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STRUCTURE OF FINITE WAVELET FRAMES OVER PRIME FIELDS

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Abstract. This article presents a systematic study for structure of finite wavelet frames over prime fields. Let p be a positive prime integer and $\mathbb{W}_p$ be the finite wavelet group over the prime field $\mathbb{Z}_p$. We study theoretical frame aspects of finite wavelet systems generated by subgroups of the finite wavelet group $\mathbb{W}_p$.

Keywords: Finite wavelet frames, finite wavelet group, prime fields.
MSC(2010): Primary: 42C15, 42C40, 65T60; Secondary: 30E05, 30E10.

1. Introduction

Signal processing of periodic signals is the basis of digital signal processing. Over the last decades, joint time-frequency (time-scale) representations of non-stationary signals have achieved significant popularity, see [10] and references therein. Time-frequency (resp. time-scale) representations are obtained by analyzing the signal with respect to an overcomplete function system whose elements are localized in time and in frequency (resp. scale). The obtained data is interpreted using frame theory. Among various types of frames, coherent or structured frames such as classic wavelet frames generated by dyadic dilations and integer translations of a window function have been proven to be particularly useful [1, 2, 5, 8, 13]. Similar to wavelet frames, Gabor frames generated by a set of modulations and translations of a given single window function have been studied at length, see [23] and references therein. Coherent frames such as wavelet frames or Gabor frames give us time-frequency (time-scale) representations and redundant time-frequency (time-scale) expansions. Finite frames have found use in a variety of applications such as digital signal processing, image analysis, filter banks, quantization, data analysis, and also compressed sensing see [3, 4, 7, 14, 23–27].

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The notion of a wavelet transform over a prime field was introduced in [11] and extended for finite fields in [9, 21]. The current paper consists of abstract aspects of nature of finite wavelet systems over prime fields. This paper aims to further develop theoretical aspects of finite wavelet frames over prime fields which has not been studied in depth when compared to finite Gabor frames.

This article contains 4 sections. Section 2 is devoted to fix notations and a brief summary of Fourier analysis on finite cyclic groups, periodic signal processing, and theory of finite frames. In Section 3, we present the notion of finite wavelet groups over prime fields. Then in Section 4 we will study theoretical frame aspects of finite wavelet systems generated by subgroups of the finite wavelet group $\mathbb{W}_p$.

2. **Preliminaries and notations**

Let G be a finite group and $\mathbb{H}$ be a finite dimensional complex inner-product space with $\dim\mathbb{H} = N$. Let $\mathcal{U}(\mathbb{H})$ be the group of all unitary operators on $\mathbb{H}$, which is, up to isomorphism of groups, the matrix group of all unitary $N \times N$-matrices with complex entries. A unitary group representation Γ of G on $\mathbb{H}$ is a mapping $\Gamma : G \to \mathcal{U}(\mathbb{H})$ such that

$$\Gamma(gg') = \Gamma(g)\Gamma(g') \quad \text{for } g, g' \in G.$$

For a finite group G, the finite dimensional complex vector function/signal space $\mathbb{C}^G = \{\mathbf{x} : G \to \mathbb{C}\}$ is a $|G|$-dimensional vector space with complex vector entries indexed by elements in the finite group G, where $|G|$ denotes the order of the group. For $\mathbb{C}^{\mathbb{Z}_N}$, where $\mathbb{Z}_N$ denotes the cyclic group of N elements $\{0, \ldots, N-1\}$, we simply write $\mathbb{C}^N$ at times. The notation $\|\mathbf{x}\|_0 = |\{k \in \mathbb{Z}_N : \mathbf{x}(k) \neq 0\}|$ counts non-zero entries in $\mathbf{x} \in \mathbb{C}^N$. The inner product of two signals $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{g\in G} \mathbf{x}(g)\overline{\mathbf{y}(g)}$, and the induced norm is the $\|.\|_2$-norm of $\mathbf{x}$, that is $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$.

Throughout this article we shall use the standard and traditional harmonic analysis modeling for the linear vector space of all periodic signals or finite size data. A given one-dimensional (1D) finite discrete data or signal $\mathbf{x}$, i.e., a signal of a given length $N \in \mathbb{N}$ denoted by $\mathbf{x} = [\mathbf{x}(0), \ldots, \mathbf{x}(N-1)]$, which is a function defined on the set $\mathbb{Z}_N = \{0, \ldots, N-1\} \subset \mathbb{Z}$. The translation operator $T_k : \mathbb{C}^N \to \mathbb{C}^N$ is $T_k\mathbf{x}(s) = \mathbf{x}(s-k)$ for $\mathbf{x} \in \mathbb{C}^N$ and $k, s \in \mathbb{Z}_N$. The modulation operator $M_\ell : \mathbb{C}^N \to \mathbb{C}^N$ is given by $M_\ell\mathbf{x}(s) = e^{-2\pi i\ell s/N}\mathbf{x}(s)$ for $\mathbf{x} \in \mathbb{C}^N$ and $\ell, s \in \mathbb{Z}_N$. The translation and modulation operators on the Hilbert space $\mathbb{C}^N$ are unitary operators in the $\|.\|_2$-norm. For $\ell, k \in \mathbb{Z}_N$ we have $(T_k)^* = (T_k)^{-1} = T_{N-k}$ and $(M_\ell)^* = (M_\ell)^{-1} = M_{N-\ell}$. The circular convolution of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ is defined by

$$\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} \mathbf{x}(s)\mathbf{y}(k-s) \text{ for } k \in \mathbb{Z}_N.$$

The circular involution or circular adjoint of $\mathbf{x} \in \mathbb{C}^N$ is given by $\mathbf{x}^*(s) = \overline{\mathbf{x}(-s)} = \overline{\mathbf{x}(N-s)}$. The complex linear space $\mathbb{C}^N$ equipped with the $\|.\|_1$-norm, the circular convolution, and involution is a Banach $*$-algebra [12]. The unitary discrete Fourier Transform (DFT) of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is defined by $\widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathbf{x}(k)\overline{\mathbf{w}_\ell(k)}$, for all $\ell \in \mathbb{Z}_N$ where for all $\ell, k \in \mathbb{Z}_N$ we have $\mathbf{w}_\ell(k) = e^{2\pi i\ell k/N}$. The set $\{\mathbf{w}_\ell : \ell \in \mathbb{Z}_N\}$ is precisely the group of all pure frequencies (characters) $\widehat{\mathbb{Z}_N}$ (i.e homomorphisms or characters into the circle group $\mathbb{T}$) on the additive group $\mathbb{Z}_N$. More precisely, the map $\ell \mapsto \mathbf{w}_\ell$ is a group isomorphism between $\mathbb{Z}_N$ and $\widehat{\mathbb{Z}_N}$. Therefore, $\mathbf{w}_{\ell+\ell'} = \mathbf{w}_\ell\mathbf{w}_{\ell'}$ and $\overline{\mathbf{w}_\ell} = \mathbf{w}_{N-\ell}$ for all $\ell, \ell' \in \mathbb{Z}_N$. Thus DFT of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ at the frequency $\ell \in \mathbb{Z}_N$ is

$$\widehat{\mathbf{x}}(\ell) = \mathcal{F}_N(\mathbf{x})(\ell) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathbf{x}(k)\overline{\mathbf{w}_\ell(k)} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathbf{x}(k)e^{-2\pi i\ell k/N}. \tag{2.1}$$

The discrete Fourier transform (DFT) is a unitary transform in $\|.\|_2$-norm, i.e., for all $\mathbf{x} \in \mathbb{C}^N$ satisfies the Parseval formula $\|\mathcal{F}_N(\mathbf{x})\|_2 = \|\mathbf{x}\|_2$. The Polarization identity implies the Plancherel formula $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}}\rangle$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$. The unitary DFT (2.1) satisfies $\widehat{T_k\mathbf{x}} = M_k\widehat{\mathbf{x}}$, $\widehat{M_\ell\mathbf{x}} = T_{N-\ell}\widehat{\mathbf{x}}$, $\widehat{\mathbf{x}^*} = \overline{\widehat{\mathbf{x}}}$, and $\widehat{\mathbf{x} * \mathbf{y}} = \widehat{\mathbf{x}}.\widehat{\mathbf{y}}$, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ and $k, \ell \in \mathbb{Z}_N$. The inverse discrete Fourier formula (IDFT) for a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is

$$\mathbf{x}(\ell) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\widehat{\mathbf{x}}(k)\mathbf{w}_\ell(k) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\widehat{\mathbf{x}}(k)e^{2\pi i\ell k/N}, \quad 0 \le \ell \le N-1. \tag{2.2}$$

A finite system (sequence) $\mathfrak{A} = \{\mathbf{y}_j : 0 \le j \le M-1\} \subset \mathbb{C}^N$ is called a frame (or finite frame) for the finite dimensional complex Hilbert space $\mathbb{C}^N$, if there exist positive constants $0 < A \le B < \infty$ such that [7]

$$A\|\mathbf{x}\|_2^2 \le \sum_{j=0}^{M-1}|\langle \mathbf{x}, \mathbf{y}_j\rangle|^2 \le B\|\mathbf{x}\|_2^2, \text{ for all } \mathbf{x} \in \mathbb{C}^N. \tag{2.3}$$

If $\mathfrak{A} = \{\mathbf{y}_j : 0 \le j \le M-1\}$ is a frame for $\mathbb{C}^N$, the synthesis operator $F : \mathbb{C}^M \to \mathbb{C}^N$ is $F\{c_j\}_{j=0}^{M-1} = \sum_{j=0}^{M-1} c_j\mathbf{y}_j$ for all $\{c_j\}_{j=0}^{M-1} \in \mathbb{C}^M$. The adjoint (analysis) operator $F^* : \mathbb{C}^N \to \mathbb{C}^M$ is $F^*\mathbf{x} = \{\langle \mathbf{x}, \mathbf{y}_j\rangle\}_{j=0}^{M-1}$ for all $\mathbf{x} \in \mathbb{C}^N$. By composing F and F^*, we get the positive and invertible frame operator $S : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\mathbf{x} \mapsto S\mathbf{x} = FF^*\mathbf{x} = \sum_{j=0}^{M-1}\langle \mathbf{x}, \mathbf{y}_j\rangle\mathbf{y}_j \text{ for all } \mathbf{x} \in \mathbb{C}^N, \tag{2.4}$$

In terms of the analysis operator we have $A\|\mathbf{x}\|_2^2 \le \|F^*\mathbf{x}\|_2^2 \le B\|\mathbf{x}\|_2^2$ for $\mathbf{x} \in \mathbb{C}^N$. If $\mathfrak{A}$ is a finite frame for $\mathbb{C}^N$, the set $\mathfrak{A}$ spans the complex Hilbert space $\mathbb{C}^N$ which implies $M \ge N$, where $M = |\mathfrak{A}|$. It should be mentioned that each finite spanning set in $\mathbb{C}^N$ is a finite frame for $\mathbb{C}^N$. The ratio between M and N is called the redundancy of the finite frame $\mathfrak{A}$ (i.e., $\mathrm{red}_{\mathfrak{A}} = M/N$), where $M = |\mathfrak{A}|$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a finite frame for $\mathbb{C}^N$, each $\mathbf{x} \in \mathbb{C}^N$ satisfies the following reconstruction formulas

$$\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, S^{-1}\mathbf{y}_j\rangle \mathbf{y}_j = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j\rangle S^{-1}\mathbf{y}_j. \tag{2.5}$$

In this case, the complex numbers $\langle \mathbf{x}, S^{-1}\mathbf{y}_j\rangle$ are called frame coefficients and the finite sequence $\mathfrak{A}^\bullet := \{S^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ which is a frame for $\mathbb{C}^N$ as well, is called the canonical dual frame of $\mathfrak{A}$. A finite frame $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ for $\mathbb{C}^N$ is called tight if we have $A = B$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a tight frame for $\mathbb{C}^N$ with frame bound A, then the canonical dual frame $\mathfrak{A}^\bullet$ is exactly $\{A^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ and for $\mathbf{x} \in \mathbb{C}^M$ we have [7]

$$\mathbf{x} = \frac{1}{A}\sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j\rangle \mathbf{y}_j. \tag{2.6}$$

3. Finite Wavelet Group over Prime Fields

In this section we briefly state structure and basic properties of cyclic dilation operators, see [11,15–19,21]. We then present the notion of finite wavelet groups over prime fields [20].

Let p be a positive prime integer. The set

$$\mathbb{U}_p := \mathbb{Z}_p - \{0\} = \{1, \ldots, p-1\}, \tag{3.1}$$

is a finite multiplicative Abelian group of order $p-1$ with respect to the multiplication module p with the identity element 1. The multiplicative inverse for $m \in \mathbb{U}_p$ (i.e., an element $m_p \in \mathbb{U}_p$ with $mm_p \overset{p}{\equiv} m_pm \overset{p}{\equiv} 1$) is m_p which satisfies $m_pm + np = 1$ for some $n \in \mathbb{Z}$, which can be done by Bezout lemma [20].

For $m \in \mathbb{U}_p$, define the **cyclic dilation operator** $D_m : \mathbb{C}^p \to \mathbb{C}^p$ by

$$D_m\mathbf{x}(k) := \mathbf{x}(m_pk)$$

for all $\mathbf{x} \in \mathbb{C}^p$ and $k \in \mathbb{Z}_p$, where m_p is the multiplicative inverse of m in $\mathbb{U}_p$.

In the following propositions we state basic algebraic properties of cyclic dilation operators.

Proposition 3.1. *Let p be a positive prime integer. Then*

(i) *For $(m,k) \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $D_mT_k = T_{mk}D_m$.*
(ii) *For $m, m' \in \mathbb{U}_p$, we have $D_{mm'} = D_mD_{m'}$.*
(iii) *For $(m,k), (m',k') \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $T_{k+mk'}D_{mm'} = T_kD_mT_{k'}D_{m'}$.*
(iv) *For $(m,\ell) \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $D_mM_\ell = M_{m_p\ell}D_m$.*

The next result states analytic properties of cyclic dilations.

Proposition 3.2. *Let p be a positive prime integer and $m \in \mathbb{U}_p$. Then*

(i) *The dilation operator* $D_m : \mathbb{C}^p \to \mathbb{C}^p$ *is a* $*$*-homomorphism.*

(ii) *The dilation operator* $D_m : \mathbb{C}^p \to \mathbb{C}^p$ *is unitary in* $\|.\|_2$*-norm and satisfies*

$$(D_m)^* = (D_m)^{-1} = D_{m_p}.$$

(iii) *For* $\mathbf{x} \in \mathbb{C}^p$*, we have* $\widehat{D_m \mathbf{x}} = D_{m_p}\widehat{\mathbf{x}}$*.*

Remark 3.3. Proposition 3.2 guarantees that cyclic dilations lead to permutation of spectra as well. This property of cyclic dilations have recently been used in implementation of algorithms for sparse fast Fourier transform (sFFT), see [28] and references therein.

The underlying set

$$\mathbb{U}_p \times \mathbb{Z}_p = \{(m, k) : m \in \mathbb{U}_p, k \in \mathbb{Z}_p\},$$

equipped with the following group operations

$$(m, k) \ltimes (m', k') := (mm', k + mk'), \quad (m, k)^{-1} := (m_p, m_p.(p - k)), \tag{3.2}$$

is a finite non-Abelian group of order $p(p-1)$ denoted by $\mathbb{W}_p$ and called as **finite affine group** on p integers or the **finite wavelet group** associated to the integer p or simply we call it as the p**-wavelet group**.

The following proposition summarizes basic properties of the finite wavelet group.

Proposition 3.4. *Let* $p > 2$ *be a positive prime integer. Then*

(i) $\mathbb{W}_p$ *is a non-Abelian group of order* $p(p-1)$ *which contains a copy of* $\mathbb{Z}_p$ *as a normal Abelian subgroup and a copy of* $\mathbb{U}_p$ *as a non-normal Abelian subgroup.*

(ii) *The map* $\sigma : \mathbb{W}_p \to \mathcal{U}_{p\times p}(\mathbb{C})$ *defined by* $(m, k) \mapsto \sigma(m, k) := T_k D_m$ *for* $(m, k) \in \mathbb{W}_p$*, is a unitary representation of the finite affine group* $\mathbb{W}_p$ *on the finite dimensional Hilbert space* $\mathbb{C}^p$*.*

Remark 3.5. In contrast to dyadic dilations, which preserve geometry of a signal, cyclic dilations destroy geometric features of signals by rearranging their entries. Thus, the matrix representation of a cyclic dilation operator has a non-localized structure. This non-localization property is not interesting from the classical geometric signal processing points of view. But from the compressive sensing approach such non-localized structure is beneficial when considered as measurement matrices, while for traditional dyadic dilations this makes the traditional discrete wavelet systems [6, 26] practically useless in compressive sensing, see [14, 22, 23].

4. Finite Wavelet Frames over Prime Fields

Throughout this section we still assume that p is a positive prime integer. A **finite wavelet system** for the complex Hilbert space $\mathbb{C}^p$ is a family or system of the form

$$\mathcal{W}(\mathbf{y},\Delta) := \{\sigma(m,k)\mathbf{y} = T_k D_m \mathbf{y} : (m,k) \in \Delta \subseteq \mathbb{W}_p\}, \tag{4.1}$$

for some window signal $\mathbf{y} \in \mathbb{C}^p$ and a subset Δ of $\mathbb{W}_p$.

If $\Delta = \mathbb{W}_p$, we put $\mathcal{W}(\mathbf{y}) := \mathcal{W}(\mathbf{y}, \mathbb{W}_p)$, and it is called a **full** finite wavelet system over $\mathbb{Z}_p$. A finite wavelet system which spans $\mathbb{C}^p$ is a frame and is referred to as a **finite wavelet frame** over the prime field $\mathbb{Z}_p$.

If $\mathbf{y} \in \mathbb{C}^p$ is a window signal, then for $\mathbf{x} \in \mathbb{C}^p$, we have

$$\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle = \langle \mathbf{x}, T_k D_m \mathbf{y}\rangle = \langle T_{p-k}\mathbf{x}, D_m \mathbf{y}\rangle, \quad \text{for } (m,k) \in \mathbb{W}_p. \tag{4.2}$$

The following proposition gives us a Fourier (resp. convolution) representation of wavelet coefficients.

Proposition 4.1. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ *and* $(m,k) \in \mathbb{W}_p$. *Then,*

(i) $\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle = \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-k)$.
(ii) $\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle = \sqrt{p}\mathbf{x} * D_m\mathbf{y}^*(k)$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ and $(m,k) \in \mathbb{W}_p$. (i) Using the Plancherel formula we have

$$\begin{aligned}
\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle &= \langle \mathbf{x}, T_k D_m \mathbf{y}\rangle \\
&= \langle \widehat{\mathbf{x}}, \widehat{T_k D_m \mathbf{y}}\rangle \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{T_k D_m \mathbf{y}}(\ell)} \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{M_k\widehat{D_m \mathbf{y}}(\ell)} \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)\mathbf{w}_k(\ell)} \\
&= \sum_{\ell=0}^{p-1} \left(\widehat{\mathbf{x}}.\overline{\widehat{D_m \mathbf{y}}}\right)(\ell)\overline{\mathbf{w}_{p-k}(\ell)} = \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-k).
\end{aligned}$$

(ii) Similarly using the Plancherel formula, we can write

$$\begin{aligned}
\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)}\mathbf{w}_k(\ell) \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)(\widehat{D_m \mathbf{y}})^*(\ell)\mathbf{w}_k(\ell) \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)(\widehat{D_m \mathbf{y}^*})(\ell)\mathbf{w}_k(\ell)
\end{aligned}$$

$$= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x} * D_m \mathbf{y}^*}(\ell)\mathbf{w}_k(\ell) = \sqrt{p}\mathbf{x} * D_m\mathbf{y}^*(k).$$

□

In the following theorem we present a concrete formulation for the $\|.\|_2$-norm of wavelet coefficients.

Theorem 4.2. *Let p be a positive prime integer, M be a divisor of $p-1$, and $\mathbb{M}$ be a multiplicative subgroup of $\mathbb{U}_p$ of order M. Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal and $\mathbf{x} \in \mathbb{C}^p$. Then,*

$$\sum_{m\in\mathbb{M}}\sum_{k\in\mathbb{Z}_p} |\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 = pM|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2$$
$$+ p\left(\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2\right) + \sum_{\ell\in\mathbb{U}_p-\mathbb{M}}\gamma_\ell(\mathbf{y},\mathbb{M})|\widehat{\mathbf{x}}(\ell)|^2\right),$$

where

$$\gamma_\ell(\mathbf{y},\mathbb{M}) := \sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2, \quad \text{for all } \ell \in \mathbb{U}_p - \mathbb{M}.$$

Proof. Let $\mathbf{y} \in \mathbb{C}^p$ be a window function, $\mathbf{x} \in \mathbb{C}^p$, and $m \in \mathbb{U}_p$. Using Proposition 4.1 and Plancherel formula, we have

$$\begin{aligned}
\sum_{k=0}^{p-1}|\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 &= p\sum_{k=0}^{p-1}\left|\mathcal{F}_p\left(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}}\right)(p-k)\right|^2 \\
&= p\sum_{k=0}^{p-1}\left|\mathcal{F}_p\left(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}}\right)(k)\right|^2 \\
&= p\sum_{\ell=0}^{p-1}\left|\left(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}}\right)(\ell)\right|^2 \\
&= p\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\overline{\widehat{D_m\mathbf{y}}(\ell)}|^2 = p\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\overline{\widehat{\mathbf{y}}}(m\ell)|^2.
\end{aligned}$$

Therefore we achieve

$$\begin{aligned}
\sum_{m\in\mathbb{M}}\sum_{k=0}^{p-1}|\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 &= p\sum_{m\in\mathbb{M}}\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\overline{\widehat{\mathbf{y}}}(m\ell)|^2 \\
&= p\sum_{\ell=0}^{p-1}\sum_{m\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2|\overline{\widehat{\mathbf{y}}}(m\ell)|^2.
\end{aligned}$$

Then we can write

$$\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right) = |\widehat{\mathbf{x}}(0)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(0)|^2\right) + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right),$$

which implies that

$$\sum_{m\in\mathbb{M}}\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2 = M|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right). \tag{4.3}$$

Changing the summation order, we have

$$\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right) = \sum_{\ell\in\mathbb{M}}\sum_{m\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2 + \sum_{\ell\in\mathbb{U}_p-\mathbb{M}}\sum_{m\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2.$$

For each $\ell\in\mathbb{M}$, replacing m by $m\ell_p$ we get

$$\begin{aligned}\sum_{m\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2 &= |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right)\\ &= |\widehat{\mathbf{x}}(\ell)|^2\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\\ &= |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\right),\end{aligned}$$

which implies that

$$\sum_{\ell\in\mathbb{M}}\sum_{m\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2 = \left(\sum_{\ell\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\right). \tag{4.4}$$

Then by (4.3) and (4.4), we obtain

$$\begin{aligned}\sum_{m\in\mathbb{M}}\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{\mathbf{y}}(m\ell)|^2 &= M|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m\ell)|^2\right)\\ &= M|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2\right)\\ &\quad + \sum_{\ell\in\mathbb{U}_p-\mathbb{M}}\gamma_\ell(\mathbf{y},\mathbb{M})|\widehat{\mathbf{x}}(\ell)|^2,\end{aligned}$$

which completes the proof. □

As an interesting consequence of the technique inside the proof of Theorem 4.2 we have the following corollary.

Corollary 4.3. *Let p be a positive prime integer. Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal and $\mathbf{x} \in \mathbb{C}^p$. Then*

$$\sum_{m=1}^{p-1}\sum_{k=0}^{p-1}|\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 = p(p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + p\left(\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\right).$$

Proof. Using equation (4.3) for $\mathbb{M} = \mathbb{U}_p$, we get the result. □

Applying Theorem 4.2, we can present the following characterization of a given window signal $\mathbf{y} \in \mathbb{C}^p$ and a subgroup of the finite wavelet group $\mathbb{W}_p$ to guarantee that generated wavelet system is a frame for $\mathbb{C}^p$.

Theorem 4.4. *Let p be a positive prime integer, M be a divisor of $p-1$, and $\mathbb{M}$ be a multiplicative subgroup of $\mathbb{U}_p$ of order M. Let $\Delta_{\mathbb{M}} := \mathbb{M} \times \mathbb{Z}_p$ and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for $\mathbb{C}^p$ if and only if the following conditions hold*

(i) $\widehat{\mathbf{y}}(0) \neq 0$.
(ii) *There exists an $m \in \mathbb{M}$ with $\widehat{\mathbf{y}}(m) \neq 0$.*
(iii) $\gamma_\ell(\mathbf{y}, \mathbb{M}) \neq 0$ *for all $\ell \in \mathbb{U}_p - \mathbb{M}$.*

Proof. Let $\mathbf{y}$ be a non-zero window signal which satisfies conditions (i), (ii) and (iii). Thus

$$\gamma(\mathbf{y}, \mathbb{M}) := \min\left\{\gamma_\ell(\mathbf{y}, \mathbb{M}) : \ell \in \mathbb{U}_p - \mathbb{M}\right\},$$

is non-zero. Let $0 < A < \infty$ be given by

$$A := \min\left\{ M\left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2, p\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2, p\gamma(\mathbf{y}, \mathbb{M})\right\}. \tag{4.5}$$

Let $\mathbf{x} \in \mathbb{C}^p$. Then using Theorem 4.2, we can write

$$\sum_{m\in\mathbb{M}}\sum_{k\in\mathbb{Z}_p}|\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 = pM|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + p\left(\sum_{m\in\mathbb{M}}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell\in\mathbb{M}}|\widehat{\mathbf{x}}(\ell)|^2\right)$$

$$
\begin{aligned}
&+p \sum_{\ell \in \mathbb{U}_p - \mathbb{M}} \gamma_\ell(\mathbf{y}, \mathbb{M}) |\widehat{\mathbf{x}}(\ell)|^2 \\
\geq \;& pM|\widehat{\mathbf{y}}(0)|^2 |\widehat{\mathbf{x}}(0)|^2 + p\left(\sum_{m\in\mathbb{M}} |\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell\in\mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2\right) \\
&+p\gamma(\mathbf{y},\mathbb{M}) \cdot \sum_{\ell\in\mathbb{U}_p-\mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2 \\
\geq \;& A|\widehat{\mathbf{x}}(0)|^2 + A\left(\sum_{\ell\in\mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2\right) + A\left(\sum_{\ell\in\mathbb{U}_p-\mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2\right) \\
= \;& A\|\widehat{\mathbf{x}}\|_2^2 = A\|\mathbf{x}\|_2^2,
\end{aligned}
$$

which implies that the finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for $\mathbb{C}^p$.

Conversely, let $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal such that the finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for $\mathbb{C}^p$. Thus there exists $A' > 0$ such that

$$\sum_{m\in\mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, \sigma(m,k)\mathbf{y}\rangle|^2 \geq A'\|\mathbf{x}\|_2, \quad \text{for all } \mathbf{x} \in \mathbb{C}^p.$$

Then by Theorem 4.2, we get

$$
\begin{aligned}
&M|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{m\in\mathbb{M}} |\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell\in\mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2\right) \\
&\qquad + \sum_{\ell\in\mathbb{U}_p-\mathbb{M}} \gamma_\ell(\mathbf{y},\mathbb{M})|\widehat{\mathbf{x}}(\ell)|^2 \geq A''\|\mathbf{x}\|_2,
\end{aligned}
\tag{4.6}
$$

for all $\mathbf{x} \in \mathbb{C}^p$, where $0 < A'' = A'/p$. Let $\mathbf{x} \in \mathbb{C}^p$ with $\widehat{\mathbf{x}}(0) = 1$ and $\widehat{\mathbf{x}}(\ell) = 1$ for $1 \leq \ell \leq p-1$. Applying the equation (4.6), we get $\widehat{\mathbf{y}}(0) \neq 0$. If $\mathbf{x} \in \mathbb{C}^p$ is a non-zero vector with $\widehat{\mathbf{x}}(\ell) = 0$ for $\ell \notin \mathbb{M}$, then equation (4.6) guarantees that $\sum_{m\in\mathbb{M}} |\widehat{\mathbf{y}}(m)|^2 \neq 0$, and hence we have $\widehat{\mathbf{y}}(m) \neq 0$ for some $m \in \mathbb{M}$. Let $\ell \in \mathbb{U}_p - \mathbb{M}$ be given. Then pick a nonzero vector $\mathbf{x} \in \mathbb{C}^p$ such that $\widehat{\mathbf{x}}(\ell) = 1$ and $\widehat{\mathbf{x}}(\ell') = 0$ for all $0 \leq \ell' \neq \ell \leq p-1$. A similar argument implies that $\gamma_\ell(\mathbf{y}, \mathbb{M}) \neq 0$. □

The following result shows that for a large class of non-zero window signals the finite wavelet system $\mathcal{W}(\mathbf{y})$ is a frame for $\mathbb{C}^p$ with redundancy $p-1$.

Corollary 4.5. *Let p be a positive prime integer and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y})$ constitutes a frame for $\mathbb{C}^p$ with the redundancy $p-1$ if and only if $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$.*

We can also deduce the following condition concerning the tight frame property for finite wavelet systems generated by subgroups of $\mathbb{W}_p$.

Proposition 4.6. *Let p be a positive prime integer, M be a divisor of $p-1$, and $\mathbb{M}$ be a multiplicative subgroup of $\mathbb{U}_p$ of order M. Let $\Delta_{\mathbb{M}} := \mathbb{M} \times \mathbb{Z}_p$ and*

$\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a tight frame for $\mathbb{C}^p$ if and only if $\widehat{\mathbf{y}}(0) \neq 0$ and

$$M|\widehat{\mathbf{y}}(0)|^2 = \sum_{m\in\mathbb{M}} |\widehat{\mathbf{y}}(m)|^2 = \gamma_\ell(\mathbf{y}, \mathbb{M}), \tag{4.7}$$

for all $\ell \in \mathbb{U}_p - \mathbb{M}$. In this case

$$\alpha_{\mathbf{y}} := pM|\widehat{\mathbf{y}}(0)|^2 = p\sum_{m\in\mathbb{M}} |\widehat{\mathbf{y}}(m)|^2$$

is the frame bound.

Proof. It can be proven by a similar argument used in Theorem 4.4. □

Corollary 4.7. *Let p be a positive prime integer and $\mathbf{y} \in \mathbb{C}^p$ be a window signal with $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. The finite wavelet system $\mathcal{W}(\mathbf{y})$ is a tight frame for $\mathbb{C}^p$ if and only if $\mathbf{y}$ satisfies $\|\mathbf{y}\|_2 = \sqrt{p}|\widehat{\mathbf{y}}(0)|$. In this case,*

$$\alpha_{\mathbf{y}} := (p-1)\left|\sum_{k=0}^{p-1} \mathbf{y}(k)\right|^2 = p(p-1)|\widehat{\mathbf{y}}(0)|^2 = p\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2, \tag{4.8}$$

is the frame bound.

Remark 4.8. It should be mentioned that Corollaries 4.3 and 4.7 coincide with direct consequences of results in [11].

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