\dim(\langle x\rangle + L^2) = m + 1$. Since $C_L(u) + L^2 = \langle u\rangle + L^2$ for all $u \in L \setminus L^2$, we have $C_L(u) \subseteq \langle u\rangle + L^2$. In particular, $C_L(x) \subseteq \langle x\rangle + L^2$. Since $\dim C_L(x) = \dim(L/[x, L]) = m + 1$, we have $C_L(x) = \langle x\rangle + L^2$. This implies that $L^2 \subseteq C_L(x)$.

First suppose that $[x, L] \cap H^2 \neq 0$. Let $0 \neq [x, l] \in [x, L] \cap H^2$. Then $[x, l] = [x, \alpha x + h] = [x, h]$ for some $h \in H \setminus L^2$ and $\alpha \in F$. Since $h \in H \setminus L^2$, it follows from Lemma 2.9 that $[h, L] = L^2$. If $[h, l']$ be an arbitrary element of L^2, then $[h, l'] = [h, \beta x + h']$ for some $h' \in H$ and $\beta \in F$. Thus,

$$[h, l'] = [h, \beta x + h'] = \beta[h, x] + [h, h'] \in H^2,$$

which implies that $L^2 \subseteq H^2$. Since the reverse inclusion is obvious, we have $L^2 = H^2$.

Now suppose that $[x, L] \cap H^2 = 0$. This implies that $H^2 \subset L^2$. If $h \in H \setminus L^2$, then by assumption, $C_L(h) \subseteq \langle h\rangle + L^2 \subseteq H$ and hence $C_H(h) = C_L(h) \cap H = C_L(h)$. Since $\dim C_L(h) = \dim(L/[h, L]) = \dim(L/L^2) = m$, we have $\dim[h, H] = \dim(H/C_H(h)) = m - 1$. This proves that $[h, H] = H^2$ since $[h, H] \subseteq H^2 \subset L^2$. Applying the Jacobi identity, one can show that H^2 is an ideal of L. Now consider the Lie algebra L/H^2. Since $L^2 \nsubseteq H^2$, L/H^2 is a non-abelian Lie algebra of dimension $m + 1$. Also, we have $(L/H^2)^2 = L^2/H^2$ and $\dim(L^2/H^2) = 1$. Thus $L^2/H^2 \subseteq Z(L/H^2)$ and hence L/H^2 has nilindex 2. Now, we show that L/H^2 is a Camina Lie algebra. Suppose on the contrary that there is some element $l + H^2 \in L/H^2 \setminus L^2/H^2$ such that $[l + H^2, L/H^2] \subset L^2/H^2$. Then $[l + H^2, L/H^2] = 0$ and so $[l, L] \subseteq H^2$. If $l \in L \setminus (\langle x\rangle + L^2)$, then $L^2 = [l, L] \subseteq H^2$, which contradicts the fact that $H^2 \subset L^2$. So, let $l \in (\langle x\rangle + L^2)$. Then $l = \lambda x + v$ for some $v \in L^2$ and $0 \neq \lambda \in F$. Since $[l, H] \subseteq H^2$, it follows that $[l, h] = [\lambda x + v, h] = \lambda[x, h] + [v, h] \in H^2$, where h is an element of $H \setminus L^2$. As $v \in L^2 \subseteq H$, we have $[v, h] \in H^2$ and hence $[x, h] \in H^2$. If $[x, h] = 0$ then $h \in C_L(x)$ so that $C_L(x) + L^2 \neq \langle x\rangle + L^2$ contrary to the hypotheses of this step. But then $0 \neq [x, h] \in [x, L] \cap H^2$ contradicts the assumption that $[x, L] \cap H^2 = 0$. Therefore, L/H^2 is a Camina Lie algebra with nilindex 2. Hence, by [4, Corollary 2.5], L/H^2 is a Heisenberg Lie algebra and consequently m is even. Since H/H^2 is an abelian ideal of L/H^2, we have $\dim(H/H^2) = m \leq m/2 + 1$, which implies that $m \leq 2$ contradicting our assumption that $m \geq 3$. Hence the latter case can not occur.

Now, from steps 1, 2 and 3, it follows that there exists a maximal ideal H of L such that $L^2 = H^2$. Let H be any maximal ideal of L such that $L^2 = H^2$. If $L^2 \subseteq Z(H)$, then $H^2 = L^2 \subseteq Z(H)$. Hence H has nilindex 2 and, by Lemma 2.4, L has nilindex 2. Therefore $L^2 \nsubseteq Z(H)$ whenever the nilindex of L is at least 3. This completes the proof. □

3. **Proof of Theorem A**

In what follows, we state some results that will be used in the proof of Theorem A.

Proposition 3.1. *Let L be an n-dimensional nilpotent Lie algebra and $\{x_1, \ldots, x_d\}$ be a minimal generating set of L. Then*

$$\dim \mathrm{Der}_c(L) \leq \sum_{i=1}^{d} \dim[x_i, L].$$

Moreover, if $\dim L^2 = m$, then

$$\dim \mathrm{Der}_c(L) \leq m(n-m). \tag{3.1}$$

Proof. If $\alpha \in \mathrm{Der}_c(L)$, then $\alpha(x_i) \in [x_i, L]$ for all $1 \leq i \leq d$. It is easy to see that the map

$$\begin{array}{rcl} \psi : \mathrm{Der}_c(L) & \longrightarrow & ([x_1, L] \oplus \cdots \oplus [x_d, L]) \\ \alpha & \longmapsto & (\alpha(x_1), \ldots, \alpha(x_d)) \end{array}$$

is an injective linear transformation. Hence the result follows.

Now if $\dim L^2 = m$, then $d = n - m$. Since $\dim[x_i, L] \leq \dim L^2$ for all $1 \leq i \leq d$, it follows that $\dim \mathrm{Der}_c(L) \leq m(n-m)$, as required. □

In the sequel, we denote the set

$$\{f \in T(L/Z(L), L^2) : f(x + Z(L)) \in [x, L], \forall x \in L\}$$

by $T_c(L/Z(L), L^2)$.

Proposition 3.2. *Let L be a nilpotent Lie algebra with nilindex 2. Then $\mathrm{Der}_c(L) \cong T_c(L/Z(L), L^2)$ as abelian Lie algebras.*

Proof. For any $\alpha \in \mathrm{Der}_c(L)$, the map $\psi_\alpha : L/Z(L) \to L^2$ defined by $\psi_\alpha(x + Z(L)) = \alpha(x)$ is a linear transformation. It is easy to see that the map $\psi : \mathrm{Der}_c(L) \to T_c(L/Z(L), L^2)$ defined by $\psi(\alpha) = \psi_\alpha$ is a Lie isomorphism, from which the result follows. □

Theorem 3.3. *Let L be a nilpotent Lie algebra. Then the equality holds in (3.1) if and only if either L is an abelian Lie algebra or it is Camina Lie algebra with nilindex 2.*

Proof. Let L be a nilpotent Lie algebra of dimension n and $\dim L^2 = m$. If L is abelian, then the equality holds. Now let L be a Camina Lie algebra with nilindex 2. Then $L^2 = Z(L)$ and $[x, L] = L^2$ for all $x \in L \setminus L^2$. Suppose that $\alpha \in T(L/Z(L), L^2)$. Then $\alpha(x + Z(L)) \in L^2 = [x, L]$ and hence $\alpha \in T_c(L/Z(L), L^2)$. Thus $T(L/Z(L), L^2) \subseteq T_c(L/Z(L), L^2)$. Since the reverse inclusion is obvious, we have $T_c(L/Z(L), L^2) = T(L/Z(L), L^2)$. Now, from Proposition 3.2, we get

$$\dim \mathrm{Der}_c(L) = \dim T(L/Z(L), L^2) = \dim(L/Z(L)) \dim L^2 = m(n - m),$$

hence the equality holds.

Conversely, suppose that L is a nilpotent Lie algebra for which the equality holds in (3.1). If L is abelian, then there is nothing to prove. So assume that L is non-abelian. We know that every element $x \in L \setminus L^2$ belongs to a minimal generating set $\{x = x_1, \ldots, x_{n-m}\}$ of L. If $[x, L] \subset L^2$, then $\dim[x, L] < \dim L^2$ and, by Proposition 3.1, $\dim \mathrm{Der}_c(L) < m(n - m)$, which is a contradiction. Therefore, $[x, L] = L^2$ for all $x \in L \setminus L^2$. This implies that L is a Camina Lie algebra and hence L has nilindex at most 3 by [4, Theorem 5.6].

Since L is a Camina Lie algebra and the equality holds in (3.1), the map ψ in Proposition 3.1 is onto. Therefore, for any minimal set of generators $\{x_1, \ldots, x_{n-m}\}$ of L and any elements $y_1, \ldots, y_{n-m} \in L^2$ (need not be distinct), there exists a derivation $\alpha \in \mathrm{Der}_c(L)$ such that $\alpha(x_i) = y_i$ for $1 \leq i \leq n - m$. In particular, we can choose α in such a way that $\alpha(x_i) = 0$, for all $1 \leq i \leq n - m - 1$ and $\alpha(x_{n-m}) = y$, where y is an arbitrary element of L^2. Suppose L is nilpotent with nilindex 3 and $\{x_1, \ldots, x_{n-m}\}$ is a minimal generating set of L. Then, by Proposition 2.6, the subalgebra H with minimal generating set $\{x_1, \ldots, x_{n-m-1}\}$ is a maximal ideal of L such that $H^2 = L^2$ and $L^2 \nsubseteq Z(H)$. We fix an element $y \in L^2 \setminus Z(H)$. Now we can choose a derivation $\alpha \in \mathrm{Der}_c(L)$ such that $\alpha(x_i) = 0$ for all $1 \leq i \leq n-m-1$ and $\alpha(x_{n-m}) = y$. Then $\alpha(h) = 0$ for all $h \in H$ and $\alpha(x_{n-m}) \in H$. Since

$$0 = \alpha([x_{n-m}, h]) = [\alpha(x_{n-m}), h] + [x_{n-m}, \alpha(h)] = [y, h]$$

for all $h \in H$, one gets $y \in Z(H)$. This contradiction proves that the nilindex of L cannot be 3. Since L is non-abelian, L has nilindex 2, which completes the proof of the theorem. □

Let L be an n-dimensional nilpotent Camina Lie algebra with nilindex 2 such that $\dim L^2 = m$. Then, by [4, Theorem 3.2], $n - m$ is even and $n - m \geq 2m$. This implies that

$$m \leq \frac{n}{3}. \tag{3.2}$$

Now, we are in a position to prove Theorem A.

Proof of Theorem A. If L is abelian, then the result holds obviously. So let L be a non-abelian Lie algebra and $\dim L^2 = m$. Since L is nilpotent, we have

$1 \leq m \leq n-2$. Therefore all possible values of $m(n-m)$ are

$$\left\{ n-1, 2(n-2), 3(n-3), \ldots, \frac{n^2}{4} \right\}$$

if n is even and

$$\left\{ n-1, 2(n-2), 3(n-3), \ldots, \frac{(n-1)(n+1)}{4} \right\}$$

if n is odd. Putting these values in (3.1), we get

$$\dim \mathrm{Der}_c(L) \leq \begin{cases} \frac{n^2}{4} & \textit{if } n \textit{ is even}, \\ \frac{n^2-1}{4} & \textit{if } n \textit{ is odd}. \end{cases}$$

If n is even and $\dim\mathrm{Der}_c(L) = n^2/4$, then $m = n/2$ and hence the equality holds in (3.1). Now, by Theorem 3.3, it follows that L is a Camina Lie algebra with nilindex 2 so that , by (3.2), $m \leq n/3$, which is a contradiction. Hence, there exists no nilpotent Lie algebra L satisfying $\dim\mathrm{Der}_c(L) = \frac{n^2}{4}$. Therefore,

$$\dim \mathrm{Der}_c(L) \leq \frac{n^2}{4} - 1 = \frac{n^2-4}{4}.$$

This proves the theorem.

4. **Proof of Theorem B**

We first give some results which will be used for the proof of Theorem B.

Lemma 4.1. *Let L be a non-abelian nilpotent Lie algebra of dimension $2m$ such that $\dim L^2 = m$ and the equality holds in Theorem A. Then L is an almost Camina Lie algebra with nilindex ≥ 3, which is not a Camina Lie algebra. Moreover, $\dim[x, L] \geq m-1$ for all $x \in L \setminus L^2$.*

Proof. If L has nilindex 2, then $L^2 \subseteq Z(L)$. Therefore $\dim C_L(x) \geq m+1$ for any element $x \in L \setminus L^2$ and hence $\dim[x, L] = \dim(L/C_L(x)) \leq m-1$ for all $x \in L \setminus L^2$. Now, from Proposition 3.1, it follows that $\dim\mathrm{Der}_c(L) \leq (2m-m)(m-1) < m^2-1$, which is a contradiction. Hence, the nilindex of L is at least 3.

Now, we show that L is not a Camina Lie algebra. Suppose on the contrary that L is a Camina Lie algebra. Then L has nilindex 3. Let $\dim(L^2/L^3) = s$ and $\dim L^3 = r$. Then, by Theorem 2.2, $L^3 = Z(L)$, $m = 2s$, and s is even. Also, $2m = \dim L = m+s+r$. Since m and s are even, it follows that r is even and hence $r \geq 2$. Let $\{x_1, \ldots, x_m\}$ be a minimal generating set for L. Then, by Proposition 2.6, the subalgebra H with minimal generating set $\{x_1, \ldots, x_{m-1}\}$ is a maximal ideal of L such that $L^2 = H^2$. Let $\mathcal{D}$ be the subalgebra of $\mathrm{Der}_c(L)$ consisting of all $\alpha \in \mathrm{Der}_c(L)$ satisfying $\alpha(h) = 0$ for all $h \in H$ and $\alpha(x_m) \in H$. For each $\alpha \in \mathcal{D}$ and $h \in H$, we have

$$0 = \alpha([x_m, h]) = [\alpha(x_m), h] + [x_m, \alpha(h)] = [\alpha(x_m), h]$$

and so $\alpha(x_m) \in Z(H)$. Since the equality holds in Theorem A, $\dim\langle\alpha(x_m) \mid \alpha \in \mathcal{D}\rangle \geq m-1$ and hence $\dim Z(H) \geq m-1$. On the other hand, by Lemma 2.7, we have $Z(L) = Z(H)$ for $r \geq 2$. Hence

$$\dim \frac{L^2}{L^3} = \dim \frac{L^2}{Z(L)} = \dim \frac{L^2}{Z(H)} = 1$$

when $\dim Z(H) = m-1$, which contradicts the fact that s is even. Thus $\dim Z(H) = m$ and so $L^2 = Z(L)$, which is impossible as the nilindex of L is 3. This proves that L is not Camina.

Since L is not a Camina Lie algebra, there exists a minimal generating set $\{x_1, \ldots, x_m\}$ for L such that $[x_i, L] \subset L^2$ for some $1 \leq i \leq m$. Suppose that there are at least two numbers $1 \leq i, j \leq m$ such that $[x_i, L], [x_j, L] \subset L^2$. Then $\dim[x_i, L]$,$\dim[x_j, L] < m$ and hence, by Proposition 3.1, $m^2 - 1 = \dim\mathrm{Der}_c(L) \leq m(m-2) + (m-1) + (m-1)$, which is impossible. Therefore, $[x_i, L] \subset L^2$ for at most one $1 \leq i \leq m$. Now, by definition of almost Camina Lie algebras, it follows that L is almost Camina.

Finally, suppose that $\dim[x_i, L] \leq m-2$ for some x_i with $1 \leq i \leq m$. Then, by Proposition 3.1, we have $m^2 - 1 = \dim\mathrm{Der}_c(L) \leq m(m-1) + (m-2)$, which is a contradiction. This completes the proof of the lemma. □

Proposition 4.2. *Let L be an almost Camina Lie algebra of dimension $2m$ ($m \geq 2$) such that $\dim L^2 = m$. If the equality holds in Theorem A, then one of the following holds:*

(i) $L = L_{4,3} = \langle x_1, \ldots, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle$.
(ii) $L = L_{6,24}(\varepsilon) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \varepsilon x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5\rangle$, *where* $\varepsilon \neq 0$.
(iii) $L = L^{(2)}_{6,8}(\eta) = \langle x_1, \ldots, x_6 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \eta x_6, [x_2, x_3] = x_6, [x_2, x_4] = x_5 + x_6\rangle$, *where* $\eta \neq 0$.

Proof. Since $\dim L^2 = m$ with $m \geq 2$, L must be non-abelian. Therefore, by Lemma 4.1, the nilindex of L is at least 3 and L is not a Camina Lie algebra. First suppose that $m = 2$. Then $\dim L = 4$ and L has nilindex 3 so that $L = L_{4,3}$ (see [2]).

Now suppose that $m \geq 3$. Since L is an almost Camina Lie algebra and it is not a Camina Lie algebra, there exists an $z \in L \setminus L^2$ such that $[z, L] \subset L^2$. Thus, by Lemma 2.9, $[y, L] = L^2$ for all $y \in L \setminus (\langle z\rangle + L^2)$. Also, by Lemma 4.1, we have $\dim[x, L] \geq m-1$ for all $x \in L \setminus L^2$ and hence $\dim[z, L] = m-1$. Therefore, $Z(L) \subset L^2$. Since the equality holds in Theorem A, for a minimal generating set $\{x_1 = z, x_2, \ldots, x_m\}$ of L, an element $y_1 \in [z, L]$ and elements $y_2, \ldots, y_m \in L^2$, there exists $\alpha \in \mathrm{Der}_c(L)$ such that

$$\alpha(x_i) = y_i, 1 \leq i \leq m.$$

Since L is nilpotent with nilindex at least 3, from Proposition 2.10, it follows that there exists a maximal ideal H of L such that $H^2 = L^2$ and $L^2 \nsubseteq Z(H)$. We consider the following cases.

Case1. $z \in H$. Since $H^2 = L^2$, the element $z \in H \setminus L^2$ can be extended to a minimal generating set $\{z = x_1, x_2, \ldots, x_m\}$ of L such that $\{x_1, \ldots, x_{m-1}\}$ is a minimal generating set of H and $x = x_m \notin H$. Since L is almost Camina and $[z, L] \subset L^2$, it follows that $[x, L] = L^2$. Choose an element $y \in L^2 \setminus Z(H)$. Then there exists $\alpha \in \mathrm{Der}_c(L)$ such that

$$\alpha(x_i) = 0, \forall 1 \leq i \leq m-1 \text{ and } \alpha(x_m) = y.$$

Then $\alpha(h) = 0$ for all $h \in H$ and $\alpha(x) \in H$, hence

$$0 = \alpha([x, h]) = [\alpha(x), h] + [x, \alpha(h)] = [y, h],$$

which implies that $y \in Z(H)$, a contradiction.

Case 2. $z \notin H$ and $[z, L] \nsubseteq Z(H)$. Let $\{x_1 = z, x_2, \ldots, x_m\}$ be a minimal generating set of L such that $\{x_2, \ldots, x_m\}$ is a minimal generating set for H. Since $[z, L] \nsubseteq Z(H)$ we may choose a non-zero element $y \in [z, L] \setminus Z(H)$. Then, there exists an element $\alpha \in \mathrm{Der}_c(L)$ such that

$$\alpha(x_1) = y, \ \alpha(x_i) = 0, \ \forall 2 \leq i \leq m.$$

Then $\alpha(h) = 0$ for all $h \in H$ and $\alpha(x_1) \in H$ so that

$$0 = \alpha([x_1, h]) = [\alpha(x_1), h] + [x, \alpha(h)] = [y, h],$$

which implies that $y \in Z(H)$, a contradiction.

Case 3. $z \notin H$ and $[z, L] \subseteq Z(H)$. Let $\{x_1 = z, x_2, \ldots, x_m\}$ be a minimal generating set of L such that $\{x_2, \ldots, x_m\}$ is a minimal generating set for H. Clearly, $[z, L] \subseteq Z(H) \cap L^2 \subset L^2$. Since $\dim[z, L] = m-1$ and $\dim L^2 = m$, we have $[z, L] = Z(H) \cap L^2 = Z(H) \cap H^2$. Hence, to prove $[z, L] = Z(H)$ it is sufficient to show that $Z(H) \subseteq H^2$. Suppose on the contrary that $Z(H) \nsubseteq H^2$. Then, there exists a non-trivial element $u \in Z(H) \setminus H^2$. Since $[z, L] \subset L^2$ and $z \notin H$, it follows that for this $u \in H \setminus H^2$ we have $[u, L] = L^2$. Therefore

$$3 \leq m = \dim L^2 = \dim[u, L] = \dim[u, \langle z \rangle + H] = 1,$$

which is a contradiction. Hence $Z(H) \subseteq H^2$.

If $C_L(z) + L^2 \neq \langle z \rangle + L^2$, then there exists an element $y \in H \setminus L^2$ such that $[y, z] = 0$. Since $[z, L] \subset L^2$ and $z \notin H$, by Lemma 2.9, it follows that $[y, L] = L^2$. Since $y \in H$, we have $Z(H) \subseteq C_H(y) \subseteq C_L(y)$. Also, $y, z \in C_L(y)$ so that $Z(H) + \langle y, z \rangle \subseteq C_L(y)$. Thus $\dim C_L(y) \geq m+1$ and hence

$$\dim L^2 = \dim[y, L] = \dim \frac{L}{C_L(y)} \leq m-1,$$

which is a contradiction. So $C_L(z) + L^2 = \langle z \rangle + L^2$. Since $\dim C_L(z) = m+1 = \dim(\langle z \rangle + L^2)$, it follows that $C_L(z) = \langle z \rangle + L^2$. Now we have $L = C_L(z) + H$ and so $[z, H] = [z, L] = Z(H)$. Set $M = [z, H]$. Since $[z, L^2] = [z, H^2] = 0$, the map $\beta : H/L^2 \to M$ defined by $\beta(h + L^2) = [z, h]$ is a well-defined Lie epimorphism. Since H/L^2 and M have dimension $m-1$, β is a Lie isomorphism and M is abelian with basis $\beta(x_i) = [z, x_i]$ for $2 \le i \le m$. If $y \in H \setminus L^2$, then $L^2 = [y, L] = [y, \langle z \rangle + H]$. Since $[y, z] \in M$, H/M is a Camina Lie algebra. Also, $\dim(H^2/M) = \dim(L^2/M) = 1$. Therefore, H/M is an m-dimensional Heisenberg Lie algebra and so m is odd. Since $H/M = H/Z(H)$ has nilindex 2, it follows that H is nilpotent with nilindex 3 and hence, by Lemma 2.4, L has the nilindex 3.

Suppose that $m > 3$. Then we may suppose that $[x_2, x_3] + M$ spans H^2/M and that $[x_2, x_i], [x_3, x_i] \in M$ for all $4 \le i \le m$. Utilizing the Jacobi identity, we have

$$[[x_2, x_3], x_i] + [[x_3, x_i], x_2] + [[x_i, x_2], x_3] = 0, \ \forall 4 \le i \le m.$$

Since $[x_2, x_i]$ and $[x_3, x_i]$ belong to $M = Z(H)$, we have $[[x_i, x_2], x_3] = [[x_3, x_i], x_2] = 0$. Therefore $[[x_2, x_3], x_i] = 0$, which implies that $[H^2, x_i] = 0$ for all $4 \le i \le m$. Since $m \ge 5$, we may assume (by re-indexing the generators if necessary) that H^2/M is spanned by $[x_4, x_5] + M$. Since $[x_i, x_j] \in M$ for $i = 2, 3$ and $j = 4, 5$, a similar argument shows that $[H^2, x_2] = [H^2, x_3] = 0$. Hence

$$H^2 \subseteq Z(\langle x_2, x_3, \ldots, x_m \rangle + L^2) = Z(H),$$

which is impossible for H has nilindex 3. Hence $m = 3$.

Now, we have $\dim M = 2$ and $\dim H^2 = 3$. Hence $\dim H = 5$ and consequently $\dim L = 6$. The only 6-dimensional almost Camina Lie algebras with $\dim L^2 = 3$ are $L_{6,1}^{(2)}$, $L_{6,24}(\varepsilon)$ and $L_{6,8}^{(2)}(\eta)$, where ε and η are non-zero (see [1]). A simple verification shows that an element of $\mathrm{Der}_c(L_{6,1}^{(2)})$ has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 & \alpha_{1,5} & \alpha_{1,6} \\ 0 & 0 & \alpha_{2,3} & 0 & \alpha_{2,5} & \alpha_{2,5} \\ 0 & 0 & 0 & 0 & \alpha_{2,3} & \alpha_{2,5} \\ 0 & 0 & 0 & 0 & -\alpha_{1,3} & \alpha_{4,6} \\ 0 & 0 & 0 & 0 & 0 & \alpha_{2,3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and for Lie algebras $L_{6,24}(\varepsilon)$ and $L_{6,8}^{(2)}(\eta)$, an arbitrary element of $\mathrm{Der}_c(L)$ has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 & \alpha_{1,5} & \alpha_{1,6} \\ 0 & 0 & \alpha_{2,3} & 0 & \alpha_{2,5} & \alpha_{2,6} \\ 0 & 0 & 0 & 0 & \alpha_{2,3} & -\alpha_{1,3} \\ 0 & 0 & 0 & 0 & \alpha_{4,5} & \alpha_{4,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the equality in Theorem A holds for Lie algebras $L_{6,24}(\varepsilon)$ and $L^{(2)}_{6,8}(\eta)$, as required. $\square$

Remark 4.3. Consider the Lie algebras $L_{3,2}$, $L_{4,3}$ and $L_{6,22}(\varepsilon)$, where $\varepsilon \neq 0$ (see [1, 2]). A simple verification shows that elements of $\mathrm{Der}_c(L)$ have the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} \\ 0 & 0 & \alpha_{2,3} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \alpha_{1,3} & \alpha_{1,4} \\ 0 & 0 & \alpha_{2,3} & 0 \\ 0 & 0 & 0 & \alpha_{2,3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha_{1,5} & \alpha_{1,6} \\ 0 & 0 & 0 & 0 & \alpha_{2,5} & \alpha_{2,6} \\ 0 & 0 & 0 & 0 & \alpha_{3,5} & \alpha_{3,6} \\ 0 & 0 & 0 & 0 & \alpha_{4,5} & \alpha_{4,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $L = L_{3,2}$, $L_{4,3}$ or $L_{6,22}(\varepsilon)$, respectively. Clearly, the equality holds in Theorem A for such Lie algebras.

Now we prove Theorem B.

Proof of Theorem B. Suppose the equality holds in Theorem A. Clearly, the equality in Theorem A implies the equality in (3.1) when n is odd while this is not always true when n is even. Hence, we consider two cases.

Case 1. Equality holds in (3.1). Then, by Theorem 3.3, L is a Camina Lie algebra with nilindex 2. Now, from (3.2), it follows that $m \leq n/3$, where $\dim L^2 = m$.

First suppose that n is even. Hence, the equality holds in Theorem A only if $m = (n-2)/2$ or $m = (n+2)/2$. Since $(n+2)/2 \nleq n/3$ we must have $m = (n-2)/2$. Since $m \leq n/3$, one gets $n \leq 6$. Therefore, $n = 4$ or 6.

If $n = 4$, then there exists no Camina Lie algebra with nilindex 2, because the only 4-dimensional nilpotent Lie algebra with nilindex 2 is $L_{4,2}$ (see [1]), which is not a Camina Lie algebra.

If $n = 6$, then among 6-dimensional nilpotent Lie algebras (see [1]), the only Camina Lie algebra with nilindex 2 is $L_{6,22}(\varepsilon)$, where $\varepsilon \neq 0$.

Next suppose that n is odd. Then the equality holds in Theorem A only if $m = (n-1)/2$ or $m = (n+1)/2$. The value $m = (n+1)/2$ contradicts $m \leq n/3$. So the only possibility is $m = (n-1)/2$. Therefore $(n-1)/2 \leq n/3$, which implies that $n \leq 3$. Since L is a non-abelian nilpotent Lie algebra, $n = 3$ and hence L is a 3-dimensional Heisenberg Lie algebra.

Case 2. Equality does not hold in (3.1). Since the equality in Theorem A implies the equality in (3.1) when n is odd, n must be even and $\dim L^2 = \frac{n}{2}$. Therefore, $\dim L = 2m$ and $\dim L^2 = m$, where $m \geq 2$. Now, by Lemma 4.1, it follows that L is an almost Camina Lie algebra and, by Proposition 4.2, either $L = L_{4,3}$, $L = L_{6,24}(\varepsilon)$ with $\varepsilon \neq 0$ or $L = L^{(2)}_{6,8}(\eta)$ with $\eta \neq 0$.

The converse is clear.

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