INHOMOGENEOUS TWO-PARAMETER ABSTRACT CAUCHY PROBLEM

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ABSTRACT. We use the semigroup theory to study the inhomogeneous two-parameter abstract Cauchy problem 2-IACP

\[
\begin{align*}
\frac{\partial}{\partial t_i}u(t_1, t_2) &= H_i u(t_1, t_2) + f(t_1, t_2) \\
& i = 1, 2 \\
& t_i \in [0, a_i) \\
u(0, 0) &= u_0, \quad u_0 \in X,
\end{align*}
\]

where \(X\) is a Banach space, \(H_i : D(H_i) \subseteq X \to X, \quad i = 1, 2\), is a densely-defined closed linear operator and \(f : [0, a_1] \times [0, a_2) \to X\) is a continuous function \((a_1, a_2 > 0)\). We discuss the existence and uniqueness of solution of 2-IACP. In fact, we claim that if \((H_1, H_2)\) is the generator of a \(C_0\)-two-parameter semigroup \(\{W(t_1, t_2)\}_{t_1, t_2 \geq 0}\), then 2-IACP with some conditions has a unique solution.

1. Introduction

Let \(X\) be a Banach space, \(B(X)\) is the Banach space of all bounded linear operators on \(X\) and \(R^*_+ = \{(t_1, t_2, \ldots, t_n) : t_i \geq 0, \quad i = 1, 2, \ldots, n\}\). By an \(n\)-parameter semigroup of operators we mean a homomorphism \(W : (R^*_+, +) \to B(X)\) for which \(W(0) = I\) and denote it by \((X, R^*_+, W)\). Let now \(\{e_i\}^n_{i=1}\) be the standard basis of \(R^n\). Trivially for \(s \in R^*_+\), the component \(u_i(s) = W(se_i)\) of \(W\) defines a one-parameter semigroup of

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operators, \( i = 1, 2, \ldots, n \). Also for each integers \( 0 \leq i, j \leq n \), the \( n \)-parameter semigroup property implies that, \( u_i(s)u_j(s') = u_j(s')u_i(s) \).
The \( n \)-parameter semigroup \( (X, \mathcal{R}^n_t, W) \) is called strongly (respectively, uniformly) continuous if for each \( i = 1, 2, \ldots, n \), the one-parameter components \( u_i(s) = W(s e_i) \) are strongly (respectively, uniformly) continuous. One can prove that the \( n \)-parameter semigroup \( (X, \mathcal{R}^n_t, W) \) is strongly continuous if and only if \( \lim_{t \in \mathbb{R}^n_t, t \to 0} W(t)x = x \), for all \( x \in X \), and it is uniformly continuous if and only if \( \lim_{t \in \mathbb{R}^n_t, t \to 0} W(t) = I \).

Consider an \( n \)-parameter semigroup of operators \( (X, \mathcal{R}^n_t, W) \) and let \( H_i, i = 1, 2, \ldots, n \), be the infinitesimal generator of the component semigroup \( \{u_i(t)\}_{t \geq 0} \) of \( W, i = 1, 2, \ldots, n \). We shall think of \( (H_1, H_2, \ldots, H_n) \) as the infinitesimal generator of \( (X, \mathcal{R}^n_t, W) \).

N-parameter semigroups of operators introduced by Hille in 1944 and one can see some of their properties in [3]. For some new results in the theory of \( n \)-parameter semigroups and their applications one can see [4] and [5].

If \( W \) is a \( C_0 \)-\( n \)-parameter semigroup of operators then by the Hille-Yosida theorem, \( H_i, i = 1, 2, \ldots, n \), is a closed and densely defined operator. Let \( D(H_i) \subseteq X \) be the domain of \( H_i, i = 1, 2, \ldots, n \). For \( x \in X_1 = \bigcap_{i=1}^n D(H_i) \) we define \( \| x \|_1 = \| x \| + \sum_{i=1}^n \| H_ix \| \).

In [1] one can see that if \( (X, \mathcal{R}^n_t, W) \) is a \( C_0 \)-\( n \)-parameter semigroup of operators with the infinitesimal generator \( (H_1, H_2, \ldots, H_n) \) then
(a) There is \( M \geq 1 \) and \( \omega_i \in \mathbb{R}, i = 1, 2, \ldots, n \), such that \( \| W(t_1, t_2, \ldots, t_n) \| \leq Me^{\sum_{i=1}^n t_i \omega_i} \). So \( \| W(t_1, \ldots, t_n) \| \) is bounded in any compact subset of \( \mathcal{R}^n_t \);
(b) If \( x \in D(H_i) \), so does \( W(t)x \), for each \( t \in \mathcal{R}^n_t \), and \( H_i W(t)x = W(t)H_ix \), \( i = 1, 2, \ldots, n \);
(c) \( X_1 = \bigcap_{i=1}^n D(H_i) \) is a dense subspace of \( X \) and moreover \( (X_1, \| \|_1) \) is a Banach space;
(d) For each integers \( 1 \leq i, j \leq n \), \( D(H_iH_j) \cap D(H_jH_i) \subseteq D(H_iH_j) \) and for every \( x \in D(H_iH_j) \cap D(H_jH_i), H_iH_jx = H_jH_ix \).

Also we have the Hille-Yosida theorem for \( n \)-parameter semigroups as follows [4]:
\( (H_1, \ldots, H_n) \) is the infinitesimal generator of a \( C_0 \)-\( n \)-parameter semigroup \( \{W(t)\}_{t \in \mathcal{R}^n_t} \) satisfying \( \| W(t_1, \ldots, t_n) \| \leq M_0 e^{\sum_{i=1}^n t_i \omega_i} \) for some \( M_0 \geq 1 \) and \( \omega_i > 0, i = 1, \ldots, n \), if and only if
(a) \( H_i \) is a closed densely defined operator, \( i = 1, \ldots, n \), and
Inhomogeneous two-parameter abstract Cauchy problem

\[ R(\lambda, H_i)R(\lambda, H_i) = R(\lambda, H_i)R(\lambda', H_j), \text{ for all } \lambda > \omega_i \ , \lambda' > \omega_j \text{ and integers } 1 \leq i, j \leq n, \]

(b) For each \(i = 1, \ldots, n\), \([\omega_i, \infty) \subseteq \rho(H_i)\) and there is \(M \geq 1\) such that

\[ \| R(\lambda, H_i)^n \| \leq \frac{M}{(\Re \lambda - \omega_i)^n}, \quad i = 1, \ldots, n, \text{ and } \Re \lambda > \omega_i. \]

Now we introduce n-ACP and its solution ([4]).

Suppose that \(X\) is a Banach space, \(H_i\) are closed linear operators from \(D(H_i) \subseteq X\) into \(X\) and \(a_i > 0, i = 1, 2, \ldots, n\). Then a continuous \(X\)-valued function \(u: [0, a_1] \times \cdots \times [0, a_n] \rightarrow X\) with continuous partial derivatives which satisfies the following n-parameter abstract Cauchy problem n-ACP

\[ \frac{\partial}{\partial t_i} u(t_1, t_2, \ldots, t_i, \ldots, t_n) = H_i u(t_1, \ldots, t_n) \]

\[ i = 1, 2, \ldots, n \quad t_i \in [0, a_i] \]

\[ u(0) = u_0 \quad u_0 \in \bigcap_{i=1}^{n} D(H_i) \]

is called a solution of the initial value problem (1.1).

It is proved ([4, Theorem. 2.1]) that if \((H_1, H_2, \ldots, H_n)\) is the infinitesimal generator of a \(C_0\)-n-parameter semigroup \((X, \mathcal{F}^n, W)\), then (1.1) has the unique solution \(u(t_1, t_2, \ldots, t_n) = W(t_1, t_2, \ldots, t_n)u_0\) for each \(u_0 \in \bigcap_{i=1}^{n} D(H_i)\), where \((t_1, t_2, \ldots, t_n) \in [0, a_1] \times \cdots \times [0, a_n]\).

For convenience we denote by \(I_a\) the positive two-cell \([0, a_1] \times [0, a_2]\) where \(a = (a_1, a_2) \in \mathbb{R}^2_+\). So one can see that for a closed linear operator \(A: D(A) \subseteq X \rightarrow X\), the two-parameter initial value problem

\[ \frac{\partial}{\partial t_1} u(t_1, t_2) - \frac{\partial}{\partial t_2} u(t_1, t_2) = Au(t_1, t_2) \quad (t_1, t_2) \in I_a \]

\[ u(0) = x \quad x \in D(A), \]

doesn't have a unique solution for each \(x \in D(A)\) in both \(I_a\) and \(I_{a'}\) (this can be proved in a similar way as in the proof of Theorem 2.5 of [4]). The initial value problem (1.2) can have a solution, for example if \((H_1, H_2)\) is the generator of a \(C_0\)-two-parameter semigroup \(\{W(t_1, t_2)\}_{t_1, t_2 \geq 0}\) and \(A = H_1 - H_2\) then obviously \(u(t_1, t_2) = W(t_1, t_2)x\) is a solution of (1.2) in any positive two-cell \(I_a\) and for the initial value \(x \in \bigcap_{i=1}^{2} D(H_i) = D(A)\).

In this paper we intend to study the inhomogeneous two-parameter abstract Cauchy problem 2-IACP.
\( \frac{\partial}{\partial t_i} u(t_1, t_2) = H_i u(t_1, t_2) + f(t_1, t_2) \quad t_i \in [0, a_i), \quad i = 1, 2, \)

\[ u(0, 0) = u_0, \quad u_0 \in \bigcap_{i=1}^{2} D(H_i), \]

where \( H_i : D(H_i) \subseteq X \to X, \ i = 1, 2, \) is a densely-defined closed linear operator and \( f : [0, a_1) \times [0, a_2) \to X \) is a continuous function and \( a_1, a_2 > 0. \)

By a (classical) solution of 2-IACP we mean a continuous \( X \)-valued function \( u : [0, a_1) \times [0, a_2) \to X \) having continuous partial derivatives such that \( u(t_1, t_2) \in \bigcap_{i=1}^{2} D(H_i) \) for all \( (t_1, t_2) \in (0, a_1) \times (0, a_2) \) and \( u \) satisfies 2-IACP. In the next section we study conditions under which 2-IACP has a unique solution.

We end this section with the definition of the Bochner line integral that we use in the next section. Suppose that \( M(x, y) \) and \( N(x, y) \) are continuous functions of two variables from the open disk \( B \) in \( \mathbb{R}^2 \) to a Banach space \( X \) and suppose also that \( C \) is a curve in \( B \) with parametric equations

\[
\begin{align*}
    x &= f(t) \\
    y &= g(t)
\end{align*}
\]

such that \( f, g \) have continuous first derivative on \([a, b]\). In this case Bochner line integral \( M(x, y)dx + N(x, y)dy \) on \( C \) which is defined by

\[
\oint_C M(x, y)dx + N(x, y)dy
\]

is

\[
\int_a^b [M(f(t), g(t)) f'(t) + N(f(t), g(t)) g'(t)] dt.
\]

Also it is easily proved that if \( \frac{\partial M}{\partial y}(x, y) \) and \( \frac{\partial N}{\partial x}(x, y) \) are continuous in \( B \) and there exists a function \( \phi \) such that \( \nabla \phi(x, y) = M(x, y) i + N(x, y) j \), where \( i \) and \( j \) are the unit vectors of axes \( X \) and \( Y \), respectively, and \( C \) is a sectionally smooth curve (it means \( f(t) \) and \( g(t) \) are differentiable functions except probably in the finite points) in \( B \) from the point \((x_1, y_1)\) to point \((x_2, y_2)\), then the Bochner line integral

\[
\oint_C M(x, y)dx + N(x, y)dy
\]
is independent of the path \( C \) and
\[
\oint_{C} M(x, y)\,dx + N(x, y)\,dy = \phi(x_2, y_2) - \phi(x_1, y_1).
\]

2. Inhomogeneous two-parameter abstract Cauchy problem

Consider 2-IACP as we mentioned before. Let \( \{W(t_1, t_2)\}_{t_1, t_2 \geq 0} \) be the \( C_0 \)-two-parameter semigroup generated by \((H_1, H_2)\) and let \( u(t_1, t_2) \) be a solution of 2-IACP. Then the \( X \)-valued function of two-variables \( g(s_1, s_2) = W(t_1 - s_1, t_2 - s_2)u(s_1, s_2) \) has partial derivatives for \( 0 < s_1 < t_1 \), \( 0 < s_2 < t_2 \) and for \( i=1,2 \), we have
\[
\frac{\partial g}{\partial s_i} = -H_i W(t_1 - s_1, t_2 - s_2)u(s_1, s_2) + W(t_1 - s_1, t_2 - s_2)H_i u(s_1, s_2) + W(t_1 - s_1, t_2 - s_2)f(s_1, s_2) = W(t_1 - s_1, t_2 - s_2)f(s_1, s_2).
\]

So one obtains
\[
(2.1) \quad dg = W(t_1 - s_1, t_2 - s_2)f(s_1, s_2)(ds_1 + ds_2).
\]

If 2-IACP has a solution and the Bochner line integral of the above assertion from the point \((0,0)\) to point \((t_1, t_2)\) exists (for example if \( f(t, t_0) \in L^1(0, a_1, X) \) for each \( t_0 \in [0, a_2) \), and \( f(t_0, t) \in L^1(0, a_2, X) \) for each \( t_0 \in [0, a_1) \)), then this Bochner line integral is independent of the path that connecting these two points to each other. So by line integrating of two-sided of the assertion (2.1) from \((0,0)\) to \((t_1, t_2)\) it yields to
\[
(2.2) \quad u(t_1, t_2) = W(t_1, t_2)u_0 + \oint_{(0,0)}^{(t_1, t_2)} W(t_1 - s_1, t_2 - s_2)f(s_1, s_2)(ds_1 + ds_2).
\]

This proves the uniqueness of the solution. Now we introduced a path that we use in the Bochner line integral
\[
v(t_1, t_2) = \oint_{(0,0)}^{(t_1, t_2)} W(t_1 - s_1, t_2 - s_2)f(s_1, s_2)(ds_1 + ds_2).
\]
This path contains two line segments
\[
\begin{cases}
  s_1 = t, & 0 \leq t \leq t_1 \\
  s_2 = 0, & \\
  s_1 = t_1, & 0 \leq t \leq t_2.
\end{cases}
\]
We calculate the Bochner line integral \( v(t_1, t_2) \) on this special path and denote it by \( V(t_1, t_2) \), so we have
\[
(2.3) \quad V(t_1, t_2) = \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt.
\]
Now if \( V(t_1, t_2) \) exists then for every \( u_0 \in X \) the 2-IACP has at most one solution and if it has a solution, then \( V(t_1, t_2) = v(t_1, t_2) \) and the solution of 2-IACP is given by (2.2). It is natural to consider the right-hand side of (2.3) as a generalized solution of 2-IACP even if it has not partial derivatives relative to \( t_1 \) or \( t_2 \), and does not strictly satisfy the equation in the sense of (classical) solution. We therefore give the following definition:

**Definition 2.1.** Let \((H_1, H_2)\) be the infinitesimal generator of a \(C_0\)-two-parameter semigroup \( \{W(t_1, t_2)\}_{t_1, t_2 \geq 0} \). Let \( u_0 \in X \) and \( f(t_1, t) \in L^1(0, a_2; X) \) and \( f(t, t_2) \in L^1(0, a_1; X) \) for each \( t_1 \in [0, a_1) \) and \( t_2 \in [0, a_2) \). The continuous function
\[
u(t_1, t_2) = W(t_1, t_2) u_0 + \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt
\]
is called the mild solution of the 2-IACP.

We will be interested in imposing further conditions on \( f \) so that for \( u_0 \in \bigcap_{i=1}^2 D(H_i) \), the mild solution becomes a (classical) one. 

Now we show that the continuity of \( f \), in general, is not sufficient to ensure the existence of solutions of 2-IACP for \( u_0 \in \bigcap_{i=1}^2 D(H_i) \). Indeed, let \((H_1, H_2)\) be the infinitesimal generator of a \(C_0\)-two-parameter semigroup \( \{W(t_1, t_2)\}_{t_1, t_2 \geq 0} \) and let \( x \in X \) be such that \( W(t_1, t_2) x \) does not belong to \( D(H_i) \) for any \( t_1, t_2 \geq 0 \). Let \( f(s_1, s_2) = W(s_1, s_2) x \). Then \( f(s_1, s_2) \) is continuous for \( s_1, s_2 \geq 0 \). Consider the following 2-IACP
\[
(2.4) \quad \begin{cases}
  \frac{\partial}{\partial t_i} u(t_1, t_2) = H_i u(t_1, t_2) + W(t_1, t_2) x \\
  u(0, 0) = 0.
\end{cases} 
\]
We claim that (2.4) has no solution even though \( u(0,0) = 0 \in \bigcap_{i=1}^2 D(H_i) \).
Indeed, the mild solution of (2.4) is
\[
  u(t_1, t_2) = \int_{(0,0)}^{(t_1, t_2)} W(t_1-s_1, t_2-s_2) W(s_1, s_2) x(ds_1 + ds_2) = (t_1 + t_2) W(t_1, t_2) x.
\]
But \((t_1 + t_2)W(t_1, t_2) x\) does not have partial derivative relative to \( t_1 \) for \( t_1 > 0 \) and therefore \( u(t_1, t_2) \) cannot be the solution of (2.4).

Thus in order to prove the existence of solutions of 2-IACP we have to require more than just the continuity of \( f \). In the following theorem we state a general criterion for the existence of solutions of 2-IACP. In fact we prove that if \( u_0 \in \bigcap_{i=1}^2 D(H_i) \) and \( f \) has some conditions then
\[
  u(t_1, t_2) = W(t_1, t_2) u_0 + \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt
\]
is a solution of 2-IACP. Our technique for proving theorem 2.2 is based on Pazy’s technique for the one-parameter case [6].

**Theorem 2.2.** Let \((H_1, H_2)\) be the infinitesimal generator of a \( C_0 \)-two-parameter semigroup \( \{W(t_1, t_2)\}_{t_1, t_2 \geq 0} \) and let \( f \) be a continuous function on \([0, a_1) \times [0, a_2)\) such that \( f(t_1, t) \in L^1(0, a_2; X)\) for each \( t_1 \in [0, a_1) \) and \( f(t, t_2) \in L^1(0, a_1; X)\) for each \( t_2 \in [0, a_2)\), \( \text{rang}(f) \subseteq \bigcap_{i=1}^2 D(H_i) \) and \( f \) has continuous partial derivatives and satisfies the following partial differential equation
\[
  \frac{\partial}{\partial t_1} f(t_1, t_2) - \frac{\partial}{\partial t_2} f(t_1, t_2) = (H_1 - H_2) f(t_1, t_2).
\]

Let
\[
  V(t_1, t_2) = \int_0^{t_1} W(t_1 - t, t_2) f(t, 0) dt + \int_0^{t_2} W(0, t_2 - t) f(t_1, t) dt
\]
for \( 0 \leq t_1 \leq a_1, 0 \leq t_2 \leq a_2 \).

Then 2-IACP has a (classical) solution \( u \) on \([0, a_1) \times [0, a_2)\) for every \( u_0 \in \bigcap_{i=1}^2 D(H_i) \) if one of the following conditions is satisfied:
(i) \( V(t_1, t_2) \) has continuous partial derivatives on \((0, a_1) \times (0, a_2)\).
(ii) \( V(t_1, t_2) \in \bigcap_{i=1}^2 D(H_i) \) for \( 0 < t_1 < a_1, 0 < t_2 < a_2 \) and \( H_1 V(t_1, t_2) \) and \( H_2 V(t_1, t_2) \) are continuous on \((0, a_1) \times (0, a_2)\).

If 2-IACP has a (classical) solution \( u \) on \([0, a_1) \times [0, a_2)\) for some \( u_0 \in
\[ \bigcap_{i=1}^{2} D(H_i), \] then \( V(t_1, t_2) = v(t_1, t_2) \) satisfies both (i) and (ii).

**Proof.** If 2-JACP has a solution \( u \) for some \( u_0 \in \bigcap_{i=1}^{2} D(H_i) \), then this solution is given by (2.2). Consequently \( v(t_1, t_2) = u(t_1, t_2) - W(t_1, t_2)u_0 \) has partial derivatives for \( t_1 > 0 \) and \( t_2 > 0 \) and we have

\[
\frac{\partial}{\partial t_i} v(t_1, t_2) = \frac{\partial}{\partial t_i} u(t_1, t_2) - W(t_1, t_2)H_iu_0, \quad i = 1, 2.
\]

Obviously the above derivatives are continuous on \((0, a_1) \times (0, a_2)\). Therefore (i) is satisfied. Also if \( u_0 \in \bigcap_{i=1}^{2} D(H_i) \), then \( W(t_1, t_2)u_0 \in \bigcap_{i=1}^{2} D(H_i) \) for \( t_1, t_2 \geq 0 \) and therefore \( v(t_1, t_2) = u(t_1, t_2) - W(t_1, t_2)u_0 \in \bigcap_{i=1}^{2} D(H_i) \) for \( t_1, t_2 > 0 \) and

\[
H_i v(t_1, t_2) = H_i u(t_1, t_2) - H_i W(t_1, t_2)u_0
\]

is continuous on \((0, a_1) \times (0, a_2)\). Thus (ii) also is satisfied.

Now we show that if \( V(t_1, t_2) \) satisfies one of the conditions (i) or (ii), then \( u(t_1, t_2) = W(t_1, t_2)u_0 + V(t_1, t_2) \) is the unique solution of 2-JACP.

For \( V(t_1, t_2) \) we have

\[
\frac{W(h, 0)}{h} V(t_1, t_2) = \frac{V(t_1 + h, t_2) - V(t_1, t_2)}{h}
\]

(2.5)

By the continuity of \( f \) the second term on the right-hand side of (2.5) tends to \( W(0, t_2)f(t_1, 0) \) when \( h \) tends to zero. Also by adding

\[
\pm \frac{1}{h} \int_0^{t_2} W(0, t_2 - t)f(t_1, t)dt
\]

to the last term of the right-hand side of (2.5), and letting \( h \) goes to zero we obtain

\[
H_1 V(t_1, t_2) = \frac{\partial}{\partial t_1} V(t_1, t_2) - W(0, t_2)f(t_1, 0)
\]

\[
- \int_0^{t_2} W(0, t_2 - t) \frac{\partial}{\partial t_1}(t_1, t)dt
\]

\[
+ \int_0^{t_2} H_1 W(0, t_2 - t)f(t_1, t)dt.
\]
So we have

\[ H_1 V(t_1, t_2) - \frac{\partial}{\partial t_1} V(t_1, t_2) = -W(0, t_2) f(t_1, 0) \]

\[ + \int_0^{t_2} W(0, t_2 - t)[H_1 f(t_1, t) - \frac{\partial f}{\partial t_1}(t_1, t) dt]. \]

Now we show that the right-hand side of (2.6) is equal to \(-f(t_1, t_2)\).

\[ W(0, t_2) f(t_1, 0) - \int_0^{t_2} W(0, t_2 - t)[H_1 f(t_1, t) - \frac{\partial f}{\partial t_1}(t_1, t) dt] \]
\[ = W(0, t_2) f(t_1, 0) - \int_0^{t_2} W(0, t_2 - t)[H_2 W(0, t_2 - t) f(t_1, t) - W(0, t_2 - t) \frac{\partial f}{\partial t_2}(t_1, t) dt] \]
\[ = W(0, t_2) f(t_1, 0) + \int_0^{t_2} \frac{\partial W(0, t_2 - t)}{\partial t} f(t_1, t) dt + W(0, t_2 - t) \frac{\partial f}{\partial t_2}(t_1, t) dt \]
\[ = W(0, t_2) f(t_1, 0) + f(t_1, t_2) - W(0, t_2) f(t_1, 0) = f(t_1, t_2). \]

So we obtain

\[ H_1 V(t_1, t_2) = \frac{\partial}{\partial t_1} V(t_1, t_2) - f(t_1, t_2). \]

On the other hand it is easy to verify that for \(h > 0\) the identity

\[ \frac{W(0, h) f V(t_1, t_2)}{h} = \frac{V(t_1, t_2 + h) - V(t_1, t_2)}{h} - \frac{1}{h} \int_{t_2}^{t_2 + h} W(0, t_2 + h - t) f(t_1, t) dt \]

holds.

By the continuity of \(f\) it is clear that the second term on the right-hand side of (2.8) has the limit \(f(t_1, t_2)\) as \(h \to 0\). So we have

\[ H_2 V(t_1, t_2) = \frac{\partial}{\partial t_2} V(t_1, t_2) - f(t_1, t_2). \]

If \(V(t_1, t_2)\) has continuous partial derivatives on \((0, a_1) \times (0, a_2)\), then it follows from (2.7) and (2.9) that \(V(t_1, t_2) \in \bigcap_{i=1}^2 D(H_i)\) for \(0 < t_1 < a_1\), \(0 < t_2 < a_2\) and since \(V(0, 0) = 0\) it follows that \(u(t_1, t_2) = W(t_1, t_2) u_0 + V(t_1, t_2)\) is the solution of 2-LACP for \(u_0 \in \bigcap_{i=1}^2 D(H_i)\).

If \(V(t_1, t_2) \in \bigcap_{i=1}^2 D(H_i)\) then it follows from (2.4) and (2.7) that \(V(t_1, t_2)\) has partial derivatives from the right at \(t_1\) and \(t_2\) and the right partial derivative \(\frac{\partial^+}{\partial t_i} V(t_1, t_2), i = 1, 2\), of \(V\) satisfies the equation \(\frac{\partial^+}{\partial t_i} V(t_1, t_2) = H_i V(t_1, t_2) + f(t_1, t_2)\). Since \(\frac{\partial^+}{\partial t_i} V(t_1, t_2), i = 1, 2\), is continuous, \(V(t_1, t_2)\) has continuous partial derivatives at \(t_1\) and \(t_2\).
\[
\frac{\partial}{\partial t_1}V(t_1, t_2) = H_i V(t_1, t_2) + f(t_1, t_2).
\]
Since \( V(0, 0) = 0 \), \( u(t_1, t_2) = W(t_1, t_2)u_0 + V(t_1, t_2) \) is the solution of 2-IACP for \( u_0 \in \bigcap_{i=1}^2 D(H_i) \) and the proof is complete. \( \Box \)

Now we can obtain \( f(t_1, t_2) \) from two-parameter initial value problem (1.2). As we mentioned two-parameter initial value problem (1.2) doesn’t have a unique solution for each \( x \in \bigcap_{i=1}^2 D(H_i) \) in both positive two-cells \( I_a \) and \( I_{d'} \), so \( f(t_1, t_2) \) is not unique for each \( x \in \bigcap_{i=1}^2 D(H_i) \) in both positive two-cells \( I_a \) and \( I_{d'} \). For example, assume that \( u_0 \in \bigcap_{i=1}^2 D(H_i) \) and \( f(t_1, t_2) = W(t_1, t_2)u_0 \). By these assumptions, the conditions of Theorem 2.2 hold and \( v(t_1, t_2) = (t_1 + t_2 + 1)W(t_1, t_2)u_0 \). So \( u(t_1, t_2) = (t_1 + t_2 + 1)W(t_1, t_2)u_0 \) becomes a classical solution of 2-IACP.

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