Title:
Application of Hopf’s lemma on contact CR-warped product submanifolds of a nearly Kenmotsu manifold

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APPLICATION OF HOPF’S LEMMA ON CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD

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Abstract. In this paper we consider contact CR-warped product submanifolds of the type $M = N_T \times f N_L$, of a nearly Kenmotsu generalized Sasakian space form $M(f_1, f_2, f_3)$ and by use of Hopf’s Lemma we show that $M$ is simply contact CR-product under certain condition. Finally, we establish a sharp inequality for squared norm of the second fundamental form and equality case is discussed. The results in this paper generalize existing results in the literature.

Keywords: Warped product, CR-submanifolds, nearly Kenmostsu manifold.


1. Introduction

The notion of CR-warped product submanifolds as a natural generalization of CR-products was introduced by B. Y. Chen (see [9,11]). Basically, Chen obtained some basic results for CR-warped product submanifolds of Kaehler manifolds and established a sharp relation ship between the warping function $f$ and squared norm of the second fundamental form. Later, I. Hesigawa and I. Mihai proved a similar inequality for contact CR-warped product submanifolds of Sasakian manifolds [12]. Moreover, I. Mihai in [16] improved same inequality for contact CR-warped product submanifolds of Sasakian space form.

Furthermore, in [2] K. Arslan et al. obtained a sharp estimation for contact CR-warped product submanifolds in the setting of Kenmotsu space form. Many geometers obtained similar estimation for different setting of almost contact metric manifolds (see references).
In the present study, we consider contact CR-warped product submanifolds of a nearly Kenmotsu generalized Sasakian space form and obtained a characterizing inequality for existence of contact CR-warped product submanifolds. Finally, we also obtained a sharp inequality for squared norm of the second fundamental form in terms of warping function. The results in this paper generalize the results of the papers (see [2, 4]).

2. Preliminaries

A $(2n + 1)$-dimensional $C^\infty$-manifold $\tilde{M}$ is said to have an almost contact structure, if there exist on $\tilde{M}$ a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying [6]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$  (2.1)

There always exists a Riemannian metric $g$ on an almost contact metric manifold $\tilde{M}$ satisfying the following conditions

$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$  (2.2)

for all $X, Y \in T\tilde{M}$.

An almost contact structure $(\phi, \xi, \eta)$ is said to be normal if the almost complex structure $J$ on the product manifold $\tilde{M} \times R$ given by

$$J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt}),$$

where $f$ is a $C^\infty$-function on $\tilde{M} \times R$, has no torsion, that is $J$ is integrable and the condition for normality in terms of $\phi, \xi$ and $\eta$ is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\tilde{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. Finally, the fundamental 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric manifold is said to be Kenmotsu manifold if [2]

$$\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for all $X, Y \in T\tilde{M}$.

An almost contact metric manifold is said to be nearly Kenmotsu manifold if [17]

$$\nabla_X\phi)Y + (\nabla_Y\phi)X = -\eta(Y)\phi X - \eta(X)\phi Y,$$  (2.3)

for all $X, Y \in T\tilde{M}$.

Equation (2.3) is equivalent to

$$\nabla_X\phi)X = -\eta(X)X,$$  (2.4)

for each $X \in T\tilde{M}$. 
Given an almost contact metric manifold $\tilde{M}$, it is said to be a generalized Sasakian space form [1], if there exist three functions $f_1$, $f_2$ and $f_3$ on $\tilde{M}$ such that
\begin{equation}
\bar{R}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\}
+ 2g(X,\phi Y)\phi Z + f_3\{\eta(X)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},
\end{equation}
for any vector fields $X,Y,Z$ on $\tilde{M}$, where $\bar{R}$ denotes the curvature tensor of $\tilde{M}$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\tilde{M}$ is Sasakian space form [6], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\tilde{M}$ is a Kenmotsu space form [13], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\tilde{M}$ is a cosymplectic space form [1].

Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric $g$, and if $\nabla$ and $\nabla^\perp$ are the induced connection on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively, then the Gauss and Weingarten formulae are given by
\begin{equation}
\nabla_X Y = \nabla_X^\perp Y + h(X,Y),
\end{equation}
\begin{equation}
\nabla_X N = -A_N X + \nabla^\perp_X N,
\end{equation}
for each $X,Y \in TM$ and $N \in T^\perp M$, where $h$ and $A_N$ are the second fundamental form and the shape operator respectively, for the immersion of $M$ in $\tilde{M}$, they are related as
\begin{equation}
g(h(X,Y),N) = g(A_N X,Y),
\end{equation}
where $g$ denotes the Riemannian metric on $\tilde{M}$ as well as on $M$.

The mean curvature vector $H$ of $M$ is given by
\begin{equation}
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i,e_i),
\end{equation}
where $n$ is the dimension of $M$ and $\{e_1,e_2,\ldots,e_n\}$ is a local orthonormal frame of vector fields on $M$. The squared norm of the second fundamental form is defined as
\begin{equation}
\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).
\end{equation}
A submanifold $M$ of $\tilde{M}$ is said to be a totally geodesic submanifold, if $h(X,Y) = 0$, for each $X,Y \in TM$, and totally umbilical submanifold if $h(X,Y) = g(X,Y)H$.

For any $X \in TM$, we write
\begin{equation}
\phi X = PX + FX,
\end{equation}
where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. 

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Similarly, for $N \in T^\perp M$, we can write
\begin{equation}
\phi N = tN + fN,
\end{equation}
where $tN$ and $fN$ are the tangential and normal components of $\phi N$ respectively.

The covariant differentiation of the tensors $\phi$, $P$, $F$, $t$ and $f$ are defined as respectively
\begin{align*}
(\nabla_X \phi)Y &= \nabla_X (\phi Y) - \phi (\nabla_X Y), \\
(\nabla_X P)Y &= \nabla_X (PY) - P\nabla_X Y, \\
(\nabla_X F)Y &= \nabla_X F Y - F\nabla_X Y, \\
(\nabla_X t)N &= \nabla_X tN - t\nabla_X^\perp N, \\
(\nabla_X f)N &= \nabla_X fN - f\nabla_X^\perp N.
\end{align*}
Furthermore, for any $X, Y \in TM$, the tangential and normal parts of $(\nabla_X \phi)Y$ are denoted by $P_X Y$ and $Q_X Y$ i.e.,
\begin{equation}
(\nabla_X \phi)Y = P_X Y + Q_X Y.
\end{equation}
By use of (2.1) and (2.12), it is easy to verify the following property
\begin{equation}
(\nabla_X \phi)\phi Y = -\phi(\nabla_X \phi Y) - \eta(\nabla_X Y)\xi.
\end{equation}
On using equations (2.6)-(2.14) and (2.17), we may obtain that
\begin{align*}
P_X Y &= (\nabla_X P)Y - AP_X Y - t(X, Y), \\
Q_X Y &= (\nabla_X F)Y + h(X, TY) - fh(X, Y).
\end{align*}
Similarly, for $N \in T^\perp M$, denoting by $P_X N$ and $Q_X N$ respectively, the tangential and normal parts of $(\nabla_X \phi)N$, we find that
\begin{align*}
P_X N &= (\nabla_X t)N + PA_N X - A f_N X, \\
Q_X N &= (\nabla_X f)N + h(tN, X) + FA_N X.
\end{align*}
On a submanifold $M$ of a nearly Kenmotsu manifold by (2.3) and (2.17)
\begin{align*}
(a) P_X Y + P_Y X &= -\eta(Y)P X - \eta(X)P Y, \\
(b) Q_X Y + Q_Y X &= -\eta(Y)F X - \eta(X)F Y,
\end{align*}
for any $X, Y \in TM$.

An $m$-dimensional Riemannian submanifold $M$ of an almost contact metric manifold $\tilde{M}$, where $\xi$ is tangent to $M$, is called contact CR-submanifold, if it admits an invariant distribution $D$ whose orthogonal complementary distribution $D^\perp$ is anti invariant, that is,
\[
TM = D \oplus D^\perp \oplus \langle \xi \rangle,
\]
where $\phi D \subseteq D$ and $\phi D^\perp \subseteq T^\perp M$ and $\langle \xi \rangle$ denotes 1-dimensional distribution which is spanned by $\xi$. 

If $\mu$ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of contact CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$ T^\perp M = \mu \oplus \phi D^\perp. $$

A contact CR-submanifold $M$ is called contact CR-product if the distribution $D$ and $D^\perp$ are parallel on $M$. In this case $M$ is foliated by the leaves of these distributions. In general, if $N_1$ and $N_2$ are Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$ respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric $g$ defined as

$$ g(X, Y) = g_1(d\pi_1 X, d\pi_1 Y) + g_2(d\pi_2 X, d\pi_2), $$

where $\pi_1$ and $\pi_2$ are the projection maps of $M$ onto $N_1$ and $N_2$, respectively, and $d\pi_1$ and $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of contact CR-product submanifold, one can consider warped product of manifolds which are defined as follows

**Definition 2.1.** Let $(B, g_B)$ and $(C, g_C)$ be two Riemannian manifolds with Riemannian metric $g_B$ and $g_C$ respectively, and $f$ be a positive differentiable function on $B$. The warped product of $B$ and $C$ is the Riemannian manifold $(B \times C, g)$, where

$$ g = g_B + f^2 g_C. $$

For a warped product manifold $N_1 \times_f N_2$, we denote by $D_1$ and $D_2$ the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, $D_1$ is obtained by the tangent vectors of $N_1$ via the horizontal lift, and $D_2$ is obtained by the tangent vectors of $N_2$ via vertical lift. In case of contact CR-warped product submanifolds $D_1$ and $D_2$ are replaced by $D$ and $D^\perp$ respectively.

The warped product manifold $(B \times C, g)$ is denoted by $B \times_f C$. If $X$ is the tangent vector field to $M = B \times_f C$ at $(p, q)$ then

$$ ||X||^2 = ||d\pi_1 X||^2 + f^2(p)||d\pi_2 X||^2. $$


**Theorem 2.2.** Let $M = B \times_f C$ be warped product manifolds. If $X, Y \in TB$ and $V, W \in TC$ then

(i) $\nabla_X Y \in TB$,

(ii) $\nabla_X V = \nabla_V X = \left(\frac{X}{f}\right)V$,

(iii) $\nabla_V W = -\frac{g(V, W)}{f} \nabla f$.

From above Theorem, for the warped product $M = B \times_f C$ it is easy to conclude that

$$ \nabla_X V = \nabla_V X = (X \ln f)V, $$

(2.27)
for any $X \in TB$ and $V \in TC$.
\(\nabla f\) is the gradient of $f$ and is defined as
\[
(2.28) \quad g(\nabla f, X) = Xf,
\]
for all $X \in TM$.

**Corollary 2.3.** On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold

(i) $N_1$ is totally geodesic in $M$,
(ii) $N_2$ is totally umbilical in $M$.

In what follows, $N_\perp$ and $N_T$ will denote an anti-invariant and invariant submanifold respectively, of an almost contact metric manifold $\tilde{M}$.

A warped product manifold is said to be trivial if its warping function $f$ is constant. More generally, a trivial warped product manifold $M = N_1 \times N_2$ is a Riemannian product $N_1 \times N_2^f$, where $N_2^f$ is the manifold with the Riemannian metric $f^2g_2$ which is homothetic to the original metric $g_2$ of $N_2$. For example, a trivial contact CR-warped product is contact CR-product.

Let $M$ be a $m$-dimensional Riemannian manifold with Riemannian metric $g$ and let $\{e_1, \ldots, e_m\}$ be an orthogonal basis of $TM$. As a consequence of (2.28), we have
\[
(2.29) \quad \|\nabla f\|^2 = \sum_{i=1}^{m} (e_i(f))^2.
\]

The Laplacian of $f$ is defined by
\[
(2.30) \quad \Delta f = \sum_{i=1}^{m} \{f(e_i)e_i - e_i e_i f\}.
\]

Now, we state the Hopf’s Lemma.

**Hopf’s Lemma** [10]. Let $M$ be a $n$-dimensional compact Riemannian manifold. If $\psi$ is differentiable function on $M$ such that $\Delta \psi \geq 0$ everywhere on $M$ (or $\Delta \psi \leq 0$ everywhere on $M$), then $\psi$ is a constant function.

### 3. Contact CR-warped product submanifolds

In this section we consider contact CR-warped product of the type $N_T \times_f N_\perp$ of the nearly Kenmotsu manifolds $\tilde{M}$, where $N_T$ and $N_\perp$ are the invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. Throughout this section, we consider $\xi$ tangent to $N_T$.

Now we obtain some basic results in the following Lemma.

**Lemma 3.1.** Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then

(i) $\xi \ln f = 1$,
(ii) $g(h(X, Z), \phi Z) = -\phi X \ln f \|Z\|^2$, 

\[(iii) \quad g(h(\phi X, Z), \phi Z) = X \ln f \|Z\|^2,\]

for any \(X \in TN_T\) and \(Z \in TN_\perp\).

Proof. Parts (i) and (ii) are the special cases of [17, Lemma 3.1]. Applying equations (2.6), (2.5) and (2.27) as follows

\[g(h(\phi X, Z), \phi Z) = g(\nabla_Z \phi X, \phi Z) = -g(\nabla_Z \phi Z, \phi X),\]

or

\[g(h(\phi X, Z) = X \ln fZ.\]

Now we will prove the following Lemma which is useful in the proof of our main theorem

**Lemma 3.2.** Let \(M = N_T \times f N_\perp\) be a contact CR-warped product submanifold of a nearly Kenmotsu manifold \(M\). Then

\[g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), Q_X Z),\]

for any \(X \in TN_T\) and \(Z \in TN_\perp\).

Proof. By (2.6) and (2.12)

\[h(\phi X, Z) = (\nabla_Z \phi)X + \phi \nabla_Z X + \phi h(X, Z) - \nabla_Z \phi X.\]

Thus by using (2.17) and (2.27)

\[h(\phi X, Z) = P_Z X + Q_Z X + X \ln f \phi Z + \phi h(X, Z) - \phi X \ln f Z.\]

Comparing normal parts

\[h(\phi X, Z) = Q_Z X + X \ln f \phi Z + \phi h_\mu(X, Z),\]

or

\[g(h(\phi X, Z), \phi h(X, Z)) = g(Q_Z X, \phi h(X, Z)) + \|h_\mu(X, Z)\|^2.\]

By using (2.23)(b), we get

\[g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), Q_X Z).\]

Now we prove the following characterization theorem

**Theorem 3.3.** Let \(M = N_T \times f N_\perp\) be a contact CR-warped product submanifold of nearly Kenmotsu generalized Sasakian space form \(\tilde{M}(f_1, f_2, f_3)\) such that \(N_T\) is compact. Then \(M\) is contact CR-product submanifold if either one of the following inequality holds

\[\text{(i) } \sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e_j)\|^2 \geq 2 \cdot p \cdot q \cdot f_2 + \sum_{i=1}^{2p} \sum_{j=1}^{q} |Q_{e_i} e_j|^2,\]
For any unit vector fields $\mathbf{e}_i, \mathbf{e}_j$ and $\mathbf{h}_\mu$ denotes the component of $\mathbf{h}$ in $\mu$, $2p+1$ and $q$ are the dimensions of $N_T$ and $N_\bot$.

**Proof.** For any unit vector fields $X$ tangent to $N_T$ and orthogonal to $\xi$, and $Z$ tangent to $N_\bot$. Then from (2.5) we have

$$
R(X, \phi X, Z, \phi Z) = -2.f_2.g(X, X) g(Z, Z).
$$

On the other hand by Coddazi equation

$$
R(X, \phi X, Z, \phi Z) = g(\nabla^i X, \phi X, Z, h) - g(h, \nabla X \phi X, Z, \phi Z)
$$

$$
- g(h, \phi X, \nabla Z, \phi Z) - g(\nabla^i X, h, X, Z, \phi Z)
$$

$$
+ g(h, \nabla \phi X, X, Z, \phi Z) + g(h, X, \nabla \phi X, Z, \phi Z).
$$

By using part (iii) of Lemma 3.3, (2.12), (2.6) and (2.17), we get

$$
g(\nabla^i X, \phi X, Z, \phi Z) = Xg(h, \phi X, Z, \phi Z) - g(h, \phi X, Z, \nabla X \phi Z)
$$

$$= Xg(\nabla X g(Z, Z)) - g(h, \phi X, Z, \nabla X) + g(h, \phi X, Z, \phi Z).
$$

On further simplification above equation yields

$$
g(\nabla^i X, \phi X, Z, \phi Z) = X^2\ln f g(Z, Z) + 2(X \ln f)^2 g(Z, Z) - g(h, \phi X, Z, Q_X Z)
$$

$$- g(h, \phi X, Z, \phi h(X, Z)) - g(h, \phi X, Z, \phi Z).
$$

By using Lemma 3.2, we have

$$
(\nabla^i X, \phi X, Z, \phi Z) = X^2\ln f g(Z, Z) + (X \ln f)^2 g(Z, Z) - \|h_\mu(X, Z)\|^2
$$

$$- g(h, \phi h(X, Z), \phi h(X, Z), Q_X Z).
$$

Further, using (2.6), (2.17), (2.23)(b) and (2.27) in the last term of above equation, we get

$$
g(\nabla^i X, \phi X, Z, \phi Z) = X^2\ln f g(Z, Z) + (X \ln f)^2 g(Z, Z)
$$

$$- \|h_\mu(X, Z)\|^2 + \|Q_X Z\|^2.
$$

Similarly, we can calculate

$$
- g(\nabla^i X, \phi X, Z, \phi Z) = (\phi X)^2\ln f g(Z, Z) + (\phi X \ln f)^2 g(Z, Z)
$$

$$- \|h_\mu(\phi X, Z)\|^2 + \|Q_{\phi X} Z\|^2.
$$

From part (iii) of Lemma 3.1, we have

$$
g(A_{\phi Z} X, \phi X) = X \ln f,
$$

replacing $X$ by $\nabla X$

$$
g(A_{\phi Z} X, \phi \nabla X) = \nabla X X \ln f.
$$

By using the Gauss formula in last equation, we get

$$
g(A_{\phi Z} X, \phi (\nabla X X - h(X, X)) = \nabla X X \ln f.
$$
By use of (2.6), (2.12), (2.4) and (2.27), it is easy to see that $h(X, X) \in \mu$, applying this fact in (3.5), we get
$$g(A\phi Z Z, \nabla_X \phi X - (\nabla_X \phi) X) = \nabla_X X \ln f.$$
In view of (2.4), the above equation reduced to
$$g(h(\nabla_X \phi X, Z), \phi Z) = \nabla_X X \ln f g(Z, Z).$$

Similarly,
$$g(h(\nabla_X X, Z), \phi Z) = -\nabla_X \phi X \ln f g(Z, Z).$$

By use of (2.27) and part (iii) of Lemma 3.1, it is easy to see the following
$$g(h(\phi X, Z), \phi Z) = (X \ln f)^2 g(Z, Z)$$
and
$$g(h(X, \nabla_X Z), \phi Z) = -(\phi X \ln f)^2 g(Z, Z).$$

Substituting (3.3), (3.4), (3.6), (3.7), (3.8) and (3.9) in (3.2), we find
$$\nabla^2 (X; \phi_X; Z; \phi_Z) = \frac{X \ln f}{2} g(Z, Z) + \frac{\phi_X}{2} \ln f g(Z, Z) \nabla X \phi X g(Z, Z)$$
and
$$\frac{\phi_X}{2} \ln f g(Z, Z) \nabla_\phi X g(Z, Z) + \frac{\phi_X}{2} \ln f g(Z, Z) \parallel h(X, Z) \parallel^2.$$
Corollary 3.4. Let $M = N_T \times f N_\perp$ be a contact CR-warped product submanifolds of a nearly Kenmotsu generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$, such that $N_T$ is compact. Then $M$ is contact CR-product if and only if,

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e_j)\|^2 = 2pqf_2 + \sum_{i=1}^{2p} \sum_{j=1}^{q} \|Qe_i e_j\|^2.$$  

Moreover, if the ambient manifold $\tilde{M}$ is a Kenmotsu space form, then from above findings we have the following.

Corollary 3.5. Let $M = N_T \times f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu space form $\tilde{M}(c)$ such that $N_T$ is compact. Then $M$ is contact CR-product submanifold if either one of the inequality

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e_j)\|^2 \geq \frac{c+1}{2}pq,$$

or

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e_j)\|^2 \leq \frac{c+1}{2}pq,$$

holds, where $h_\mu$ denotes the component of $h$ in $\mu$, $2p+1$ and $q$ are the dimensions of $N_T$ and $N_\perp$, respectively.

Corollary 3.6. Let $M = N_T \times f N_\perp$ be a contact CR-warped product submanifolds of a Kenmotsu space form $\tilde{M}(c)$ such that $N_T$ is compact. Then $M$ is contact CR-product if and only if

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e_j)\|^2 = \frac{c+1}{2}pq.$$  

4. Another inequality

In the present section, we estimate the squared norm of the second fundamental form in terms of warping function.

Theorem 4.1. Let $\tilde{M}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional nearly Kenmotsu generalized Sasakian space form and $M = N_T \times f N_\perp$ be an $m$-dimensional contact CR-warped product submanifold, such that $N_1$ is $(2p+1)$-dimensional invariant submanifold tangent to $\xi$ and $N_\perp$ be a $q$-dimensional anti-invariant submanifold of $\tilde{M}(f_1, f_2, f_3)$. Then

(i) The squared norm of the second fundamental form $h$ satisfies

$$\|h\|^2 \geq q\|\nabla \ln f\|^2 - \Delta \ln f - 1 + 2pqf_2 + \|QD D^\perp\|^2,$$

where $\Delta$ denotes the Laplace operator on $N_T$.  


(ii) The equality sign of (4.1) holds identically if and only if we have

(a) $N_T$ is totally geodesic invariant submanifold of $\tilde{M}(f_1, f_2, f_3)$. Hence $N_T$ is a nearly Kenmotsu generalized Sasakian space form.

(b) $N_\perp$ is a totally umbilical anti-invariant submanifold of $\tilde{M}(f_1, f_2, f_3)$.

Proof. For any $X \in TN_T$ and $Z \in TN_\perp$, from (2.9)(b) and parts (ii) and (iii) of Lemma 3.1, we have

$$g(h(\xi, Z), \phi Z) = 0,$$

and

$$g(h(\phi X, Z), \phi Z) = X \ln f \|Z\|^2.$$ 

Since $\ln f = 1$, then combining this with above two equations, we have

$$\sum_{i=0}^{2p} \sum_{j=1}^q \|h_{\phi D} (e_i, e_j)\|^2 = q \|\nabla \ln f\|^2 - 1.$$ 

We use the following notation

$$\sum_{i=1}^{2p} \sum_{j=1}^q \|\phi e_i e_j\|^2 = \|\phi D D\|^2.$$ 

Substituting above notation in (4.3) and combining it with (4.2), we obtain the inequality (4.1).

Let $h''$ be the second fundamental form of $N_\perp$ in $M$. Then, we have

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -X \ln f g(Z, W),$$

on using (2.28), we get

$$h''(Z, W) = -g(Z, W) \nabla \ln f.$$ 

If the equality sign of (4.1) holds identically, then we obtain

$$h(D, D) = 0, \ h(D^\perp, D^\perp) = 0.$$ 

The first condition of (4.5) implies that $N_T$ is totally geodesic in $M$. On the other hand, one has

$$g(h(X, \phi Y), \phi Z) = g(\phi_X \phi Y, \phi Z) = -g(\phi Y, (\phi_X \phi)Z).$$
By use of (2.12) and (2.6) we get the following equation
\[ g(\phi Y, (\nabla_Z \phi)X) = g(\phi Y, \nabla_Z \phi X) - g(Y, \nabla_Z X), \]
in view of (2.27) the above equation reduced to
(4.7) \[ g(\phi Y, (\nabla_Z \phi)X) = 0. \]
From (4.6), (4.7) and (2.23)(a) we have
(4.8) \[ g(h(X, \phi Y), \phi Z) = -g(\phi Y, (\nabla_X \phi)Z + (\nabla_Z \phi)X) = 0. \]
From (4.8), it is evident that \( N_T \) is totally geodesic in \( \tilde{M}(f_1, f_2, f_3) \) and hence is a nearly Kenmotsu generalized Sasakian space form.
The second condition of (4.5) and (4.4) imply that \( N_\perp \) is totally umbilical in \( \tilde{M}(f_1, f_2, f_3) \).

In the last we have the following corollary which can be deduced from inequality (4.1).

**Corollary 4.2.** Let \( M = N_T \times_f N_\perp \) be a contact CR-warped product submanifold of a Kenmotsu space form \( \tilde{M}(c) \), then squared norm of the second fundamental form satisfies
\[ \|h\|^2 \geq q\|\nabla \ln f\|^2 - \Delta \ln f - 1 \] + \( \frac{c+1}{2} \) \( p \cdot q \)
where \( \Delta \) is the Laplace operator on \( N_T \), and \( 2p+1 \) and \( q \) are the dimensions of \( N_T \) and \( N_\perp \) respectively.

**Remark 4.3.** Inequality (4.1) is the generalization of the inequality obtained in [2].

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