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HYERS-ULAM-RASSIAS STABILITY OF A COMPOSITE FUNCTIONAL EQUATION IN VARIOUS NORMED SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam (or Hyers-Ulam-Rassias) stability of the following composite functional equation

$$f(f(x)-f(y))+f(x)+f(y)=f(x+y)+f(x-y),$$

in various normed spaces.

1. Introduction and preliminaries

Let Γ^+ denote the set of all probability distribution functions $F:\mathbb{R}\cup[-\infty,+\infty]\to[0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0)=0, F(+\infty)=1$. It is clear that the set $D^+=\{F\in\Gamma^+ : l^-F(-\infty)=1\}$, where $l^-f(x)=\lim_{t\to x^-}f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F\leq G$ if and only if $F(t)\leq G(t)$ for all $t\in\mathbb{R}$. For any $a\geq 0$, the element $H_a(t)$ of D^+ is defined by $H_a(t)=\begin{cases} 0, & \text{if} \quad t\leq a, \\ 1, & \text{if} \quad t>a \end{cases}$.

Definition 1.1. A function $T:[0,1]^2\to[0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;

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(b) T is continuous;
(c) $T(x,1) = x$ for all $x \in [0,1]$;
(d) $T(x,y) \le T(z,w)$ whenever $x \le z$ and $y \le w$ for all $x,y,z,w \in [0,1]$.

Definition 1.2. A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \ge 0$;
(c) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \ge 0$.

Definition 1.3. *By a* non-Archimedean field *we mean a field* $\mathbb{K}$ *equipped with a function (valuation)* $|\cdot| : \mathbb{K} \to [0,\infty)$ *such that for all* $r,s \in \mathbb{K}$, *the following conditions hold:*
(i) $|r| = 0$ *if and only if* $r = 0$;
(ii) $|rs| = |r||s|$;
(iii) $|r+s| \le max\{|r|,|s|\}$.

Remark 1.4. *Clearly* $|1| = |-1| = 1$ *and* $|n| \le 1$ *for all* $n \in \mathbb{N}$.

Definition 1.5. *Let* X *be a vector space over a scalar field* $\mathbb{K}$ *with a non-Archimedean non-trivial valuation* $|\cdot|$. *A function* $||\cdot|| : X \to \mathbb{R}$ *is a* non-Archimedean norm *(valuation) if it satisfies the following conditions:*
(i) $||x|| = 0$ *if and only if* $x = 0$;
(ii) $||rx|| = |r|||x||$ $(r \in \mathbb{K}, x \in X)$;
(iii) *The strong triangle inequality (ultrametric); namely* $||x+y|| \le max\{||x||,||y||\},\ x,y \in X$.
Then $(X, ||\cdot||)$ *is called a non-Archimedean space.*

Definition 1.6. *A sequence* $\{x_n\}$ *is* Cauchy *if and only if* $\{x_{n+1} - x_n\}$ *converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.*

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x,y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$".

Example 1.7. *Fix a prime number* p. *For any nonzero rational number* x, *there exists a unique integer* $n_x \in \mathbb{Z}$ *such that* $x = \frac{a}{b}p^{n_x}$, *where* a *and* b *are integers not divisible by* p. *Then* $|x|_p := p^{-n_x}$ *defines a non-Archimedean norm on* $\mathbb{Q}$. *The completion of* $\mathbb{Q}$ *with respect to the metric*

$d(x,y)=|x-y|_p$ *is denoted by* $\mathbb{Q}_p$ *which is called the* p-adic number field. *In fact,* $\mathbb{Q}_p$ *is the set of all formal series* $x=\sum_{k\geq n_x}^{\infty} a_k p^k$ *where* $|a_k|\leq p-1$ *are integers. The addition and multiplication between any two elements of* $\mathbb{Q}_p$ *are defined naturally. The norm* $|\sum_{k\geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ *is a non-Archimedean norm on* $\mathbb{Q}_p$ *and it makes* $\mathbb{Q}_p$ *a locally compact filed.*

Arriola and Beyer [1] investigated the Hyers-Ulam stability of approximate additive functions $f:\mathbb{Q}_p\to\mathbb{R}$. They showed that if $f:\mathbb{Q}_p\to\mathbb{R}$ is a continuous function for which there exists a fixed ϵ:

$$|f(x+y)-f(x)-f(y)|\leq\epsilon$$

for all $x,y\in\mathbb{Q}_p$, then there exists a unique additive function $T:\mathbb{Q}_p\to\mathbb{R}$ such that

$$|f(x)-T(x)|\leq\epsilon$$

for all $x\in\mathbb{Q}_p$.
However, the following example shows that similar result is not true in non-Archimedean normed spaces.

Example 1.8. *Let* $p>2$ *and let* $f:\mathbb{Q}_p\to\mathbb{Q}_p$ *be defined by* $f(x)=2$. *Then for* $\epsilon=1$,

$$|f(x+y)-f(x)-f(y)|=1\leq\epsilon$$

for all $x,y\in\mathbb{Q}_p$. *However, the sequences* $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ *and* $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ *are not Cauchy. In fact, by using the fact that* $|2|=1$, *we have*

$$\left|\frac{f(2^n x)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}}\right| = |2^{-n}\cdot 2 - 2^{-(n+1)}\cdot 2| = |2^{-n}| = 1$$

and

$$\left|2^n f\left(\frac{x}{2^n}\right)-2^{n+1}f\left(\frac{x}{2^{n+1}}\right)\right| = |2^n.2-2^{(n+1)}.2| = |2^{n+1}| = 1$$

for all $x,y\in\mathbb{Q}_p$ *and* $n\in\mathbb{N}$. *Hence these sequences are not convergent in* $\mathbb{Q}_p$.

Definition 1.9. *Let* X *be a set. A function* $d:X\times X\to[0,\infty]$ *is called a generalized metric on* X *if* d *satisfies the following conditions:*
(a) $d(x,y)=0$ *if and only if* $x=y$ *for all* $x,y\in X$;
(b) $d(x,y)=d(y,x)$ *for all* $x,y\in X$;
(c) $d(x,z)\leq d(x,y)+d(y,z)$ *for all* $x,y,z\in X$.

Theorem 1.10. *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.1}$$

for all nonnegative integers n, or there exists a positive integer n_0 such that
(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;
(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

A classical question in the theory of functional equations is the following:"When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem admits a solution, we say that the equation is *stable.* The first stability problem concerning group homomorphisms was raised by Ulam [31] in 1940. In the following year, Hyers [10] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [19] proved a generalization of Hyers' theorem for additive mappings. The result of Rassias has provided a significant influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias's theorem was obtained by Găvruta [8] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. In 1897, Hensel [9] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [11, 12].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([2]- [8], [14]– [29]).

In Sections 2 and 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [30].

In this paper, we prove the Hyers-Ulam-Rassias stability of the functional equation

$$f(f(x)-f(y))+f(x)+f(y)=f(x+y)+f(x-y) \tag{1.2}$$

in random and non-Archimedean normed spaces.

2. Random stability of the functional equation (1.2): a direct method

In this section, using a direct method, we prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in random normed spaces.

Theorem 2.1. *Let X be a real linear space, $(Z,\mu',\min)$ an RN-space and $\varphi: X^2 \to Z$ a function such that there exists $0<\alpha<\frac{1}{2}$ with*

$$\mu'_{\varphi\left(\frac{x}{2},\frac{y}{2}\right)}(t) \geq \mu'_{\alpha\varphi(x,y)}(t) \tag{2.1}$$

for all $x \in X$ and $t>0$ and

$$\lim_{n\to\infty} \mu'_{\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\left(\frac{t}{2^n}\right)=1$$

for all $x,y \in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f: X \to Y$ is a mapping such that

$$\mu_{f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{2.2}$$

for all $x,y \in X$ and $t>0$. Then the limit

$$A(x)=\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$ and defines a unique additive mapping $A: X \to Y$ such that and

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right). \tag{2.3}$$

for all $x \in X$ and $t>0$.

Proof. Putting $y=x$ in (2.2), we see that

$$\mu_{f(2x)-2f(x)}(t) \geq \mu'_{\varphi(x,x)}(t). \tag{2.4}$$

Replacing x by $\frac{x}{2}$ in (2.4), we obtain

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \geq \mu'_{\varphi\left(\frac{x}{2},\frac{x}{2}\right)}(t) \tag{2.5}$$

for all $x \in X$. Replacing x by $\frac{x}{2^n}$ in (2.5) and using (2.1), we obtain

$$\mu_{2^{n+1}f(\frac{x}{2^{n+1}})-2^n f(\frac{x}{2^n})}(t) \geq \mu'_{\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)}\left(\frac{t}{2^n}\right) \geq \mu'_{\varphi(x,x)}\left(\frac{t}{2^n\alpha^{n+1}}\right)$$

and so

$$\begin{aligned}\mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}\left(\sum_{k=0}^{n-1} 2^k\alpha^{k+1}t\right) &= \mu_{\sum_{k=0}^{n-1} 2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)}\left(\sum_{k=0}^{n-1} 2^k\alpha^{k+1}t\right)\\ &\geq T_{M_{k=0}^{n-1}}\left(\mu_{2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)}(2^k\alpha^{k+1}t)\right)\\ &\geq T_{M_{k=0}^{n-1}}\left(\mu'_{\varphi(x,x)}(t)\right) = \mu'_{\varphi(x,x)}(t).\end{aligned}$$

This implies that

$$\mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t) \geq \mu'_{\varphi(x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k\alpha^{k+1}}\right). \tag{2.6}$$

Replacing x by $\frac{x}{2^p}$ in (2.6), we obtain

$$\begin{aligned}\mu_{2^{n+p}f\left(\frac{x}{2^{n+p}}\right)-2^p f\left(\frac{x}{2^p}\right)}(t) &\geq \mu'_{\varphi(x,x)}\left(\frac{t}{\sum_{k=p}^{n+p-1} 2^k\alpha^{k+1}}\right)\\ &\to 1 \text{ when } n \to +\infty,\end{aligned} \tag{2.7}$$

so $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $A(x) \in Y$ such that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x).$$

Fix $x \in X$ and put $p = 0$ in (2.7). Then we obtain

$$\mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t) \geq \mu'_{\varphi(x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k\alpha^{k+1}}\right)$$

and so, for any $\delta > 0$,

$$\begin{aligned}\mu_{A(x)-f(x)}(t+\delta) &\geq T\left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\delta), \mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t)\right)\\ &\geq T\left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\delta), \mu'_{\varphi(x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k\alpha^{k+1}}\right)\right).\end{aligned} \tag{2.8}$$

Taking $n \to \infty$ in (2.8), we get

$$\mu_{A(x)-f(x)}(t+\delta) \geq \mu'_{\varphi(x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right). \tag{2.9}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (2.9), we get

$$\mu_{A(x)-f(x)}(t) \geq \mu'_{\varphi(x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right).$$

Replacing x and y by $\frac{x}{2^n}$ and $\frac{y}{2^n}$ in (2.2), respectively, we get

$$\mu_{2^n\left[f\left(f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)+f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)\right]}(t) \geq \mu'_{\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\left(\frac{t}{2^n}\right)$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\left(\frac{t}{2^n}\right) = 1$, we conclude that A satisfies (1.2). On the other hand,

$$2A\left(\frac{x}{2}\right) - A(x) = \lim_{n\to\infty} 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = 0.$$

This implies that $A : X \to Y$ is an additive mapping. To prove the uniqueness of the additive mapping A, assume that there exists another additive mapping $L : X \to Y$ which satisfies (2.3). Then we have

$$\begin{aligned}\mu_{A(x)-L(x)}(t) &= \lim_{n\to\infty} \mu_{2^n A\left(\frac{x}{2^n}\right)-2^n L\left(\frac{x}{2^n}\right)}(t) \\ &\geq \lim_{n\to\infty} \min\left\{\mu_{2^n A\left(\frac{x}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)}\left(\frac{t}{2}\right), \mu_{2^n f\left(\frac{x}{2^n}\right)-2^n L\left(\frac{x}{2^n}\right)}\left(\frac{t}{2}\right)\right\} \\ &\geq \lim_{n\to\infty} \mu'_{\varphi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}\left(\frac{(1-2\alpha)t}{2^{n+1}\alpha}\right) \geq \lim_{n\to\infty} \mu'_{\varphi(x,x)}\left(\frac{(1-2\alpha)t}{2^{n+1}\alpha^{n+1}}\right).\end{aligned}$$

Since $\lim_{n\to\infty} \frac{(1-2\alpha)t}{2^{n+1}\alpha^{n+1}} = \infty$, we get $\lim_{n\to\infty} \mu'_{\varphi(x,x)}\left(\frac{(1-2\alpha)t}{2^{n+1}\alpha^{n+1}}\right) = 1$. Therefore, it follows that $\mu_{A(x)-L(x)}(t) = 1$ for all $t > 0$ and so $A(x) = L(x)$. This completes the proof. □

Corollary 2.2. *Let X be a real normed linear space, $(Z, \mu', \min)$ an RN-space and $(Y, \mu, \min)$ a complete RN-space. Let r be a positive real number with $r > 1$, $z_0 \in Z$ and $f : X \to Y$ a mapping satisfying*

$$\mu_{f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)}(t) \geq \mu'_{(\|x\|^r+\|y\|^r)z_0}(t) \tag{2.10}$$

for all $x, y \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^p z_0}\left(\frac{(2^r-2)t}{2}\right)$$

for all $x \in X$ *and* $t > 0$.

Proof. Let $\alpha = 2^{-r}$ and let $\varphi : X^2 \to Z$ be a mapping defined by $\varphi(x,y) = (\|x\|^r + \|y\|^r)z_0$. Then, from Theorem 2.1, the conclusion follows. □

Theorem 2.3. *Let* X *be a real linear space,* $(Z, \mu', \min)$ *an RN-space and* $\varphi : X^2 \to Z$ *a function such that there exists* $0 < \alpha < 2$ *such that* $\mu'_{\varphi(2x,2y)}(t) \geq \mu'_{\alpha\varphi(x,y)}(t)$ *for all* $x \in X$ *and* $t > 0$ *and* $\lim_{n\to\infty} \mu'_{\varphi(2^n x, 2^n y)}(2^n t) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *Let* $(Y, \mu, \min)$ *be a complete RN-space. If* $f : X \to Y$ *is a mapping satisfying (2.2). Then the limit* $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that and*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x)}((2-\alpha)t) \tag{2.11}$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = x$ in (2.2), we see that

$$\mu_{\frac{f(2x)}{2} - f(x)}(t) \geq \mu'_{\varphi(x,x)}(2t). \tag{2.12}$$

Replacing x by $2^n x$ in (2.12), we obtain that

$$\mu_{\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}}(t) \geq \mu'_{\varphi(2^n x, 2^n x)}(2^{n+1} t) \geq \mu_{\varphi(x,x)}\left(\frac{2^{n+1} t}{\alpha^n}\right). \tag{2.13}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let* X *be a real normed linear space,* $(Z, \mu', \min)$ *an RN-space and* $(Y, \mu, \min)$ *a complete RN-space. Let* r *be a positive real number with* $0 < r < 1$, $z_0 \in Z$ *and* $f : X \to Y$ *a mapping satisfying (2.10). Then the limit* $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^p z_0}\left(\frac{(2-2^r)t}{2}\right)$$

for all $x \in X$ *and* $t > 0$.

Proof. Let $\alpha = 2^r$ and let $\varphi : X^2 \to Z$ be a mapping defined by $\varphi(x,y) = (\|x\|^r + \|y\|^r)z_0$. Then, from Theorem 2.3, the conclusion follows. □

3. **Random stability of the functional equation (1.2): a fixed point method**

Throughout this section, using a fixed point method, we prove Hyers-Ulam-Rassias stability of functional equation (1.2) in RN-spaces.

Theorem 3.1. *Let X be a linear space, (Y, μ, T_M) a complete RN-space and Φ a mapping from X^2 to D^+ such that there exists $0 < \alpha < \frac{1}{2}$ such that*

$$\Phi_{2x,2y}(t) \leq \Phi_{x,y}(\alpha t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$ ($\Phi(x, y)$is denoted by$\Phi_{x,y}$). Let $f : X \to Y$ be a mapping satisfying

$$\mu_{f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)}(t) \geq \Phi_{x,y}(t) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,x}\left(\frac{(1-2\alpha)t}{\alpha}\right) \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = x$ in (3.2) and replacing x by $\frac{x}{2}$, we have

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \geq \Phi_{\frac{x}{2},\frac{x}{2}}(t) \tag{3.4}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by

$$d(f, g) = \inf_{u\in(0,\infty)} \left\{\mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x}(t),\, \forall x \in X,\, t > 0\right\}, \tag{3.5}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [14], Lemma 2.1). Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{3.6}$$

for all $x \in X$.

First, we prove that J is a strictly contractive mapping with the Lipschitz constant 2α. In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,x}(t)$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(2\alpha\epsilon t) = \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(2\alpha\epsilon t) &= \mu_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha\epsilon t)\\ &\geq \Phi_{\frac{x}{2},\frac{x}{2}}(\alpha t)\\ &\geq \Phi_{x,x}(t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) < \epsilon$ implies that $d(Jg, Jh) < 2\alpha\epsilon$. This means that $d(Jg, Jh) \leq 2\alpha d(g,h)$ for all $g, h \in S$. It follows from (3.4) that

$$d(f, Jf) \leq \alpha.$$

By Theorem 1.10, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A\Big(\frac{x}{2}\Big) = \frac{1}{2}A(x) \tag{3.7}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (3.7) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) = A(x)$$

for all $x \in X$.
(3) $d(f, A) \leq \frac{d(f,Jf)}{1-2\alpha}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{\alpha}{1-2\alpha}$$

and so

$$\mu_{f(x)-A(x)}\left(\frac{\alpha t}{1-2\alpha}\right) \geq \Phi_{x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.3) holds. On the other hand, replacing x, y by $\frac{x}{2^n}$ and $\frac{y}{2^n}$, respectively, in (3.2), we have

$$\mu_{2^n\left[f\left(f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)+f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)\right]}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n}}\left(\frac{t}{2^n}\right)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (3.1), it follows that

$$\Phi_{\frac{x}{2^n},\frac{y}{2^n}}\left(\frac{t}{2^n}\right) \geq \Phi_{x,y}\left(\frac{t}{2^n\alpha^n}\right) \to 1 \quad \text{as} \quad n \to +\infty$$

for all $x, y \in X$ and $t > 0$. Therefore

$$\mu_{A(A(x)-A(y))-A(x+y)-A(x-y)+A(x)+A(y)}(t) = 1$$

for all $x, y \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ satisfies (1.2). Furthermore, since for all $x, y \in X$, we have

$$\begin{aligned} A(2x) - 2A(x) &= \lim_{n\to\infty} 2^n f\left(\frac{x}{2^{n-1}}\right) - 2\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \\ &= 2\left[\lim_{n\to\infty} 2^{n-1} f\left(\frac{x}{2^{n-1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)\right] \\ &= 0, \end{aligned}$$

we conclude that $A : X \to Y$ is additive. This completes the proof. □

Corollary 3.2. *Let X be a real normed space, $\theta \geq 0$ and let r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)}(t) \geq \frac{t}{t + \theta\big(\|x\|^r + \|y\|^r\big)} \tag{3.8}$$

for all $x, y \in X$ and $t > 0$. Then $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + 2\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta\big(\|x\|^r + \|y\|^r\big)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-r}$, then we get the desired result. □

Theorem 3.3. *Let X be a linear space, (Y, μ, T_M) a complete RN-space and Φ a mapping from X^2 to D^+ such that for some $0 < \alpha < 2$, $\Phi_{\frac{x}{2},\frac{y}{2}}(t) \leq \Phi_{x,y}(\alpha t)$ for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying (3.2). Then the limit $A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,x}((2 - \alpha)t) \tag{3.9}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = x$ in (3.2), we have

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq \Phi_{x,x}(2t) \tag{3.10}$$

for all $x \in X$ and $t > 0$. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := \frac{1}{2}h(2x)$ for all $x \in X$. It follows from (3.10) that

$$d(f, Jf) \leq \frac{1}{2}.$$

By Theorem 1.10, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{3.11}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (3.11) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x}(t)$ for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.
(3) $d(f, A) \leq \frac{d(f,Jf)}{1-\frac{\alpha}{2}}$ with $f \in \Omega$, which implies the inequality

$$\mu_{f(x)-A(x)}\left(\frac{t}{2-\alpha}\right) \geq \Phi_{x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.9) holds. The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let X be a real normed space, $\theta \geq 0$ and let r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying (3.8). Then the limit $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2-2^r)t}{(2-2^r)t + 2\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.3 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^r$, then we get the desired result. □

4. Non-Archimedean stability of functional equation (1.2): a fixed point method

In this section, using a fixed point approach, we prove the Hyers-Ulam-Rassias stability of functional equation (1.2) in non-Archimedean normed spaces.

Throughout this section, X is a non-Archimedean normed spaces and that Y is a complete non-Archimedean normed spaces. Also we assume that $|2| \neq 1$.

Theorem 4.1. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists $L < 1$ with*

$$|2|\zeta\left(\frac{x}{2}, \frac{y}{2}\right) \leq L\zeta(x, y) \tag{4.1}$$

for all $x, y \in X$. If $f : X \to Y$ is a mapping satisfying

$$\Big\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\Big\| \leq \zeta(x, y) \tag{4.2}$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L\zeta(x, x)}{|2| - |2|L}. \tag{4.3}$$

Proof. Putting $y = x$ in (4.2), we have

$$\Big\|f(2x) - 2f(x)\Big\| \leq \zeta\,(x, x) \tag{4.4}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (4.4), we obtain

$$\Big\|2f\left(\frac{x}{2}\right) - f(x)\Big\| \leq \zeta\left(\frac{x}{2}, \frac{x}{2}\right) \tag{4.5}$$

for all $x \in X$. Consider the set $S^* := \{g : X \to Y\}$ and the generalized metric d^* in S^* defined by

$$d^*(f, g) = \inf\Big\{\mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu\zeta(x, x),\ \forall x \in X\Big\}, \tag{4.6}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S^*, d^*) is complete (see [14], Lemma 2.1). Now, we consider a linear mapping $J^* : S^* \to S^*$ such that

$$J^*h(x) := 2h\left(\frac{x}{2}\right) \tag{4.7}$$

for all $x \in X$. Let $g, h \in S^*$ be such that $d^*(g,h) = \epsilon$. Then we have $\|g(x) - h(x)\| \le \epsilon\zeta(x,x)$ for all $x \in X$ and so

$$\begin{aligned}\|J^*g(x) - J^*h(x)\| = \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| &\le |2|\epsilon\zeta\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le |2|\epsilon\frac{L}{|2|}\zeta(x,x)\end{aligned}$$

for all $x \in X$. Thus $d^*(g,h) = \epsilon$ implies that $d^*(J^*g, J^*h) \le L\epsilon$. This means that $d^*(J^*g, J^*h) \le Ld^*(g,h)$ for all $g, h \in S^*$. It follows from (4.5) that

$$d^*(f, J^*f) \le \frac{L}{|2|}. \tag{4.8}$$

By Theorem 1.10, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J^*, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{4.9}$$

for all $x \in X$. The mapping A is a unique fixed point of J^* in the set $\Omega = \{h \in S^* : d^*(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (4.9) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - A(x)\| \le \mu\zeta(x,x)$ for all $x \in X$.
(2) $d^*(J^{*n}f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$.
(3) $d^*(f, A) \le \frac{d^*(f,J^*f)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d^*(f, A) \le \frac{L}{|2| - |2|L}. \tag{4.10}$$

This implies that the inequality (4.3) holds. By (4.2), we have

$$\begin{aligned}&\left\|2^n\left[f\left(f\left(\tfrac{x}{2^n}\right) - f\left(\tfrac{y}{2^n}\right)\right) - f\left(\tfrac{x+y}{2^n}\right) - f\left(\tfrac{x-y}{2^n}\right) + f\left(\tfrac{x}{2^n}\right) + f\left(\tfrac{y}{2^n}\right)\right]\right\| \\ &\le |2|^n\zeta\left(\tfrac{x}{2^n}, \tfrac{y}{2^n}\right) \le |2|^n \cdot \tfrac{L^n}{|2|^n}\zeta(x,y)\end{aligned}$$

for all $x, y \in X$ and $n \geq 1$ and so $\|f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y)\| = 0$ for all $x, y \in X$. On the other hand

$$2A\left(\frac{x}{2}\right) - A(x) = \lim_{n\to\infty} 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = 0.$$

Therefore, the mapping $A : X \to Y$ is additive. This completes the proof. □

Corollary 4.2. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(f(x) - f(y)) - f(x+y) - f(x-y) + f(x) + f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{4.11}$$

for all $x, y \in X$. Then the limit $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{2|2|\theta\|x\|^p}{|2|^{p+1} - |2|^2}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 if we take $\zeta(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. In fact, if we choose $L = |2|^{1-p}$, then we get the desired result. □

Similarly, we have the following results for which we sketch the proofs.

Theorem 4.3. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\zeta(2x, 2y) \leq |2|L\zeta(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (4.2). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{\zeta(x, x)}{|2| - |2|L}.$$

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{f(2x)}{2}\right\| \leq \frac{\zeta(x, x)}{|2|}$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (4.11). Then the limit $A(x) =$*

$\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in X$ *and* $A : X \to Y$ *is a unique additive mapping such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta\|x\|^p}{|2| - |2|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 if we take $\zeta(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. In fact, if we choose $L = |2|^{p-1}$, then we get the desired result. □

5. Non-Archimedean stability of functional equation (1.2): a direct method

In this section, using a direct method, we prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in non-Archimedean space. Throughout this section, G is an additive semigroup and X is a non-Archimedean Banach space.

Theorem 5.1. *Let* $\zeta : G \times G \to [0, +\infty)$ *be a function such that*

$$\lim_{n\to\infty} |2|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0 \tag{5.1}$$

for all $x, y \in G$. *Suppose that, for any* $x \in G$, *the limit*

$$\Psi(x) = \lim_{n\to\infty} \max\left\{|2|^{k+1}\zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) :\ 0 \leq k < n\right\} \tag{5.2}$$

exists and $f : G \to X$ *is a mapping satisfying*

$$\Big\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\Big\| \leq \zeta(x, y). \tag{5.3}$$

Then, for all $x \in G$, $T(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists and satisfies the inequality*

$$\|f(x) - T(x)\| \leq \frac{1}{|2|}\Psi(x). \tag{5.4}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{|2|^{k+1}\zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) :\ j \leq k < n + j\right\} = 0, \tag{5.5}$$

then T *is the unique additive mapping satisfying* (5.4).

Proof. By (4.5), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \zeta\left(\frac{x}{2}, \frac{x}{2}\right) \tag{5.6}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in (5.6), we obtain

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\| \le |2|^n \zeta\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right). \tag{5.7}$$

Thus, it follows from (5.1) and (5.7) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n\ge1}$ is a Cauchy sequence. Since X is complete, it follows that $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n\ge1}$ is convergent. Set $T(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. By induction, one can show that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \le \frac{\max\left\{|2|^{k+1}\zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) :\ 0 \le k < n\right\}}{|2|} \tag{5.8}$$

for all $n \ge 1$ and $x \in G$. By taking $n \to \infty$ in (5.8) and using (5.2), one obtains (5.4). By (5.1) and (5.3), we get

$$\begin{aligned}
&\left\|T(T(x) - T(y)) - T(x+y) - T(x-y) + T(x) + T(y)\right\| \\
&= \lim_{n\to\infty} \|2^n \left[f\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right)\right. \\
&\left. +f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right)\right] \\
&\le \lim_{n\to\infty} |2|^n \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in G$. Therefore, the mapping $T : G \to X$ satisfies (1.2). To prove the uniqueness property of T, let S be another mapping satisfying (5.4). Then we have

$$\begin{aligned}
\left\|T(x) - S(x)\right\| &= \lim_{j\to\infty} |2|^j \left\|T\left(\frac{x}{2^j}\right) - S\left(\frac{x}{2^j}\right)\right\| \\
&\le \lim_{j\to\infty} |2|^j \max\left\{\left\|T\left(\frac{x}{2^j}\right) - f\left(\frac{x}{2^j}\right)\right\|, \left\|f\left(\frac{x}{2^j}\right) - S\left(\frac{x}{2^j}\right)\right\|\right\} \\
&\le \lim_{j\to\infty} \lim_{n\to\infty} \frac{1}{|2|} \max\left\{|2|^{k+1}\zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) :\ j \le k < n+j\right\} \\
&= 0
\end{aligned}$$

for all $x \in G$. Therefore, $T = S$. This completes the proof. □

Corollary 5.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|2|}\right) \le \xi\left(\frac{1}{|2|}\right)\xi(t), \quad \xi\left(\frac{1}{|2|}\right) < \frac{1}{|2|}$$

for all $t \ge 0$. Let $\kappa > 0$ and $f : G \to X$ be a mapping such that

(5.9)
$$\Big\|f(f(x)-f(y))-f(x+y)-f(x-y)+f(x)+f(y)\Big\| \le \kappa(\xi(|x|)+\xi(|y|))$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \le 2\kappa\frac{\xi(|x|)}{|2|}.$$

Proof. If we define $\zeta : G \times G \to [0, \infty)$ by $\zeta(x, y) := \kappa(\xi(|x|) + \xi(|y|))$, then we have

$$\lim_{n\to\infty} |2|^n \zeta\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) \le \lim_{n\to\infty} \left(|2|\xi\left(\frac{1}{|2|}\right)\right)^n \Big[\kappa(\xi(|x|)+\xi(|y|))\Big] = 0$$

for all $x, y \in G$. On the other hand, for all $x \in G$,

$$\begin{aligned} \Psi(x) &= \lim_{n\to\infty} \max\Big\{|2|^{k+1}\zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : \ 0 \le k < n\Big\} \\ &= |2|\zeta\left(\frac{x}{2}, \frac{x}{2}\right) = 2\kappa\xi(|x|) \end{aligned}$$

exists. Also, we have

$$\begin{aligned} &\lim_{j\to\infty}\lim_{n\to\infty} \max\Big\{|2|^{k+1}\zeta\Big(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\Big); \ j \le k < n+j\Big\} \\ &= \lim_{j\to\infty} |2|^{j+1}\zeta\Big(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\Big) = 0. \end{aligned}$$

Thus, applying Theorem 5.1, we have the conclusion. This completes the proof. ☐

Theorem 5.3. *Let $\zeta : G \times G \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\zeta(2^n x, 2^n y)}{|2|^n} = 0 \tag{5.10}$$

for all $x, y \in G$. Suppose that, for every $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} \max\left\{\frac{\zeta(2^k x, 2^k x)}{|2|^k} : \ 0 \le k < n\right\} \tag{5.11}$$

exists and let $f : G \to X$ be a mapping satisfying (5.3), then, the limit $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and satisfies the inequality

$$\|f(x) - T(x)\| \leq \frac{1}{|2|}\Psi(x). \tag{5.12}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{\zeta(2^k x, 2^k x)}{|2|^k};\ j \leq k < n+j\right\} = 0, \tag{5.13}$$

then T is the unique mapping satisfying (5.12).

Proof. By (4.4), we have

$$\left\|f(x) - \frac{f(2x)}{2}\right\| \leq \frac{\zeta(x,x)}{|2|} \tag{5.14}$$

for all $x \in G$. Replacing x by $2^n x$ in (5.14), we obtain

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right\| \leq \frac{\zeta(2^n x, 2^n x)}{|2|^{n+1}}. \tag{5.15}$$

Thus it follows from (5.10) and (5.15) that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}_{n\geq 1}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$. On the other hand, it follows from (5.15) that

$$\begin{aligned}
\left\|\frac{f(2^p x)}{2^p} - \frac{f(2^q x)}{2^q}\right\| &= \left\|\sum_{k=p}^{q-1} \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1}x)}{2^{k+1}}\right\| \\
&\leq \max\left\{\left\|\frac{f(2^k x)}{2^k} - \frac{f(2^{k+1}x)}{2^{k+1}}\right\|\ :\ p \leq k < q\right\} \\
&\leq \frac{1}{|2|}\max\left\{\frac{\zeta(2^k x, 2^k x)}{|2|^k} : p \leq k < q\right\}
\end{aligned}$$

for all $x \in G$ and all integers $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (5.11), we obtain (5.12).

The rest of the proof is similar to the proof of Theorem 5.1. This completes the proof. □

Corollary 5.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t), \quad \xi(|2|) < |2|$$

for all $t \geq 0$. Let $\kappa > 0$ and let $f : G \to X$ be a mapping satisfying (5.9). Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\kappa\xi(|x|)}{|2|}.$$

Proof. If we define $\zeta : G \times G \to [0, \infty)$ by $\zeta(x, y) := \kappa(\xi(|x|) + \xi(|y|))$ and apply Theorem 5.3, then we get the conclusion. □

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