

SHEN'S PROCESSES ON FINSLERIAN CONNECTIONS

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ABSTRACT. We discuss the invariant properties of curvatures effected by the Matsumoto's C or L -process. We find equivalent conditions on curvatures by comparing the difference between the corresponding curvatures of closely related connections. As an application, Matsumoto's L -process on Randers manifold is studied. Shen's connection can not be obtained by using Matsumoto's processes from other connections. This leads us to two new processes which we call Shen's C and L -processes. We study the invariant properties of curvatures under Shen's processes.

1. Introduction

After Einstein's formulation of general relativity, Riemannian geometry became fashionable and one of the connections, namely Levi-Civita connection, came to forefront. This connection is both torsion-free and metric-compatible. On the other hand, Finslerian geometry is a natural extension of Riemannian geometry. Likewise, connections in Finslerian geometry can be prescribed on the pulled-back bundle π^*TM . Examples of such were proposed by Synge, Taylor, Berwald, Cartan, Hashiguchi, Chern and Shen (see [1-6], [8], [14], [16]).

Recently, the first author with his collaborators have defined a general class of Finslerian connections which lead to a general representation of some Finslerian connections in Finslerian geometry and yielded a

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classification of Finslerian connections into the three classes, namely Berwald-type, Cartan-type and Shen-type connections ([3], [4], [16]). However, there are four well-known connections in Finslerian geometry which may be considered “natural” in some sense: the Berwald, Cartan, Hashiguchi and Chern connections.

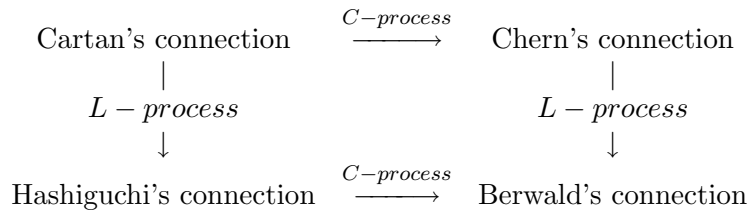
In [10], Matsumoto introduced a satisfactory and truly aesthetical axiomatic description of Cartan’s connection in the sixties. After the Cartan connection has been constructed, easy processes, baptized by Matsumoto “ L -process” and “ C -process”, yielded the Chern, the Hashiguchi and the Berwald connections.

Here, we show that the vv -curvature of connections is invariant under the Matsumoto’s L -process. Comparing the corresponding curvatures of related connections obtained by this transformation, we obtain equivalent conditions on curvatures.

Theorem 1.1. *Let (M, F) be a Finslerian manifold. Suppose that ∇ and $\tilde{\nabla}$ are two connections on M and $\tilde{\nabla}$, obtained from ∇ by Matsumoto’s L -process. Then, we have the followings:*

- (1) *Their vv -curvatures coincide;*
- (2) *If their hh -curvatures coincide, then F is a generalized Landsberg metric. Moreover, if M is a compact manifold, then F reduces to a Landsberg metric;*
- (3) *If their hh -curvatures coincide and F is of non-zero scalar flag curvature, then F is a Randers metric;*
- (4) *Their hv -curvatures coincide if and only if F is a Landsberg metric.*

It is well known that vanishing hv -curvatures of Cartan and Berwald’s connections characterize Landsberg metrics and Berwald metrics, respectively. Recently, Shen introduced a new connection in Finslerian geometry, in which vanishing hv -curvature of the connection characterized Riemannian metrics [14]. On the other hand, the Chern, Berwald, and Hashiguchi connections are obtained from the Cartan connection by Matsumoto’s processes, as depicted as follows:



However, the Shen connection can not be constructed by Matsumoto's processes from these well-known connections. Therefore, it is natural to find some kinds of processes on one of these connections, say Chern's connection, which yields the Shen connection. Here, we introduce two new processes on connections called Shen's C and L -processes. We show that the Shen connection is obtained from the Chern connection by Shen's C -process. Studying curvature tensors of two connections related by this process leads us to the following result.

Theorem 1.2. *Let (M, F) be a Finslerian manifold. Suppose that ∇ and $\tilde{\nabla}$ are two connections on M and $\tilde{\nabla}$, obtained from ∇ by Shen's C -process. Then, we have the followings:*

- (1) *If their hh-curvature coincide, then F is a Landsberg metric;*
- (2) *Their hv-curvature coincide if and only if F is Riemannian;*
- (3) *Their vv-curvature coincide.*

Throughout this paper, we use the Cartan's connection on Finslerian manifolds. The h- and v- covariant derivatives are denoted by “;” and “,” respectively. Furthermore, we suppose that the horizontal distribution of connections are the same as Cartan's connection horizontal distribution.

2. Preliminaries

Let M be an n -dimensional C^∞ manifold. Denote by $T_x M$, the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_x M$, the tangent bundle of M .

A Finslerian metric on M is a function $F : TM \rightarrow [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM , and (iii) for each $y \in T_x M$, the following form g_y on $T_x M$ is positive definite:

$$g_y(u, v) := \frac{1}{2} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_x M.$$

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of F_x , define $\mathbf{C}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_x M.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_x M_0$, define the mean Cartan torsion \mathbf{I}_y by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$

and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke's Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [7].

For $y \in T_x M_0$, define the Matsumoto torsion $\mathbf{M}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \left\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \right\},$$

and $h_{ij} := F F_{y^i y^j} = g_{ij} - \frac{1}{F^2} g_{ip} y^p g_{jq} y^q$. A Finslerian metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity was first introduced by Matsumoto [9]. Matsumoto proved that every Randers metric satisfies $\mathbf{M}_y = 0$. Later on, Matsumoto-Hōjō proved that the converse was true, as well [12].

Proposition 2.1. ([11], [12]) A Finslerian metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of \mathbf{C} and \mathbf{I} along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ and the mean Landsberg curvature $\mathbf{J}_y : T_x M \rightarrow \mathbb{R}$, defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k, \quad \text{and} \quad \mathbf{J}_y(u) := J_i(y)u^i,$$

where $L_{ijk} := C_{ijk|s} y^s$, $J_i := I_i|_s y^s$, $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The families $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ and $\mathbf{J} := \{\mathbf{J}_y\}_{y \in TM_0}$ are called the Landsberg curvature and the mean Landsberg curvature. A Finslerian metric is called Landsberg metric and weakly Landsberg metric if $\mathbf{L} = 0$ and $\mathbf{J} = 0$, respectively [17].

The rate of change of \mathbf{L} along geodesics is measured by the generalized Landsberg curvature $\bar{\mathbf{L}}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$, which is defined by $\bar{\mathbf{L}}_y(u, v, w) := \bar{L}_{ijk}(y)u^i v^j w^k$, where $\bar{L}_{ijk} := L_{ijk|s} y^s$.

The geodesics of Finslerian metric F are characterized by the following system of second order ordinary differential equations in local coordinates $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where $G^i(x, y) := \frac{1}{4} g^{il}(x, y) \{ [F^2]_{x^k y^l} y^k - [F]_{x^i}^2 \}$. These local functions G^i define a global vector field on TM_0 as follows:

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$

For $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by

$$\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x,$$

where $B^i_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y)$. \mathbf{B} is called the Berwald curvature. A Finslerian metric is called a Berwald metric if $\mathbf{B} = 0$ ([15], [18]). It is well known that every Berwald metric is a Landsberg metric.

The notion of Riemann's curvature is extended to Finslerian metrics. For $y \in T_x M_0$, the Riemann's curvature $\mathbf{R}_y : T_x M \rightarrow T_x M$ is defined by $\mathbf{R}_y(u) = R^i_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i_k(y) := 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

Taking an arbitrary plane $P \subset T_x M$ (flag) and a non-zero vector $y \in P$ (flag pole), the flag curvature $K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(\mathbf{R}_y(v), v)}{g_y(y, y)g_y(v, v) - g_y(v, y)g_y(v, y)}.$$

We say that a Finslerian metric F is of a scalar flag curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on TM_0 . If K is constant, then F is said to be of a constant flag curvature.

Consider the pull-back tangent bundle π^*TM over TM_0 , defined by $\pi^*TM = \{(u, v) \in TM_0 \times TM_0 | \pi(u) = \pi(v)\}$. Take a local coordinate system (x^i) in M . Then, the local natural frame $\{\frac{\partial}{\partial x^i}\}$ of $T_x M$ determines a local natural frame $\partial_i|_v$ for π_v^*TM the fibers of π^*TM , where $\partial_i|_v = (v, \frac{\partial}{\partial x^i}|_x)$, and $v = y^i \frac{\partial}{\partial x^i}|_x \in TM_0$. The fiber π_v^*TM is isomorphic to $T_{\pi(v)}M$, where $\pi(v) = x$. There is a canonical section ℓ of π^*TM , defined by $\ell_v = (v, v)/F(v)$.

Let TTM be the tangent bundle of TM and ρ be the canonical linear mapping $\rho : TTM_0 \rightarrow \pi^*TM$, defined by $\rho(\hat{X}) = (z, \pi_*(\hat{X}))$, where $\hat{X} \in T_z TM_0$ and $z \in TM_0$. The bundle map ρ satisfies $\rho(\frac{\partial}{\partial x^i}) = \partial_i$ and $\rho(\frac{\partial}{\partial y^i}) = 0$. Let $V_z TM$ be the set of vertical vectors at z , that is, the set of vectors tangent to the fiber through z , or equivalently $V_z TM = \ker \rho$, called the vertical space.

Let ∇ be a linear connection on π^*TM . Consider the linear mapping $\mu_z : T_z TM_0 \rightarrow T_{\pi z} M$, defined by $\mu_z(\hat{X}) = \nabla_{\hat{X}} F \ell$, where $\hat{X} \in T_z TM_0$. The connection ∇ is called a Finslerian connection if for every $z \in TM_0$, μ_z defines an isomorphism of $V_z TM_0$ onto $T_{\pi z} M$. Therefore, the tangent space TTM_0 in z is decomposed as $T_z TM_0 = H_z TM \oplus V_z TM$, where $H_z TM = \ker \mu_z$ is called the horizontal space defined by ∇ . Indeed, any

tangent vector $\hat{X} \in T_z TM_0$ in z decomposes to $\hat{X} = H\hat{X} + V\hat{X}$, where $H\hat{X} \in H_z TM$ and $V\hat{X} \in V_z TM$.

The structural equations of the Finslerian connection ∇ are:

$$(2.1) \quad \mathcal{T}(\hat{X}, \hat{Y}) = \nabla_{\hat{X}} \hat{Y} - \nabla_{\hat{Y}} \hat{X} - \rho[\hat{X}, \hat{Y}],$$

$$(2.2) \quad \Omega(\hat{X}, \hat{Y})Z = \nabla_{\hat{X}} \nabla_{\hat{Y}} Z - \nabla_{\hat{Y}} \nabla_{\hat{X}} Z - \nabla_{[\hat{X}, \hat{Y}]} Z,$$

where $X = \rho(\hat{X})$, $Y = \rho(\hat{Y})$ and $Z = \rho(\hat{Z})$. The tensors \mathcal{T} and Ω are called respectively the torsion and curvature tensors of ∇ . Three curvature tensors are defined by $R(X, Y) := \Omega(H\hat{X}, H\hat{Y})$, $P(X, \hat{Y}) := \Omega(H\hat{X}, V\hat{Y})$ and $Q(\hat{X}, \hat{Y}) := \Omega(V\hat{X}, V\hat{Y})$, where $\hat{X} = \mu(\hat{X})$ and $\hat{Y} = \mu(\hat{Y})$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to g) frame field for the pulled-back bundle $\pi^* TM$ such that $e_n = \ell$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. One readily finds that $\omega^n := \frac{\partial F}{\partial y^n} dx^n = \omega$, which is called Hilbert form, and $\omega(\ell) = 1$. Put $\nabla e_i = \omega_i^j \otimes e_j$ and $\Omega e_i = 2\Omega_i^j \otimes e_j$, where $\{\Omega_i^j\}$ and $\{\omega_i^j\}$ are called respectively, the curvature forms and connection forms of ∇ with respect to $\{e_i\}$. By definition, $\rho = \omega^i \otimes e_i$ and $\mu := \nabla F \ell = F\omega^{n+i} \otimes e_i$, where $\omega^{n+i} := \omega_n^i + d(\log F)\delta_n^i$. It is easy to show that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. In a natural coordinate, we can expand connection forms ω_i^j as follows:

$$\omega_i^j := \Gamma_{ik}^j dx^k + F_{ik}^j dy^k,$$

where $\nabla_{\frac{\partial}{\partial x^i}} \partial_j = \Gamma_{ij}^k \partial_k$ and $\nabla_{\frac{\partial}{\partial y^i}} \partial_j = F_{ij}^k \partial_k$. In the rest of the paper,

we suppose that all connections satisfy $F_{ij}^k y^i = F_{ij}^k y^j = 0$.

Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM$, $\dot{e}_i \in VTM$, such that $\rho(\bar{e}_i) = e_i$, $\mu(\dot{e}_i) = Fe_i$. Then, equations (2.1) and (2.2) are equivalent to:

$$(2.3) \quad d\omega^i - \omega^j \wedge \omega_j^i = \frac{1}{2} S_{kl}^i \omega^k \wedge \omega^l + T_{kl}^i \omega^k \wedge \omega^{n+l},$$

$$(2.4) \quad d\omega_i^j - \omega_i^k \wedge \omega_k^j = \Omega_i^j,$$

where $T_{kl}^i := \omega^i(\mathcal{T}(\dot{e}_l, \bar{e}_k))$ and $S_{kl}^i := \omega^i(\mathcal{T}(\bar{e}_k, \bar{e}_l))$. Since the Ω_j^i are 2-forms on TM_0 , they can be expanded as:

$$(2.5) \quad \Omega_i^j = \frac{1}{2} R_{i\ kl}^j \omega^k \wedge \omega^l + P_{i\ kl}^j \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_{i\ kl}^j \omega^{n+k} \wedge \omega^{n+l}.$$

The objects R , P and Q are called, respectively, the hh-, hv- and vv-curvature tensors of ∇ with the components $R(\bar{e}_k, \bar{e}_l)e_i = R_{i\ kl}^j e_j$,

$P(\bar{e}_k, \dot{e}_l)e_i = P_i^j{}_{kl}e_j$ and $Q(\dot{e}_k, \dot{e}_l)e_i = Q_i^j{}_{kl}e_j$. By (2.5), we have $R_i^j{}_{kl} = -R_i^j{}_{lk}$ and $Q_i^j{}_{lk} = -Q_i^j{}_{kl}$.

3. Matsumoto's C and L -processes

Matsumoto introduced two processes in connection theory that by them, one can construct the Berwald, Hashiguchi and Chern connections from Cartan's connection [10]. The space of all connections makes an affine space modeled on the space of $(1, 2)$ -tensors over pulled-back bundle π^*TM . It means that adding a $(1, 2)$ -tensor to a connection makes a new connection. A Finslerian metric F gives us two natural $(1, 2)$ -tensors with components $C^i{}_{jk}$ ($=g^{il}C_{ljk}$) and $L^i{}_{jk}$ ($=g^{il}L_{ljk}$). These two $(1, 2)$ -tensors play key roles in Matsumoto's processes, and in what we call Shen's processes, here. The C -processes use Cartan's tensor, and the L -processes use Landsberg's tensor.

Let (M, F) be a Finslerian manifold. Suppose that ∇ is a connection with connection forms ω_j^i . Define

$$(3.1) \quad \tilde{\omega}_j^i := \omega_j^i - C^i{}_{jk}\omega^{n+k}.$$

Then, $\tilde{\omega}_j^i$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from ∇ by Matsumoto's C -process. Similarly, define

$$(3.2) \quad \tilde{\omega}_j^i := \omega_j^i + L^i{}_{jk}\omega^k.$$

Then, $\tilde{\omega}_j^i$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from ∇ by Matsumoto's L -process. The Chern and Hashiguchi connections are obtained from Cartan's connection by Matsumoto's C -process, and Matsumoto's L -process, respectively.

3.1. Proof of Theorem 1.1. First, we recall the following well-known result from [9].

Lemma 3.1. *Let (M, F) be a Finslerian manifold and the Cartan tensor satisfy $C_{ijk} = B_i h_{jk} + B_j h_{ik} + B_k h_{ij}$ such that $y^i B_i = 0$. Then, F is a Randers metric.*

To prove Theorem 1.1, we need the following.

Proposition 3.2. *Let (M, F) be a generalized Landsberg space. Suppose $c(t)$ is a geodesic. Put $\mathbf{C}(t) = \mathbf{C}_c(U(t), V(t), W(t))$, where $U(t), V(t)$*

and $W(t)$ are the parallel vector fields along c . Then, the following equation holds:

$$(3.3) \quad \mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

Proof. Let p be an arbitrary point of M , $y, u, v, w \in T_pM$ and $c : (-\infty, \infty) \rightarrow M$ be the unit speed geodesic passing from p and $\frac{dc}{dt}(0) = y$. For $U(t), V(t)$ and $W(t)$ are the parallel vector fields along c with $U(0) = u, V(0) = v$ and $W(0) = w$, we put

$$\mathbf{L}(t) = \mathbf{L}_c(U(t), V(t), W(t)).$$

By definition of Landsberg's curvature, we have

$$(3.4) \quad \mathbf{L}(t) = \mathbf{C}'(t).$$

Let

$$(3.5) \quad \bar{\mathbf{L}}(t) = \bar{\mathbf{L}}_c(U(t), V(t), W(t)).$$

From the definition of $\bar{\mathbf{L}}_y$, we have

$$(3.6) \quad \bar{\mathbf{L}}(t) = \mathbf{L}'(t).$$

Since F is the generalized Landsberg metric, then we have

$$(3.7) \quad \mathbf{L}'(t) = 0,$$

which implies that $\mathbf{L}(t) = \mathbf{L}(0)$. By (3.4), the proof is complete. \square

Proof of Theorem 1.1: Let $\tilde{\nabla}$ be obtained from ∇ by Matsumoto's L -process,

$$\tilde{\omega}_j^i = \omega_j^i + L^i_{jk}\omega^k.$$

An exterior differential of the above relation yields

$$(3.8) \quad d\tilde{\omega}_j^i = d\omega_j^i + dL^i_{jk} \wedge \omega^k + L^i_{jk}d\omega^k.$$

On the other hand, we know that

$$(3.9) \quad dL^i_{jk} + L^s_{jk}\omega_s^i - L^i_{sk}\omega_j^s - L^i_{js}\omega_k^s = L^i_{jk|s}\omega^s + L^i_{jk.s}\omega^{n+s},$$

where “|” and “.” denote the horizontal and vertical derivatives with respect to ∇ , respectively. Using (2.3), (2.4), (3.8) and (3.9), we have

$$(3.10) \quad \begin{aligned} \tilde{\Omega}_j^i &= \Omega_j^i + (L^i_{jk|s}\omega^s + L^i_{jk.s}\omega^{n+s} - L^s_{jk}\omega_s^i + L^i_{sk}\omega_j^s + L^i_{js}\omega_k^s) \wedge \omega^k \\ &\quad - L^i_{ju} \left(\frac{1}{2} S^u_{kl}\omega^l + T^u_{kl}\omega^{n+l} \right) \wedge \omega^k + L^i_{jk}\omega^s \wedge \omega_s^k \\ &\quad - (\omega_j^k + L^k_{ju}\omega^u) \wedge (\omega_k^i + L^i_{km}\omega^m) + \omega_j^k \wedge \omega_k^i. \end{aligned}$$

Replacing (2.5) in (3.10) yields:

$$(3.11) \quad \tilde{R}_{j\ kl}^i = R_{j\ kl}^i - (L_{jk|l}^i - L_{jl|k}^i) - (L_{jk}^m L_{ml}^i - L_{jl}^m L_{mk}^i) + L_{ju}^i S_{kl}^u,$$

$$(3.12) \quad \tilde{P}_{j\ kl}^i = P_{j\ kl}^i - L_{jk;l}^i + L_{ju}^i T_{kl}^u,$$

$$(3.13) \quad \tilde{Q}_{j\ kl}^i = Q_{j\ kl}^i.$$

Immediately, we have the proof of part 1. It results that if $\tilde{\nabla}$ is obtained from ∇ by Matsumoto's L -process, then ∇ is torsion-free if and only if $\tilde{\nabla}$ is torsion-free.

Proof of part 2. Let $\tilde{R} = R$. By (3.11), we have

$$(3.14) \quad L_{jk|l}^i = L_{jl|k}^i - L_{jk}^m L_{ml}^i + L_{jl}^m L_{mk}^i + L_{ju}^i S_{kl}^u.$$

Regularity of ∇ results in $y^l|_k = 0$. Therefore, by contracting (3.14) with y^l , we get $L_{jk|l}^i y^l = 0$. By our assumption on connections, we have $L_{jk|l}^i y^l = L_{jk;l}^i y^l$. Hence, F is a generalized Landsberg metric.

Now, suppose that M is a compact manifold. By Proposition 3.2, we have

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

Since M is compact, then the Cartan tensor is bounded. Using $\|\mathbf{C}\| < \infty$, and letting $t \rightarrow +\infty$ or $t \rightarrow -\infty$, we get $L(0) = L(u, v, w) = 0$. It means that F is a Landsberg metric.

Proof of part 3. From [15], Finslerian manifolds of scalar flag curvature satisfy

$$L_{ijk|m} y^m = \frac{-F^2}{3} \{K_{.i} h_{jk} + K_{.j} h_{ik} + K_{.k} h_{ij} + 3KC_{ijk}\}.$$

By part 2, F is a generalized Landsberg metric. Then, we get

$$C_{ijk} = \frac{-1}{3K} \{K_{.i} h_{jk} + K_{.j} h_{ik} + K_{.k} h_{ij}\}.$$

By Lemma 3.1, it turns out that F is a C -reducible metric. Thus, by Proposition 2.1, F is a Randers metric.

Proof of part 4. Suppose $\tilde{P} = P$. By (3.12), we have $L_{jk;l}^i = L_{ju}^i T_{kl}^u$. Contracting with y^k , yields $L_{jk}^i = 0$, since Landsberg's tensor is positively homogeneous of degree zero and $T_{kl}^u y^k = 0$. This completes the proof. \square

Corollary 3.3. *Let $\tilde{\nabla}$ be obtained from ∇ by Matsumoto's L -process. Then, the hv-curvature of them under the L -process is invariant if and only if ∇ coincides with $\tilde{\nabla}$.*

It is obvious that any Landsberg metric is a generalized Landsberg metric but the converse is still an open problem. Following corollary 3.3 throws a light into this problem.

Corollary 3.4. *Let $\tilde{\nabla}$ be obtained from ∇ by Matsumoto's L -process. Suppose that their Riemannian curvature coincide. If their hv-curvatures are not equal, then F is a generalized Landsberg metric which is not Landsbergian.*

Now, we consider Matsumoto's C -process. By the same argument and technique used in the proof of Theorem 1.1, one can obtain the following theorem.

Theorem 3.5. *Let (M, F) be a Finslerian manifold. Suppose that ∇ and $\tilde{\nabla}$ are two connections on M . Suppose that $\tilde{\nabla}$ is obtained from ∇ by Matsumoto's C -process. Then, we have the followings:*

$$(3.15) \quad \tilde{R}_{j \ kl}^i = R_{j \ kl}^i - C_{ju}^i R_{n \ lk}^u,$$

$$(3.16) \quad \tilde{P}_{j \ kl}^i = P_{j \ kl}^i - C_{jl|k}^i - C_{ju}^i P_{n \ kl}^u,$$

$$(3.17) \quad \tilde{Q}_{j \ kl}^i = Q_{j \ kl}^i + (C_{jk.l}^i - C_{jl.k}^i) + (C_{jl}^{ru} C_{uk}^i - C_{jk}^{ru} C_{ul}^i) - C_{ju}^i Q_{n \ kl}^u.$$

3.2. Matsumoto's L -process on Randers Manifolds. An (α, β) -metric is a scalar function on TM defined by

$$F := \alpha \phi\left(\frac{\beta}{\alpha}\right), \quad s = \beta/\alpha,$$

where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$. With certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M . Randers metrics are special (α, β) -metrics which are closely related to Riemannian metrics defined by $\phi = 1 + s$; i.e., $F = \alpha + \beta$ and have important applications both in mathematics and physics [13].

Here, we study the Matsumoto's L -process on Randers manifolds equipped with connections whose hh-torsion vanish. We show that the Riemannian curvature of these connections is invariant under Matsumoto's L -process on a Randers manifold (M, F) , if and only if F is a Berwald metric. To prove this result, we need the following.

Lemma 3.6. *Let (M, F) be a Finslerian manifold and ∇ be a connection on M satisfying $g_{ij|k} = 0$. Suppose that $\tilde{\nabla}$ is obtained from ∇ by Matsumoto's L -process. Then, $R = \tilde{R}$ if and only if the following equations hold:*

$$(3.18) \quad L_{isk}L^s_{jl} - L_{isl}L^s_{jk} = 0,$$

$$(3.19) \quad L_{ijl|k} - L_{ijk|l} = 0.$$

Proof. Fix k and l and put

$$Q_{ij} := L_{ijl|k} - L_{ijk|l} + L_{isk}L^s_{jl} - L_{isl}L^s_{jk}.$$

One can write

$$Q_{ij} := Q_{ij}^s + Q_{ij}^a,$$

where,

$$Q_{ij}^s := \frac{1}{2}(Q_{ij} + Q_{ji}), \quad \text{and} \quad Q_{ij}^a := \frac{1}{2}(Q_{ij} - Q_{ji}).$$

It is easy to see that $Q_{ij} = 0$ if and only if $Q_{ij}^s = 0$ and $Q_{ij}^a = 0$. On the other hand, we have

$$\begin{aligned} Q_{ji} &= L_{jil|k} - L_{jik|l} + L_{jks}L^s_{il} - L_{jsl}L^s_{ik} \\ &= L_{ijl|k} - L_{ijk|l} + L^s_{jk}L_{sil} - L^s_{jl}L_{sik}. \end{aligned}$$

Hence,

$$Q_{ij}^s = L_{ijl|k} - L_{ijk|l},$$

and consequently,

$$Q_{ij}^a = L_{isk}L^s_{jl} - L_{isl}L^s_{jk}.$$

This proves the lemma. \square

Now, we are ready to prove the indicated fact.

Theorem 3.7. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$. Suppose that ∇ has vanishing hh-torsion and $g_{ij|k} = 0$. Let $\tilde{\nabla}$ be obtained from ∇ by Matsumoto's L -process. Then, their hh-curvatures are the same if and only if F is a Berwald metric.*

Proof. Using the assumptions $S_{kl}^i = 0$ and $R = \tilde{R}$ in (3.11) imply:

$$(3.20) \quad L^i_{jl|k} - L^i_{jk|l} + L^i_{sk}L^s_{jl} - L^i_{sl}L^s_{jk} = 0.$$

Using the assumption $g_{ij|k} = 0$, and lowering indices by g_{ij} imply that (3.20) is equivalent to:

$$(3.21) \quad L_{ijl|k} - L_{ijk|l} + L_{isk}L^s_{jl} - L_{isl}L^s_{jk} = 0.$$

By Lemma 3.6, we have

$$(3.22) \quad L_{isk}L^s_{jl} - L_{isl}L^s_{jk} = 0,$$

$$(3.23) \quad L_{ijl|k} - L_{ijk|l} = 0.$$

A direct computation yields:

$$(3.24) \quad h_i^s J_s = J_i,$$

$$(3.25) \quad h_i^s h_{js} = h_{ij},$$

$$(3.26) \quad g^{ij} h_{ij} = n - 1.$$

Since F is a Randers metric, then it is C-reducible; i.e.,

$$(3.27) \quad C_{ijk} = \frac{1}{1+n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}.$$

Taking a horizontal covariant derivative from the above relation, we get

$$(3.28) \quad L_{ijk} = \frac{1}{1+n} \{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}.$$

Substituting (3.28) into (3.22), one obtains:

$$(3.29) \quad \{h_{jl}h_{ki} - h_{jk}h_{li}\}J^s J_s + \{h_{jl}J_k - h_{jk}J_l\}J_i + \{h_{ki}J_l - h_{li}J_k\}J_j = 0.$$

Contracting (3.29) with $g^{il}g^{jk}$ and using the relations (3.24), (3.25) and (3.26), we conclude:

$$(3.30) \quad (n+1)(n-2)J^s J_s = 0.$$

Since F is positive definite and $n > 2$, then we have

$$(3.31) \quad J_s = 0.$$

By (3.28) and (3.31), we conclude that F is a Landsberg metric. It is proved that $F = \alpha + \beta$ is a Landsberg metric if and only if F is a Berwald metric [9]. This completes the proof. \square

4. Shen's C and L -processes

Recently, Shen introduced a new torsion-free and almost metric - compatible connection and proved that the hv-curvature of his connection vanishes if and only if the Finslerian structure is Riemannian [14]. However, the hv-curvature tensor of the Berwald, Cartan, Hashiguchi or the Chern connections does not characterize Riemannian structures. Shen's connection can not be constructed by Matsumoto's processes from the Cartan or Chern connection. Therefore, it is natural to find a kind of process on the Chern connection which yields the Shen connection. This problem leads us to find two new processes which we call them Shen's C and L -processes.

Let (M, F) be a Finslerian manifold. Suppose that ∇ is a connection with connection forms ω_j^i . Define

$$(4.1) \quad \tilde{\omega}_j^i := \omega_j^i - C^i_{jk} \omega^k.$$

Then, the $\tilde{\omega}_j^i$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from ∇ by Shen's C -process. Similarly, we can define

$$(4.2) \quad \tilde{\omega}_j^i := \omega_j^i - L^i_{jk} \omega^{n+k}.$$

Then, $\tilde{\omega}_j^i$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from ∇ by Shen's L -process.

Theorem 4.1. *Shen's connection is obtained from the Chern connection by Shen's C -process.*

4.1. Proof of Theorem 1.2. Let $\tilde{\nabla}$ be obtained from ∇ by Shen's C -process. Taking exterior differential from (4.1) yields

$$(4.3) \quad d\tilde{\omega}_j^i = d\omega_j^i - dC^i_{jk} \wedge \omega^k - C^i_{jk} d\omega^k.$$

On the other hand, we have

$$(4.4) \quad dC^i_{jk} + C^s_{jk} \omega_s^i - C^i_{sk} \omega_j^s - C^i_{js} \omega_k^s = C^i_{jk|s} \omega^s + C^i_{jk.s} \omega^{n+s},$$

where “|” and “.” denote the horizontal and vertical derivatives with respect to ∇ , respectively. Substituting (4.4) into (4.3), and using (2.3)

and (2.4) we get

$$\begin{aligned} \tilde{\Omega}_j^i &= \Omega_j^i - (C^i_{jk|s}\omega^s + C^i_{jk.s}\omega^{n+s} - C^s_{jk}\omega_s^i + C^i_{sk}\omega_j^s + C^i_{js}\omega_k^s) \wedge \omega^k \\ &\quad - (\omega_j^k - C^k_{jm}\omega^m) \wedge (\omega_k^i - C^i_{kl}\omega^l) + \omega_j^k \wedge \omega_k^i - C^i_{jk}\omega^s \wedge \omega_s^k \\ (4.5) \quad &+ C^i_{ju}(\frac{1}{2}S^u_{kl}\omega^l + T^u_{kl}\omega^{n+l}) \wedge \omega^k. \end{aligned}$$

Now, by decomposing $\tilde{\Omega}_j^i$ and Ω_j^i as in (2.5), one obtain:

$$(4.6) \quad \tilde{R}^i_{j\ kl} = R^i_{j\ kl} + (C^m_{jk}C^i_{ml} - C^m_{jl}C^i_{mk}) + (C^i_{jk|l} - C^i_{j|lk}) - C^i_{ju}S^u_{kl},$$

$$(4.7) \quad \tilde{P}^i_{j\ kl} = P^i_{j\ kl} - C^i_{jk.l} - C^i_{jm}T^m_{kl},$$

$$(4.8) \quad \tilde{Q}^i_{j\ kl} = Q^i_{j\ kl}.$$

Proof of part 1. Suppose $R = \tilde{R}$. Then, by (4.6) we have

$$(4.9) \quad C^m_{jk}C^i_{ml} - C^m_{jl}C^i_{mk} = C^i_{j|lk} - C^i_{jk|l} + C^i_{ju}S^u_{kl}.$$

Contracting with y^l yields $L^i_{jk} = 0$. This means that F is Landsbergian.

Proof of part 2. Suppose $P = \tilde{P}$. Then, from (4.7) we conclude:

$$(4.10) \quad C^i_{jk.l} + C^i_{jm}T^m_{kl} = 0.$$

Using the positively homogeneities of Cartan's tensor, and contracting (4.10) with y^l , yield $C^i_{jk} = 0$. Therefore, by Deicke's theorem F is Riemannian.

Finally, from (4.8), we see that their vv-curvatures are the same and ∇ is torsion-free, if and only if $\tilde{\nabla}$ is torsion-free. \square

We have some kind of rigidity on Shen's C -process.

Corollary 4.2. *Let $\tilde{\nabla}$ be obtained from ∇ by Shen's C -process. Then, the hv-curvature is invariant under Shen's C -process if and only if $\nabla = \tilde{\nabla}$.*

Now, we study Shen's L -process and get the following result.

Theorem 4.3. *Let (M, F) be a Finslerian manifold. Suppose ∇ and $\tilde{\nabla}$ are two connections on M and ∇ , obtained from $\tilde{\nabla}$ by Shen's L -process. Then, we have the followings:*

$$(4.11) \quad \tilde{R}^i_{j\ kl} = R^i_{j\ kl} - L^i_{ju}R^u_{n\ lk},$$

$$(4.12) \quad \tilde{P}^i_{j\ kl} = P^i_{j\ kl} - L^i_{j|lk} - L^i_{ju}P^u_{n\ kl},$$

$$(4.13) \quad \tilde{Q}^i_{j\ kl} = Q^i_{j\ kl} + (L^i_{jk.l} - L^i_{j|lk}) + (L^u_{jl}L^i_{uk} - L^u_{jk}L^i_{ul}) - L^i_{ju}Q^u_{n\ kl}.$$

Corollary 4.4. *If torsion-free connection ∇ on the Finslerian manifold (M, F) remains torsion-free under Shen's L -process, then F is a Landsberg metric. Hence, Shen's L -process acts on the set of all torsion-free connections identically.*

Corollary 4.5. *Let ∇ be obtained from $\tilde{\nabla}$ by Shen's L -process and $\tilde{\nabla}$ be not torsion-free. If their $h\nu$ -curvature are equal to zero, then F is a generalized Landsberg metric which is not Landsbergian.*

4.2. Shen's C -process on Berwald Connection. By Theorem 4.1, applying Shen's C -process on Chern's connection gives Shen's connection. It is natural to study the effect of Shen's C -process on the other well-known connections. Here, we study Shen's C -process on Berwald's connection.

Theorem 4.6. *Let (M, F) be a Finslerian manifold. Suppose that ∇ is the Berwald's connection on M and $\tilde{\nabla}$ is obtained from ∇ by Shen's C -process. Then, the $h\nu$ -curvature of $\tilde{\nabla}$ vanishes if and only if F is Riemannian.*

Proof. The structure equation of $\tilde{\nabla}$ is given by:

$$(4.14) \quad d\omega^i = \omega^j \wedge \omega_j^i,$$

$$(4.15) \quad dg_{ij} = g_{kj}\omega_i^k + g_{ik}\omega_j^k + 2\{A_{ijk} - L_{ijk}\}\omega^k + 2A_{ijk}\omega^{n+k},$$

where $A_{ijk} := FC_{ijk}$. Differentiating (4.15) and using (2.4), (4.14) and (4.15), we have

$$(4.16) \quad \begin{aligned} g_{kj}\Omega_i^k + g_{ik}\Omega_j^k = & -2A_{ijk}\Omega_n^k - 2A_{ijk|l}\omega^k \wedge \omega^l + 2A_{ijk.l}\omega^{n+k} \wedge \omega^{n+l} \\ & - 2\{A_{ijk.l} - A_{ijk|l}\}\omega^k \wedge \omega^{n+l} \\ & + (L_{ijk|l}\omega^l + L_{ijk.l}\omega^{n+l}) \wedge \omega^k. \end{aligned}$$

Using (2.5), yields:

$$(4.17) \quad R_{ijkl} + R_{jikl} = -2A_{ijs}R_n^s{}_{kl},$$

$$(4.18) \quad P_{ijkl} + P_{jikl} = -2L_{ijk.l} + 2\{A_{ijk.l} - A_{ijl|k}\} - 2A_{ijs}P_n^s{}_{kl},$$

$$(4.19) \quad A_{ijk.l} = A_{ijl.k}.$$

Permuting i, j, k in (4.18) implies:

$$(4.20) \quad \begin{aligned} P_{ijkl} = & -L_{ijk.l} + A_{ijk.l} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ & + A_{kis}P_n^s{}_{jl} - A_{jks}P_n^s{}_{il} - A_{ijs}P_n^s{}_{kl}. \end{aligned}$$

Multiplying (4.20) by y^i and using $P_{njnl} = 0$, yield:

$$(4.21) \quad P_{njkl} = -A_{jkl}.$$

By (4.21), we get the proof. \square

4.3. Shen's C -process on Cartan's Connection. Here, we study the effect of Shen's C -process on the Cartan connection.

Theorem 4.7. *Let (M, F) be a Finslerian manifold. Suppose that ∇ is the Cartan's connection on M and $\tilde{\nabla}$ is obtained from ∇ by Shen's C -process. Then, we have*

- (1) *if hh-curvature of $\tilde{\nabla}$ vanishes, then F is a Landsberg metric;*
- (2) *the hv-curvature of $\tilde{\nabla}$ vanishes, if and only if F is Riemannian.*

Proof. The structure equation of $\tilde{\nabla}$ is given by:

$$(4.22) \quad d\omega^i = \omega^j \wedge \omega_j^i - A^i_{kl} \omega^k \wedge \omega^{n+l},$$

$$(4.23) \quad dg_{ij} = g_{kj} \omega_i^k + g_{ik} \omega_j^k + 2A_{ijk} \omega^k.$$

Differentiating (4.23) and using (2.4), (4.22) and (4.23) lead to:

$$(4.24) \quad \begin{aligned} g_{kj} \Omega_i^k + g_{ik} \Omega_j^k = & - 2(A_{ijk|s} \omega^s + 2A_{ijk.s} \omega^{n+s}) \wedge \omega^k \\ & - 2A_{ijs} A^s_{kl} \omega^k \wedge \omega^{n+l}. \end{aligned}$$

Using (2.5), yields:

$$(4.25) \quad R_{ijkl} + R_{jikl} = 2(A_{ijk|l} - A_{ijl|k}),$$

$$(4.26) \quad P_{ijkl} + P_{jikl} = 2(A_{ijk.l} - A_{ijs} A^s_{kl}),$$

$$(4.27) \quad Q_{ijkl} + Q_{jikl} = 0.$$

If the hh-curvature of $\tilde{\nabla}$ vanishes, then by (4.25) we have $A_{ijk|l} = A_{ijl|k}$, which implies that F is a Landsberg metric.

Now, let the hv-curvature of $\tilde{\nabla}$ vanish. By (4.26), we get

$$(4.28) \quad A_{ijk.l} = A_{ijs} A^s_{kl}.$$

Contracting (4.28) with y^l yields that F is Riemannian. \square

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