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APPLICATION OF HOPF'S LEMMA ON CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD

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Abstract. In this paper we consider contact CR-warped product submanifolds of the type $M = N_T \times_f N_\perp$, of a nearly Kenmotsu generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ and by use of Hopf's Lemma we show that M is simply contact CR-product under certain condition. Finally, we establish a sharp inequality for squared norm of the second fundamental form and equality case is discussed. The results in this paper generalize existing results in the literature.

Keywords: Warped product, CR-submanifolds, nearly Kenmostsu manifold.
MSC(2010): Primary: 53C25; Secondary: 53C40, 53C42, 53D15.

1. Introduction

The notion of CR-warped product submanifolds as a natural generalization of CR-products was introduced by B. Y. Chen (see [9, 11]). Basically, Chen obtained some basic results for CR-warped product submanifolds of Kaehler manifolds and established a sharp relation ship between the warping function f and squared norm of the second fundamental form. Later, I. Hesigawa and I. Mihai proved a similar inequality for contact CR-warped product submanifolds of Sasakian manifolds [12]. Moreover, I. Mihai in [16] improved same inequality for contact CR-warped product submanifolds of Sasakian space form.

Furthermore, in [2] K. Arslan et al. obtained a sharp estimation for contact CR-warped product submanifolds in the setting of Kenmotsu space form. Many geometers obtained similar estimation for different setting of almost contact metric manifolds (see references).

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In the present study, we consider contact CR-warped product submanifolds of a nearly Kenmotsu generalized Sasakian space form and obtained a characterizing inequality for existence of contact CR-warped product submanifolds. Finally, we also obtained a sharp inequality for squared norm of the second fundamental form in terms of warping function. The results in this paper generalize the results of the papers (see [2, 4]).

2. Preliminaries

A $(2n+1)-$dimensional $C^\infty-$manifold $\bar{M}$ is said to have an almost contact structure, if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying [6]

$$\phi^2 = -I + \eta \oplus \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$ satisfying the following conditions

$$\eta(X) = g(X,\xi),\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(X), \tag{2.2}$$

for all $X, Y \in T\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a $C^\infty-$function on $\bar{M} \times R$, has no torsion, that is J is integrable and the condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the *fundamental 2-form* Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$.

An almost contact metric manifold is said to be Kenmotsu manifold if [2]

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for all $X, Y \in T\bar{M}$.

An almost contact metric manifold is said to be nearly Kenmotsu manifold if [17]

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{2.3}$$

for all $X, Y \in T\bar{M}$.

Equation (2.3) is equivalent to

$$(\bar{\nabla}_X \phi)X = -\eta(X)X, \tag{2.4}$$

for each $X \in T\bar{M}$.

Given an almost contact metric manifold $\bar{M}$, it is said to be a generalized Sasakian space form [1], if there exist three functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$
\begin{aligned}
(2.5)\quad \bar{R}(X,Y)Z \;=\;& f_1\{g(Y,Z)X - g(X,Z)Y\} \\
&+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\
&+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\
&- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},
\end{aligned}
$$

for any vector fields X, Y, Z on $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}$ is Sasakian space form [6], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}$ is a Kenmotsu space form [13], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}$ is a cosymplectic space form [1].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g, and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then the Gauss and Weingarten formulae are given by

$$(2.6)\qquad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$(2.7)\qquad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator respectively, for the immersion of M in $\bar{M}$, they are related as

$$(2.8)\qquad g(h(X,Y),N) = g(A_N X, Y),$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.
The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where n is the dimension of M and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form is defined as

$$(2.9)\qquad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)).$$

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold, if $h(X,Y) = 0$, for each $X, Y \in TM$, and totally umbilical submanifold if $h(X,Y) = g(X,Y)H$.

For any $X \in TM$, we write

$$(2.10)\qquad \phi X = PX + FX,$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for $N \in T^{\perp}M$, we can write

$$\phi N = tN + fN, \tag{2.11}$$

where tN and fN are the tangential and normal components of ϕN respectively.

The covariant differentiation of the tensors ϕ, P, F, t and f are defined as respectively

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \tag{2.12}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.13}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.14}$$

$$(\bar{\nabla}_X t)N = \nabla_X tN - t\nabla^{\perp}_X N, \tag{2.15}$$

$$(\bar{\nabla}_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N. \tag{2.16}$$

Furthermore, for any $X, Y \in TM$, the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$ are denoted by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ i.e.,

$$(\bar{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y. \tag{2.17}$$

By use of (2.1) and (2.12), it is easy to verify the following property

$$(\bar{\nabla}_X \phi)\phi Y = -\phi(\bar{\nabla}_X)\phi Y) - \eta(\nabla_X Y)\xi. \tag{2.18}$$

On using equations (2.6)-(2.14) and (2.17), we may obtain that

$$\mathcal{P}_X Y = (\bar{\nabla}_X P)Y - A_{FY}X - th(X,Y), \tag{2.19}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X,TY) - fh(X,Y). \tag{2.20}$$

Similarly, for $N \in T^{\perp}M$, denoting by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively, the tangential and normal parts of $(\bar{\nabla}_X \phi)N$, we find that

$$\mathcal{P}_X N = (\bar{\nabla}_X t)N + PA_N X - A_{fN}X, \tag{2.21}$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X f)N + h(tN,X) + FA_N X. \tag{2.22}$$

On a submanifold M of a nearly Kenmotsu manifold by (2.3) and (2.17)

(2.23)

$$(a)\mathcal{P}_X Y + \mathcal{P}_Y X = -\eta(Y)PX - \eta(X)PY, (b)\mathcal{Q}_X Y + \mathcal{Q}_Y X = -\eta(Y)FX - \eta(X)FY,$$

for any $X, Y \in TM$.

An m-dimensional Riemannian submanifold M of an almost contact metric manifold $\bar{M}$, where ξ is tangent to M, is called *contact CR-submanifold*, if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti invariant, that is,

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle,$$

where $\phi D \subseteq D$ and $\phi D^{\perp} \subseteq T^{\perp}M$ and $\langle \xi \rangle$ denotes 1-dimensional distribution which is spanned by ξ.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \mu \oplus \phi D^{\perp}. \tag{2.24}$$

A contact CR-submanifold M is called contact CR-product if the distribution D and $D^{\perp}$ are parallel on M. In this case M is foliated by the leaves of these distributions. In general, if N_1 and N_2 are Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric g defined as

$$g(X, Y) = g_1(d\pi_1 X, d\pi_1 Y) + g_2(d\pi_2 X, d\pi_2), \tag{2.25}$$

where π_1 and π_2 are the projection maps of M onto N_1 and N_2, respectively, and $d\pi_1$ and $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of contact CR-product submanifold, one can consider warped product of manifolds which are defined as follows

Definition 2.1. Let (B, g_B) and (C, g_C) be two Riemannian manifolds with Riemannian metric g_B and g_C respectively, and f be a positive differentiable function on B. The warped product of B and C is the Riemannian manifold $(B \times C, g)$, where

$$g = g_B + f^2 g_C.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift, and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In case of contact CR-warped product submanifolds D_1 and D_2 are replaced by D and $D^{\perp}$ respectively.

The warped product manifold $(B \times C, g)$ is denoted by $B \times_f C$. If X is the tangent vector field to $M = B \times_f C$ at (p, q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2. \tag{2.26}$$

R. L. Bishop and B. O'Neill [5] proved the following ¿

Theorem 2.2. *Let $M = B \times_f C$ be warped product manifolds. If $X, Y \in TB$ and $V, W \in TC$ then*

(i) $\nabla_X Y \in TB$,
(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,
(iii) $\nabla_V W = \frac{-g(V,W)}{f} \nabla f$.

From above Theorem, for the warped product $M = B \times_f C$ it is easy to conclude that

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{2.27}$$

for any $X \in TB$ and $V \in TC$.
∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf, \tag{2.28}$$

for all $X \in TM$.

Corollary 2.3. *On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold*

(i) N_1 *is totally geodesic in* M,
(ii) N_2 *is totally umbilical in* M.

In what follows, $N_\perp$ and N_T will denote an anti-invariant and invariant submanifold respectively, of an almost contact metric manifold $\bar{M}$.

A warped product manifold is said to be trivial if its warping function f is constant. More generally, a trivial warped product manifold $M = N_1 \times N_2$ is a Riemannian product $N_1 \times N_2^f$, where N_2^f is the manifold with the Riemannian metric $f^2 g_2$ which is homothetic to the original metric g_2 of N_2. For example, a trivial contact CR-warped product is contact CR-product.

Let M be a $m-$dimensional Riemannian manifold with Riemannian metric g and let $\{e_1, \ldots, e_m\}$ be an orthogonal basis of TM. As a consequence of (2.28), we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m} (e_i(f))^2. \tag{2.29}$$

The Laplacian of f is defined by

$$\Delta f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i) f - e_i e_i f\}. \tag{2.30}$$

Now, we state the Hopf's Lemma.
Hopf's Lemma [10]. Let M be a n-dimensional compact Riemannian manifold. If ψ is differentiable function on M such that $\Delta\psi \geq 0$ everywhere on M (or $\Delta\psi \leq 0$ everywhere on M), then ψ is a constant function.

3. Contact CR-warped product submanifolds

In this section we consider contact CR-warped product of the type $N_T \times_f N_\perp$ of the nearly Kenmotsu manifolds $\bar{M}$, where N_T and $N_\perp$ are the invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Throughout, this section, we consider ξ tangent to N_T.

Now we obtain some basic results in the following Lemma.

Lemma 3.1. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly Kenmotsu manifold $\bar{M}$. Then*

(i) $\xi \ln f = 1$,
(ii) $g(h(X, Z), \phi Z) = -\phi X \ln f \|Z\|^2$,

(iii) $g(h(\phi X, Z), \phi Z) = X \ln f \|Z\|^2$,

for any $X \in TN_T$ *and* $Z \in TN_\perp$.

Proof. Parts *(i)* and *(ii)* are the special cases of [17, Lemma 3.1.]. Applying equations (2.6), (2.5) and (2.27) as follows

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g(\bar{\nabla}_Z \phi X, \phi Z) = -g(\bar{\nabla}_Z \phi Z, \phi X), \\ &= -g((\bar{\nabla}_Z \phi)Z, \phi X) - g(\nabla_Z Z, X), \end{aligned}$$

or

$$g(h(\phi X, Z) = X \ln f Z.$$

□

Now we will prove the following Lemma which is useful in the proof of our main theorem

Lemma 3.2. *Let* $M = N_T \times_f N_\perp$ *be a contact CR-warped product submanifold of a nearly Kenmotsu manifold* $\bar{M}$. *Then*

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), \mathcal{Q}_X Z),$$

for any $X \in TN_T$ *and* $Z \in TN_\perp$.

Proof. By (2.6) and (2.12)

$$h(\phi X, Z) = (\bar{\nabla}_Z \phi)X + \phi \nabla_Z X + \phi h(X, Z) - \nabla_Z \phi X.$$

Thus by using (2.17) and (2.27)

$$h(\phi X, Z) = \mathcal{P}_Z X + \mathcal{Q}_Z X + X lnf \phi Z + \phi h(X, Z) - \phi X lnf Z.$$

Comparing normal parts

$$h(\phi X, Z) = \mathcal{Q}_Z X + X lnf \phi Z + \phi h_\mu(X, Z),$$

or

$$g(h(\phi X, Z), \phi h(X, Z)) = g(\mathcal{Q}_Z X, \phi h(X, Z)) + \|h_\mu(X, Z)\|^2.$$

By using $(2.23)(b)$, we get

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), \mathcal{Q}_X Z).$$

□

Now we prove the following characterization theorem

Theorem 3.3. *Let* $M = N_T \times_f N_\perp$ *be a contact CR-warped product submanifold of nearly Kenmotsu generalized Sasakian space form* $\bar{M}(f_1, f_2, f_3)$ *such that* N_T *is compact. Then* M *is contact CR-product submanifold if either one of the following inequality holds*

(i) $\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geq 2.p.q.f_2 + \sum_{i=1}^{2p} \sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2)$,

(ii) $\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i,e^j)\|^2 \leq 2.p.q.f_2$,

where h_μ denotes the component of h in μ, $2p+1$ and q are the dimensions of N_T and $N_\perp$.

Proof. For any unit vector fields X tangent to N_T and orthogonal to ξ, and Z tangent to $N_\perp$. Then from (2.5) we have

$$\bar{R}(X,\phi X,Z,\phi Z) = -2.f_2.g(X,X)\ g(Z,Z). \tag{3.1}$$

On the other hand by Coddazi equation

$$\begin{aligned}\bar{R}(X,\phi X,Z,\phi Z) &= g(\nabla^\perp_X h(\phi X,Z),\phi Z) - g(h(\nabla_X\phi X,Z),\phi Z)\\ &\quad -g(h(\phi X,\nabla_X Z),\phi Z) - g(\nabla^\perp_{\phi X}h(X,Z),\phi Z)\\ &\quad +g(h(\nabla_{\phi X}X,Z),\phi Z) + g(h(X,\nabla_{\phi X}Z),\phi Z).\end{aligned} \tag{3.2}$$

By using part (iii) of Lemma 3.3, (2.12), (2.6) and (2.17) , we get

$$\begin{aligned}g(\nabla^\perp_X h(\phi X,Z),\phi Z) &= Xg(h(\phi X,Z),\phi Z) - g(h(\phi X,Z),\bar{\nabla}_X\phi Z)\\ &= X(Xlnfg(Z,Z)) - g(h(\phi X,Z),(\bar{\nabla}_X\phi)Z + \phi\bar{\nabla}_X Z).\end{aligned}$$

On further simplification above equation yields

$$\begin{aligned}g(\nabla^\perp_X h(\phi X,Z),\phi Z) &= X^2lnfg(Z,Z) + 2(Xlnf)^2g(Z,Z) - g(h(\phi X,Z),\mathcal{Q}_X Z)\\ &\quad -g(h(\phi X,Z),\phi h(X,Z)) - Xlnfg(h(\phi X,Z),\phi Z).\end{aligned}$$

By using Lemma 3.2, we have

$$\begin{aligned}(\nabla^\perp_X h(\phi X,Z),\phi Z) &= X^2lnfg(Z,Z) + (Xlnf)^2g(Z,Z) - \|h_\mu(X,Z)\|^2\\ &\quad -g(\phi h(X,Z) - h(\phi X,Z),\mathcal{Q}_X Z).\end{aligned}$$

Further, using (2.6), (2.17), (2.23)(b) and (2.27) in the last term of above equation, we get

$$\begin{aligned}g(\nabla^\perp_X h(\phi X,Z),\phi Z) &= X^2lnfg(Z,Z) + (Xlnf)^2g(Z,Z)\\ &\quad -\|h_\mu(X,Z)\|^2 + \|\mathcal{Q}_X Z\|^2.\end{aligned} \tag{3.3}$$

Similarly, we can calculate

$$\begin{aligned}-g(\nabla^\perp_{\phi X}h(X,Z),\phi Z) &= (\phi X)^2lnfg(Z,Z) + (\phi Xlnf)^2g(Z,Z)\\ &\quad -\|h_\mu(\phi X,Z)\|^2 + \|\mathcal{Q}_{\phi X}Z\|^2.\end{aligned} \tag{3.4}$$

From part (iii) of Lemma 3.1, we have

$$g(A_{\phi Z}Z,\phi X) = Xlnf,$$

replacing X by $\nabla_X X$

$$g(A_{\phi Z}Z,\phi\nabla_X X) = \nabla_X Xlnf.$$

By using the Gauss formula in last equation, we get

$$g(A_{\phi Z}Z,\phi(\bar{\nabla}_X X - h(X,X)) = \nabla_X Xlnf. \tag{3.5}$$

By use of (2.6), (2.12), (2.4) and (2.27), it is easy to see that $h(X,X) \in \mu$, applying this fact in (3.5), we get

$$g(A_{\phi Z}Z, \bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X) = \nabla_X X lnf.$$

In view of (2.4), the above equation reduced to

$$g(h(\nabla_X \phi X, Z), \phi Z) = \nabla_X X lnf g(Z,Z). \tag{3.6}$$

Similarly,

$$g(h(\nabla_{\phi X} X, Z), \phi Z) = -\nabla_{\phi X} \phi X lnf g(Z,Z). \tag{3.7}$$

By use of (2.27) and part (iii) of Lemma 3.1, it is easy to see the following

$$g(h(\phi X, \nabla_X Z), \phi Z) = (X lnf)^2 g(Z,Z) \tag{3.8}$$

and

$$g(h(X, \nabla_{\phi X} Z), \phi Z) = -(\phi X lnf)^2 g(Z,Z). \tag{3.9}$$

Substituting (3.3), (3.4), (3.6), (3.7), (3.8) and (3.9) in (3.2), we find

(3.10)
$$\begin{aligned}\bar{R}(X,\phi X, Z, \phi Z) = {} & X^2 lnf g(Z,Z) + (\phi X)^2 lnf g(Z,Z) \\ & -\nabla_X X lnf g(Z,Z) - \nabla_{\phi X}\phi X g(Z,Z) - \|h_\mu(X,Z)\|^2 \\ & -\|h(\phi X, Z)\|^2 + \|\mathcal{Q}_X Z\|^2 + \|\mathcal{Q}_{\phi X} Z\|^2,\end{aligned}$$

Let $\{e_0 = \xi, e_1, e_2, \dots, e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, \dots, e_{2p} = \phi e_p, e^1, e^2, \dots, e^q\}$ be an orthonormal frame of TM such that $\{e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p\}$ are tangent to TN_T and $\{e^1, e^2, \dots, e^q\}$ are tangent to $TN_\perp$.
Using (3.1) and (2.30) in (3.10) and summing over $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, we get

$$q\Delta lnf = 2.p.q.f_2 - \sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 + \sum_{i=1}^{2p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2. \tag{3.11}$$

From Hopf's Lemma and (3.11), if

$$\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geq 2.p.q.f_2 + \sum_{i=1}^{2p} \|\mathcal{Q}_{e_i} e^j\|^2,$$

or

$$\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \leq 2.p.q.f_2,$$

then the warping function f is constant on M i.e., M is simply contact CR-product submanifold, which proves the theorem completely. □

Now we have the following Corollary, which can be verified easily.

Corollary 3.4. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifolds of a nearly Kenmotsu generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, such that N_T is compact. Then M is contact CR-product if and only if,*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i, e^j)\|^2 = 2.p.q.f_2 + \sum_{i=1}^{2p}\sum_{j=1}^{q}\|\mathcal{Q}_{e_i}e^j\|^2.$$

Moreover, if the ambient manifold $\bar{M}$ is a Kenmotsu space form, then from above findings we have the following.

Corollary 3.5. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu space form $\bar{M}(c)$ such that N_T is compact. Then M is contact CR-product submanifold if either one of the inequality*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i, e^j)\|^2 \geq \frac{c+1}{2}p.q,$$

or

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i, e^j)\|^2 \leq \frac{c+1}{2}p.q,$$

holds, where h_μ denotes the component of h in μ, $2p+1$ and q are the dimensions of N_T and $N_\perp$, respectively.

Corollary 3.6. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifolds of a Kenmotsu space form $\bar{M}(c)$ such that N_T is compact. Then M is contact CR-product if and only if*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i, e^j)\|^2 = \frac{c+1}{2}p.q.$$

4. **Another inequality**

In the present section, we estimate the squared norm of the second fundamental form in terms of warping function.

Theorem 4.1. *Let $\bar{M}(f_1, f_2, f_3)$ be a $(2n+1)-$dimensional nearly Kenmotsu generalized Sasakian space form and $M = N_T \times_f N_\perp$ be an m-dimensional contact CR-warped product submanifold, such that N_1 is $(2p+1)-$dimensional invariant submanifold tangent to ξ and $N_\perp$ be a $q-$dimensional anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Then*

(i) The squared norm of the second fundamental form h satisfies

$$\|h\|^2 \geq q[\|\nabla lnf\|^2 - \Delta lnf - 1] + 2.p.q.f_2 + \|\mathcal{Q}_D D^\perp\|^2, \tag{4.1}$$

where Δ denotes the Laplace operator on N_T.

(ii) The equality sign of (4.1) holds identically if and only if we have

(a) N_T is totally geodesic invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Hence N_T is a nearly Kenmotsu generalized Sasakian space form.

(b) $N_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$.

Proof. For any $X \in TN_T$ and $Z \in TN_\perp$, from (2.9)(b) and parts (ii) and (iii) of Lemma 3.1, we have

$$g(h(\xi, Z), \phi Z) = 0,$$

and

$$g(h(\phi X, Z), \phi Z) = X lnf \|Z\|^2.$$

Since $\xi lnf = 1$, then combining this with above two equations, we have

$$\sum_{i=0}^{2p}\sum_{j=1}^{q} \|h_{\phi D^\perp}(e_i, e^j)\|^2 = q[\|\nabla lnf\|^2 - 1]. \tag{4.2}$$

Again from (3.11)

$$\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 = 2.p.q.f_2 - q\Delta lnf + \sum_{i=1}^{2p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2. \tag{4.3}$$

We use the following notation

$$\sum_{i=1}^{2p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2 = \|\mathcal{Q}_D D^\perp\|^2.$$

Substituting above notation in (4.3) and combining it with (4.2), we obtain the inequality (4.1).

Let h'' be the second fundamental form of $N_\perp$ in M. Then, we have

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -X lnf g(Z, W),$$

on using (2.28), we get

$$h''(Z, W) = -g(Z, W)\nabla lnf. \tag{4.4}$$

If the equality sign of (4.1) holds identically, then we obtain

$$h(D, D) = 0, \ \ h(D^\perp, D^\perp) = 0. \tag{4.5}$$

The first condition of (4.5) implies that N_T is totally geodesic in M. On the other hand, one has

$$g(h(X, \phi Y), \phi Z) = g(\bar{\nabla}_X \phi Y, \phi Z) = -g(\phi Y, (\bar{\nabla}_X \phi) Z). \tag{4.6}$$

By use of (2.12) and (2.6) we get the following equation

$$g(\phi Y, (\bar{\nabla}_Z \phi)X) = g(\phi Y, \nabla_Z \phi X) - g(Y, \nabla_Z X),$$

in view of (2.27) the above equation reduced to

$$g(\phi Y, (\bar{\nabla}_Z \phi)X) = 0. \tag{4.7}$$

From (4.6), (4.7) and (2.23)(a) we have

$$g(h(X, \phi Y), \phi Z) = -g(\phi Y, (\bar{\nabla}_X \phi)Z + (\bar{\nabla}_Z \phi)X) = 0. \tag{4.8}$$

From (4.8), it is evident that N_T is totally geodesic in $\bar{M}(f_1, f_2, f_3)$ and hence is a nearly Kenmotsu generalized Sasakian space form.

The second condition of (4.5) and (4.4) imply that $N_\perp$ is totally umbilical in $\bar{M}(f_1, f_2, f_3)$. □

In the last we have the following corollary which can be deduced from inequality (4.1).

Corollary 4.2. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu space form $\bar{M}(c)$, then squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq q[\|\nabla lnf\|^2 - \Delta lnf - 1] + \frac{c+1}{2}.p.q,$$

where Δ is the Laplace operator on N_T, and $2p+1$ and q are the dimensions of N_T and $N_\perp$ respectively.

Remark 4.3. Inequality (4.1) is the generalization of the inequality obtained in [2].

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