A NEW BLOCK BY BLOCK METHOD FOR SOLVING TWO-DIMENSIONAL LINEAR AND NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST AND SECOND KINDS

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ABSTRACT. In this paper, we propose a new method for the numerical solution of two-dimensional linear and nonlinear Volterra integral equations of the first and second kinds, which avoids from using starting values. An existence and uniqueness theorem is proved and convergence is verified by using an appropriate variety of the Gronwall inequality. Application of the method is demonstrated for solving the useful telegraph equation.

1. Introduction

Consider the second kind Volterra integral equation (VIE) of the form

$$(1.1) \qquad f(t,s) = g(t,s) + \lambda \int_0^t \int_0^s K(t,s,x,y,f(x,y)) dy dx,$$

and the first kind VIE

$$(1.2) \qquad H(t,s) = \int_0^t \int_0^s K(t,s,x,y,f(x,y)) dx dy$$

where g, H and K are given (real-valued) continuous functions defined respectively, on $D := [0, T] \times [0, S] \subset R^2$ and
$W := \{(t, s, x, y, f) : 0 \le x \le t \le T, 0 \le y \le s \le S, -\infty < f < \infty\}$.
Also, we assume that $K(t, s, x, y, f(x, y))$ satisfies in Lipschitz condition with respect to $f(x, y)$ with the Lipschitz constant L.

Recently more significant progress has been made in the numerical solution of one-dimensional Volterra integral equations [1, 4, 5, 9]. However, there are a few works to study numerical methods for two-dimensional integral equations. Here, we recall some published works on this subject: In [2], Beltyukov and Kuznechikhina introduce a class of explicit Runge-Kutta-type methods of order 3 (without analyzing their convergence), while Singh [12] verified a method based on bivariate cubic spline functions of full continuity. Brunner and Kauthen analyzed the polynomial spline collocation and iterated collocation methods and their local and global convergence properties for two-dimensional linear VIE [3]. Han and Zhang [8] obtained an asymptotic error expansion of the iterated collocation solution for two-dimensional linear VIE and in [7] Guoqiang, Hayami, Sugihara and Jiong, discussed collocation and iterated collocation methods for nonlinear VIE. They also obtained an asymptotic error expansion for the iterated collocation solution. An Euler-type method was introduced for two-dimensional VIE of the first kind in [10]. In 2009, the two-dimensional differential transform method was proposed for solving a class of two-dimensional linear and nonlinear VIE, where the kernel function had degenerate form [13].

In this paper, we introduce a block by block method with Romberg quadrature rule for solving (1.1) and (1.2), which has the following advantages:

1. Simple structure for application.

2. Useful performance for large intervals.

3. At least six order of convergence, that is, for the given step sizes (h, k), the order of convergence is at least $h^6 + k^6$.

The concept of block by block method for integral equations seems to be described for the first time by Young [14]. A similar technique for differential equations was given by Milne [11]. This method is essentially an extrapolation procedure with the advantages of being self starting and producing a block of values at a time which is effective for the large intervals.

The method of present paper has the following extra advantages:
1. In each step, we solve a system of algebraic equations with 16 equations and 16 unknowns. Dimension of the system dose not change by changing number of the steps.
2. At the first step of Romberg rule, one can use Simpson rule instead of trapezoidal rule for increasing the order of convergence.
3. In comparison with known methods, the computation time of this method is low.

The rest of the paper is organized as follows. In section 2, we prove existence of a unique continuous solution for (1.1). In section 3, general framework of the method will be described. Application of the present method for (1.2) will be given in section 4. The sixth-order convergence result for the method is proved by using discrete Gronwall inequality in section 5. Finally, accuracy and convergence results are illustrated by numerical examples in section 6.

2. **Existence and uniqueness of solution**

In this section, we follow the Picard iterative procedure to obtain an existence condition of solution for (1.1). For a given value of f_0 the Picard iteration for (1.1) is defined by

$$(2.1) \quad f_{n+1}(t,s) \quad = \quad g(t,s) + \lambda \int_0^t \int_0^s K(t,s,x,y,f_n(x,y)) dy dx$$

$$n = 0, 1, ...$$

Since $K(t,s,x,y,f(x,y))$ is continuous on the set W, there exists a constant $M > 0$ such that $|K(t,s,x,y,f(x,y))| \leq M$ if $f(x,y)$ is bounded, say, $c < f(x,y) < d$. To ensure that the output $f_{n+1}(t,s)$ remains bounded within the range $[c,d]$ for $c < f_n(x,y) < d$,

$$(2.2)$$

$$|f_{n+1}(t,s) - g(t,s)| \leq |\lambda| \int_0^t \int_0^s |K(t,s,x,y,f_n(x,y))| dy dx \leq |\lambda| MTS.$$

This means that if our input estimate, $f_n(x,y)$ in (2.1), is bounded, then the output $f_{n+1}(t,s)$ can also be bounded within the same rang by restricting the value of λ in (2.2) and by taking into the bounds of g, $m_1 \leq g \leq m_2$. Thus, we must have

$$c < m_1 - |\lambda| MTS \leq f_{n+1}(t,s) \leq m_2 + |\lambda| MTS < d$$

which leads in choosing λ as

$$|\lambda| < min(\frac{m_1 - c}{MTS}, \frac{d - m_2}{MTS}).$$

Now, we follow a procedure to get convergence of the sequence $f_n(t, s)$ to the unique solution of (1.1). To this end, for $n = 1$ we have from (2.1)

$$
\begin{aligned}
|f_2(t, s) - f_1(t, s)| &\leq |\lambda| L \int_0^t \int_0^s |f_1(x, y) - f_0(x, y))| \, dy dx \\
&\leq |\lambda||d - c| L t s.
\end{aligned}
$$

Let the inequity holds for $n = m$, i.e.

$$(2.3) \qquad |f_{m+1}(t, s) - f_m(t, s)| \leq |d - c| \frac{(|\lambda| L t s)^m}{(m)!(m)!}$$

for $n = m + 1$, we have

$$
\begin{aligned}
|f_{m+2}(t, s) - f_{m+1}(t, s)| &\leq |\lambda| L \int_0^t \int_0^s |f_{m+1}(x, y) - f_m(x, y)| dy dx \\
&\leq |d - c| \frac{(|\lambda| L)^{m+1}}{(m)!(m)!} \int_0^t \int_0^s (xy)^m dy dx
\end{aligned}
$$

by using (2.3). Hence

$$|f_{n+1}(t, s) - f_n(t, s)| \leq |d - c| \frac{(|\lambda| L s t)^n}{n! n!} \quad \forall n \in \mathbb{N}$$

which implies that the series

$$(2.4) \qquad \sum_{n=1}^{\infty} [f_{n+1}(t, s) - f_n(t, s)]$$

is absolutely and uniformly convergent, since it is dominated by an uniformly convergent series. On the other hand, $f_n(t, s)$ can be written as

$$f_n(t, s) = f_1(t, s) + \sum_{i=1}^{n-1} [f_{i+1}(t, s) - f_i(t, s)]$$

therefore from uniform convergence of the series (2.4), we conclude that $\lim_{n \to \infty} f_n(t, s)$ exists for all $(t, s) \in [0, T] \times [0, S]$. Let $\lim_{n \to \infty} f_n(t, s) = f(t, s)$. Then by the continuity of $K(t, s, x, y, f_n(x, y))$ in $f_n(x, y)$, we have

$$\lim_{n \to \infty} K(t, s, x, y, f_n(x, y)) = K(t, s, x, y, f(x, y))$$

and so

$$\lim_{n\to\infty} f_n(t,s) = g(t,s) + \lambda \int_0^t \int_0^s K(t,s,x,y,f(x,y))dydx = f(t,s)$$

that is, $f(t,s)$ is the unique solution of (1.1).

3. Description of the method

In this section, we state principals of the Romberg quadrature, and then we describe the block by block method for solving (1.1).

To recall the Romberg quadrature rule, we denote the trapezoidal rule with the step sizes h, $\frac{h}{2}$ and $\frac{h}{4}$ respectively with T^{00}, T^{01} and T^{02} to approximate the integral $\int_\beta^\alpha V(x)dx$ for the given function V. i.e.

$$T^{00} := \frac{\alpha-\beta}{2}\left[V(\alpha) + V(\beta)\right],$$

$$T^{01} := \frac{1}{2}T^{00} + \frac{\alpha-\beta}{2}V(\frac{\alpha+\beta}{2}),$$

$$T^{02} := \frac{1}{2}T^{01} + \frac{\alpha-\beta}{4}\left[V(\frac{\alpha+3\beta}{4}) + V(\frac{3\alpha+\beta}{4})\right].$$

Thus the two stage Romberg quadrature rule can be written as

$$\int_\beta^\alpha V(x)dx \approx \frac{64T^{02} - 20T^{01} + T^{00}}{45}$$

$$(3.1) \qquad = \frac{\alpha-\beta}{90}\left[7(V_\alpha + V_\beta) + 12V_{\frac{\alpha+\beta}{2}} + 32(V_{\frac{\alpha+3\beta}{4}} + V_{\frac{3\alpha+\beta}{4}})\right]$$

and

$$\int_\eta^\gamma \int_\beta^\alpha U(x,y)dxdy \approx \frac{(\alpha-\beta)(\gamma-\eta)}{8100}\left[49(U_{\alpha,\gamma} + U_{\beta,\gamma} + U_{\alpha,\eta} + U_{\beta,\eta}) \right.$$

$$+ 84(U_{\frac{\alpha+\beta}{2},\gamma} + U_{\frac{\alpha+\beta}{2},\eta} + U_{\alpha,\frac{\gamma+\eta}{2}} + U_{\beta,\frac{\gamma+\eta}{2}}) + 144U_{\frac{\alpha+\beta}{2},\frac{\gamma+\eta}{2}}$$

$$+ 224(U_{\frac{\alpha+3\beta}{4},\gamma} + U_{\frac{3\alpha+\beta}{4},\gamma} + U_{\frac{\alpha+3\beta}{4},\eta} + U_{\frac{3\alpha+\beta}{4},\eta} + U_{\alpha,\frac{\gamma+3\eta}{4}}$$

$$+ U_{\beta,\frac{\gamma+3\eta}{4}} + U_{\alpha,\frac{3\gamma+\eta}{4}} + U_{\beta,\frac{3\gamma+\eta}{4}})$$

$$+ 384(U_{\frac{3\alpha+\beta}{4},\frac{\gamma+\eta}{2}} + U_{\frac{\alpha+3\beta}{4},\frac{\gamma+\eta}{2}} + U_{\frac{\alpha+\beta}{2},\frac{3\gamma+\eta}{4}} + U_{\frac{\alpha+\beta}{2},\frac{\gamma+3\eta}{4}})$$

$$(3.2) \qquad + 1024(U_{\frac{3\alpha+\beta}{4},\frac{\gamma+3\eta}{4}} + U_{\frac{\alpha+3\beta}{4},\frac{\gamma+3\eta}{4}} + U_{\frac{3\alpha+\beta}{4},\frac{3\gamma+\eta}{4}} + U_{\frac{\alpha+3\beta}{4},\frac{3\gamma+\eta}{4}})\Big]$$

for the single and double integrals respectively, where $V_i := V(i)$ and $U_{i,j} := U(i,j)$.

3.1. **Description of block by block method by Romberg rule.**
Let

$$\Omega_{h,k} := \{(t_i, s_j) : t_i = t_{i-1} + h, i = 1, 2, ...M, t_0 = 0, t_M = T,$$
$$s_j = s_{j-1} + k, j = 1, 2, ..., N, s_0 = 0, s_N = S\}$$

be a set of grid points on $[0, T] \times [0, S]$ and $F_{i,j} := f(t_i, s_j)$ be a solution
of (1.1) at $t = t_i$ and $s = s_j$. Then $F_{0,0} = g(0,0)$, $F_{0,j} = g(0, s_j)$
and $F_{i,0} = g(t_i, 0)$, for $i = 0, 1, ..., M$, $j = 0, 1, ..., N$ (note that M and
N must be multiple of 4). For simplifying notation, we define $K_{i,j} :=$
$K(t_{4m+p}, s_{4n+q}, t_i, s_j, F_{i,j})$ for fixed m, n, p, q.
Now, we set $(t, s) = (t_{4m+p}, s_{4n+q})$ in (1.1) for $m = 0, 1, ..., M/4 - 1$,
$n = 0, 1, ..., N/4 - 1$ and $p, q = 1, 2, 3, 4$, to compute $f(t, s)$ at the grid
points, i.e.

$$
\begin{aligned}
f(t_{4m+p}, s_{4n+q}) \quad = \quad & g(t_{4m+p}, s_{4n+q}) \\
+ \quad & \int_0^{t_{4m}} \int_0^{s_{4n}} K(t_{4m+p}, s_{4n+q}, x, y, f(x, y)) dy dx \\
+ \quad & \int_0^{t_{4m}} \int_{s_{4n}}^{s_{4n+q}} K(t_{4m+p}, s_{4n+q}, x, y, f(x, y)) dy dx \\
+ \quad & \int_{t_{4m}}^{t_{4m+p}} \int_0^{s_{4n}} K(t_{4m+p}, s_{4n+q}, x, y, f(x, y)) dy dx \\
+ \quad & \int_{t_{4m}}^{t_{4m+p}} \int_{s_{4n}}^{s_{4n+q}} K(t_{4m+p}, s_{4n+q}, x, y, f(x, y)) dy dx.
\end{aligned}
$$

(3.3)

We use (3.2) to approximate these double integrals. We denote these
approximations respectively by $R_1, ..., R_4$. For example

$$
\begin{aligned}
R_2 \quad := \quad & \int_0^{t_{4m}} \int_{s_{4n}}^{s_{4n+q}} K(t_{4m+p}, s_{4n+q}, x, y, f(x, y)) dy dx \\
\approx \quad & \frac{s_q \times t_{4m}}{8100} \Big[49(K_{0,4n} + K_{4m,4n} + K_{0,4n+q} + K_{4m,4n+q}) \\
+ \quad & 144 K_{2m,4n+\frac{q}{2}} + 84(K_{2m,4n} + K_{2m,4n+q} + K_{0,2n+\frac{q}{2}} + K_{4m,4n+\frac{q}{2}}) \\
+ \quad & 384(K_{m,4n+\frac{q}{2}} + K_{3m,4n+\frac{q}{2}} + K_{2m,4n+\frac{3q}{4}} + K_{2m,4n+\frac{q}{4}}) \\
+ \quad & 1024(K_{m,4n+\frac{q}{4}} + K_{3m,4n+\frac{q}{4}} + K_{m,4n+\frac{3q}{4}} + K_{3m,4n+\frac{3q}{4}}) \\
+ \quad & 224(K_{m,4n} + K_{3m,4n} + K_{m,4n+q} + K_{3m,4n+q} + K_{4m,4n+\frac{q}{4}} \\
+ \quad & K_{0,4n+\frac{q}{4}} + K_{4m,4n+\frac{3q}{4}} + K_{0,4n+\frac{3q}{4}}) \Big]
\end{aligned}
$$

(3.4)

for $m = 0, 1, ..., M/4 - 1$, $n = 0, 1, ..., N/4 - 1$ and $p, q = 1, 2, 3, 4$.

If $\frac{jp}{4}$ or $\frac{jq}{4}$ $(j = 1, 2, 3)$ are not integers, then $F_{4m+\frac{jp}{4}, 4n+\frac{jq}{4}}$ will be unknown which leads to a difficulty in computing R_2, R_3 and R_4. In this case, we will use bivariate Lagrange interpolation at the points (t_{4m}, s_{4n}), (t_{4m+1}, s_{4n+1}), (t_{4m+2}, s_{4n+2}), (t_{4m+3}, s_{4n+3}) and (t_{4m+4}, s_{4n+4}) to remove this difficulty; that is

(3.5)

$$
F_{u,v} \simeq \begin{cases}
\sum\limits_{i=4m}^{4m+p} \sum\limits_{j=4n}^{4n+q} F_{i,j} U_i(t_u) V_j(s_v), & \text{if } u \text{ and } v \text{ are not integers}, \\[2ex]
\sum\limits_{j=4n}^{4n+q} F_{u,j} V_j(s_v), & \text{if } u \text{ is integer and } v \text{ is not integer}, \\[2ex]
\sum\limits_{i=4m}^{4m+p} F_{i,v} U_i(t_u), & \text{if } u \text{ is not integer and } v \text{ is integer}
\end{cases}
$$

where

$$
U_i(t) = \prod_{\substack{j=4m \\ j \neq i}}^{4m+4} \frac{t - t_j}{t_i - t_j}, \qquad V_i(s) = \prod_{\substack{j=4n \\ j \neq i}}^{4n+4} \frac{s - s_j}{s_i - s_j}
$$

are the univariate Lagrange polynomials and the error of the bivariate interpolation will be at least $O((h + k)^5)$ [1], thus, from (3.3) we obtain

(3.6) $\qquad F_{4m+p, 4n+q} = g(t_{4m+p}, s_{4n+q}) + R_1 + R_2 + R_3 + R_4$

from which a system of 16 equations with the same number of unknowns will be created in each step for $p, q = 1, 2, 3, 4$ and the system will be linear or nonlinear respectively for the linear or nonlinear integral equations. For the linear case, it is solved via a direct method but for the nonlinear case, the system may be solved by using an iterative method or by using a suitable software package such as Maple. The following algorithm describes solving the system by using the Maple package.

Algorithm 1.
- *Get a set of grid points from $[0, T] \times [0, S]$.*
- *Set $F_{0,0} = g(0, 0)$, $F_{i,0} = g(t_i, 0)$, $i = 0, 1, ..., M$ and $F_{0,j} = g(0, s_j)$, $j = 0, 1, ..., N$.*
- *Start from $m = 0$, while $4m + 4 \leq M$ do*
- *start from $n = 0$, while $4n + 4 \leq N$ do*
- *for p=1 to 4 do*
- *for $q = 1$ to 4 do*
- *approximate the integrals in (3.3) by using the two-step Romberg*

rule

-          *if $\frac{jp}{4}$ or $\frac{jq}{4}$ are not integers, use an interpolation formula*
-             *end do*
-          *end do*
-          *form the system (3.6)*
-          *solve the system*
-       *end do*
-    *end do.*

4. The first kind VIE

In this section we discuss the results of the previous sections for the first kind Volterra equations. Consider a two-dimensional first kind Volterra integral equation of the form

$$(4.1) \qquad H(t,s) = \int_0^t \int_0^s K(t,s,x,y,f(x,y))dxdy, \qquad (t,s) \in D$$

where K and H are defined as before. It is evident from (4.1) that

$$H(t,0) \equiv 0, \qquad H(0,s) \equiv 0, \qquad (t,s) \in D.$$

Let K and H are smooth and

$$K(t,s,t,s,f) \neq 0 \qquad \forall (t,s) \in D.$$

Differentiating both sides of (4.1) with respect to t and s, leads to the two-dimensional second-kind Volterra integral equation

$$
\begin{aligned}
Z(t,s,f(t,s)) \;&=\; G(t,s) \\
&+\; \int_0^s K_1(t,s,y,f(t,y))dy + \int_0^t K_2(t,s,x,f(x,s))dx \\
(4.2) \qquad &+\; \int_0^t \int_0^s K_3(t,s,x,y,f(x,y))dydx
\end{aligned}
$$

with

$$K_1(t, s, y, f(t, y)) := -\frac{\partial K}{\partial s}(t, s, t, y, f(t, y)),$$

$$K_2(t, s, x, f(x, s)) := -\frac{\partial K}{\partial t}(t, s, x, s, f(x, s)),$$

$$K_3(t, s, x, y, f(x, y)) := -\frac{\partial^2 K}{\partial t \partial s}(t, s, x, y, f(x, y)),$$

$$G(t, s) := \frac{\partial^2 H}{\partial t \partial s}(t, s),$$

$$Z(t, s, f(t, s)) := K(t, s, t, s, f(t, s)).$$

Now, we solve (4.2) by the same way as we did for (1.1). Considering the set $\Omega_{h,k}$ of grid points, one can write

$$Z(t_{4m+p}, s_{4n+q}, f(t_{4m+p}, s_{4n+q})) = G(t_{4m+p}, s_{4n+q})$$

$$+ \int_0^{s_{4n+q}} K_1(t_{4m+p}, s_{4n+q}, y, f(t_{4m+p}, y))dy$$

$$+ \int_0^{t_{4m+p}} K_2(t_{4m+p}, s_{4n+q}, x, f(x, s_{4n+q}))dx$$

$$+ \int_0^{t_{4m+p}} \int_0^{s_{4n+q}} K_3(t_{4m+p}, s_{4n+q}, x, y, f(x, y))dydx,$$

$$m = 0, 1, ..., M/4 - 1, n = 0, 1, ..., N/4 - 1,$$

$$p, q = 1, 2, 3, 4.$$

By solving the equations

$$G(t_0, s_0) - Z(t_0, s_0, F_{0,0}) = 0$$

$$R_5 := \int_0^{s_{4n+q}} K_1(t_{4m+p}, s_{4n+q}, y, f(t_{4m+p}, y))dy,$$

$$R_6 := \int_0^{t_{4m+p}} K_2(t_{4m+p}, s_{4n+q}, x, f(x, s_{4n+q}))dx$$

by defining $K_{i,j}^1 := K_1(t_{4m+p}, s_{4n+q}, s_j, F_{i,j})$, we have from (3.1)

$$R_5 \approx \frac{s_{4n}}{90}\left[7(K_{4m+p,0}^1 + K_{4m+p,4n}^1) + 12K_{4m+p,2n}^1\right.$$

$$+ \quad 32(K_{4m+p,n}^1 + K_{4m+p,3n}^1)\right] + \frac{s_q}{90}\left[7(K_{4m+p,4n}^1 + K_{4m+p,4n+q}^1)\right.$$

$$+ \quad 12K_{4m+p,4n+q/2}^1 + 32(K_{4m+p,4n+q/4}^1 + K_{4m+p,4n+3q/4}^1)\right]$$

also, we approximate R_6 by the same way. If $\frac{jp}{4}$ and $\frac{jq}{4}(j = 1, 2, 3)$ are not integers, then we use the same technique as we described after the formula (3.4). Therefore
(4.3)
$$Z(t_{4m+p}, s_{4n+q}, F_{4m+p,4n+q}) = G(t_{4m+p}, s_{4n+q}) + R_1 + R_2 + R_3 + R_4 + R_5 + R_6$$

where $R_1, ..., R_4$ gives similar results as in the previous section. For $p, q = 1, 2, ..., 4$, (4.3) forms a system of equations which is solved for $F_{4m+1,4n+1}$, $F_{4m+2,4n+1}$, ... , $F_{4m+4,4n+4}$. A procedure similar to what described in the previous section can be use to resolve this system.

5. Convergence analysis

In this section, we prove that the method of the previous sections is convergent and its order of convergence is at least 6. To do this, we define $e_{n,m} := |f(t_n, s_m) - F_{n,m}|$ and use the triangular inequality repeatedly to obtain

$$
\begin{aligned}
e_{4m+4,4n+4} \quad =& hk \left| \sum_{i=0}^{4m} \sum_{j=0}^{4n} w_{i,j}^{(1)} K_{i,j} + \sum_{i=0}^{4m} \sum_{j=4n}^{4n+4} w_{i,j}^{(2)} K_{i,j} + \right. \\
(5.1) \qquad &+ \sum_{i=4m}^{4m+4} \sum_{j=0}^{4n} w_{i,j}^{(3)} K_{i,j} + \sum_{i=4m}^{4m+4} \sum_{j=4n}^{4n+4} w_{i,j}^{(4)} K_{i,j} \\
&\left. - \int_0^{4m+4} \int_0^{4n+4} K(t_{4m+4}, s_{4n+4}, x, y, f(x, y)) dy dx \right| \\
=& hk \left| \sum_{i=0}^{4m} \sum_{j=0}^{4n+4} w_{i,j}^{(5)} K_{i,j} + \sum_{i=4m}^{4m+4} \sum_{j=0}^{4n+4} w_{i,j}^{(6)} K_{i,j} \right. \\
&\left. - \int_0^{4m+4} \int_0^{4n+4} K(t_{4m+4}, s_{4n+4}, x, y, f(x, y)) dy dx \right| \\
\leq& hk \sum_{i=0}^{4m+4} \sum_{j=0}^{4n+4} |w_{i,j}'| \, |K_{i,j} - K(t_{4m+4}, s_{4n+4}, t_i, s_j, f(t_i, s_j))| \\
&+ \left| \sum_{i=0}^{4m+4} \sum_{j=0}^{4n+4} w_{i,j}' K(t_{4m+4}, s_{4n+4}, t_i, s_j, f(t_i, s_j)) \right. \\
&\left. - \int_0^{4m+4} \int_0^{4n+4} K(t_{4m+4}, s_{4n+4}, x, y, f(x, y)) dy dx \right|
\end{aligned}
$$

where

$$
w_{i,j}^{(5)} = \begin{cases} w_{i,j}^{(1)} & 0 \le j \le 4n-1, \\ w_{i,j}^{(1)} + w_{i,j}^{(2)} & j = 4n, \\ w_{i,j}^{(2)} & 4n+1 \le j \le 4n+4, \end{cases}
$$

$$
w_{i,j}^{(6)} = \begin{cases} w_{i,j}^{(3)} & 0 \le j \le 4n-1, \\ w_{i,j}^{(3)} + w_{i,j}^{(4)} & j = 4n, \\ w_{i,j}^{(4)} & 4n+1 \le j \le 4n+4, \end{cases}
$$

$$
w_{i,j}' = \begin{cases} w_{i,j}^{(5)} & 0 \le i \le 4m-1, \\ w_{i,j}^{(5)} + w_{i,j}^{(6)} & i = 4m, \\ w_{i,j}^{(6)} & 4m+1 \le i \le 4m+4. \end{cases}
$$

Therefore, by using the Lipschitz condition for K, we get

$$
e_{4m+4,4n+4} \le hk \sum_{i=0}^{4m+4} \sum_{j=0}^{4n+4} \left| w_{i,j}' \right| L_{i,j} \left| F_{i,j} - f(t_i, s_j) \right| + R
$$

where R is the error of the numerical integration. Hence for sufficiently small h and k

$$
e_{4m+4,4n+4} \le \frac{hkc}{1-hkc'} \sum_{i=0}^{4m+3} e_{i,4n+4} + \frac{hkc}{1-hkc'} \sum_{j=0}^{4n+3} e_{4m+4,j}
$$

(5.2)
$$
+ \frac{hkc}{1-hkc'} \sum_{i=0}^{4m+3} \sum_{j=0}^{4n+3} e_{i,j} + \frac{R}{1-hkc'}
$$

1 where $c = \max\limits_{i,j}\{|w_{i,j}'| L_{i,j}\}$ and $c' = |w_{4m+4,4n+4}'| L_{4m+4,4n+4}$. For other values of p and q, we use the bivariate Lagrange interpolation, (3.5), to find undetermined values of F in (5.1). Finally, we will have a relation similar to (5.2) with different c and c'. Then we obtain

$$
e_{4m+4,4n+4} \le \frac{hkc}{1-hkc'} \sum_{i=0}^{4m+p-1} e_{i,4n+q} + \frac{hkc}{1-hkc'} \sum_{j=0}^{4n+q-1} e_{4m+q,j}
$$

$$
+ \frac{hkc}{1-hkc'} \sum_{i=0}^{4m+p-1} \sum_{j=0}^{4n+q-1} e_{i,j} + \frac{R}{1-hkc'},
$$

$$
p, q = 1, 2, 3, 4
$$

and using the Gronwall inequality [10], yields

$$e_{4m+p,4n+q} \leq \frac{R}{1 - hkc'} \exp(\gamma(Nh + Mk))$$

where

$$\gamma = \frac{1}{2} \frac{c(h + k) + \sqrt{(kc + hc)^2 + 4c - 4hkcc'}}{1 - hkc'}.$$

Thus $e_{4m+p,4n+q} \to 0$ as $(h, k) \to (0, 0)$. Since R is the error of two stages Romberg quadrature rule, we have $R = O(h^6 + k^6)$ for the functions K and f with at least sixth order derivatives and so $\| e \| = O(h^6 + k^6)$.

6. Numerical results

In this section, we test the following problems to show the convergence and error bound of the presented method. All results computed by programming in Maple 10.

A)[3]

$$
\begin{aligned}
f(t, s) &= exp(-at)[cos(bs) - \frac{ct}{2b}(sin(bs) + bscos(bs))] \\
&+ \int_0^t \int_0^s c.exp(-a(t - x)).cos(b(s - y))f(x, y)dydx, \\
&\quad t, s \in [0, 2] \times [0, 2]
\end{aligned}
$$

with the exact solution $f(t, s) = exp(-at)cos(bs)$.

B)

$$f(t, s) = g(t, s) + \int_0^t \int_0^s (sx^2 + ty)sin(f(x, y))dydx, \quad t, s \in [0, 1] \times [0, 1]$$

with the exact solution $f(t, s) = t + s$.

C) [10] We set

$$K(t, s, x, y) = (sin(s + x) + sin(t + y) + 3)f(x, y)$$

and choose $g(t, s)$ such that the exact solution of the (1.2) with $T = S = 2$ to be $f(t, s) = cos(t + s)$.

D) The generalized telegraph equation

Consider the nonlinear Cauchy problem [6]

$$(6.1) \qquad \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}(f(x, t, u)) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(g(x, t, u)) + h(x, t, u)$$

subject to

$$(x, t) \in W := \{(x, t) : x - t \geq 0, \quad x + t \geq 0\},$$

$$u(x, x) = \alpha(x), \qquad u(t, -t) = \beta(t).$$

In the special case $f(x, t, w) \equiv w$ and $g(x, t, w) \equiv 0$, we obtain the 'forced' telegraph equation: this arises from the electromagnetic waves in a conducting medium and it can be derived directly from Maxwell's equations [10]. Performing the change of variables: $x' = x + t$ and $t' = x - t$, we obtain

$$4\frac{\partial^2 u}{\partial x' \partial t'} + \frac{\partial}{\partial x'}(-f + g) + \frac{\partial}{\partial t'}(f + g) = -h$$

or equivalently

(6.2) $$\frac{\partial^2 U}{\partial x' \partial t'} - \frac{\partial F}{\partial x'} - \frac{\partial G}{\partial t'} = H$$

where

$$
\begin{aligned}
U(x', t') &:= u((x' + t')/2, (x' - t')/2), \\
F(x', t', U) &:= (f(x', t', U) - g(x', t', U))/4, \\
G(x', t', U) &:= -(f(x', t', U) + g(x', t', U))/4, \\
H(x', t', U) &:= -h(x', t', U)/4.
\end{aligned}
$$

Integrating (6.2) with respect to x' and t' yields

$$
\begin{aligned}
U(X, T) &= \int_0^T \int_0^X H(x', t', U(x', t'))dx'dt' \\
&+ \int_0^X (G(x', T, U(x', T)) - G(x', 0, U(x', 0)))dx' \\
&+ \int_0^T (F(X, t', U(X, t')) - F(0, t', U(0, t')))dt' \\
&+ U(X, 0) + U(0, T) - U(0, 0)
\end{aligned}
$$

which is a two-dimensional Volterra integral equation of the form

$$
\begin{aligned}
U(X, T) &= \int_0^T \int_0^X H(x', t', U(x', t'))dx'dt' + \int_0^X G(x', T, U(x', T))dx' \\
&+ \int_0^T F(X, t', U(X, t'))dt' + R(X, T)
\end{aligned}
$$

with

$$
\begin{aligned}
R(X, T) &:= -\int_0^X G(x', 0, U(x', 0))dx' - \int_0^T F(0, t', U(0, t'))dt' \\
&+ U(X, 0) + U(0, T) - U(0, 0).
\end{aligned}
$$

Note that $R(X, T)$ is a known function, since $U(x', 0) = u(x'/2, x'/2) = \alpha(x'/2)$ and $U(0, t') = u(t'/2, -t'/2) = \beta(t'/2)$ are given by the initial conditions. Thus we have an integral equation for solving the nonlinear partial differential equation (6.1). To verify the efficiency of the method we analysis the following simple example.

We consider

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial cos(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial sin(u)}{\partial x} - sin(u)cos(t)sin(x)$$

(6.3) $$\qquad\qquad\qquad - \quad cos(u)sin(t)cos(x)$$

with the initial conditions given on $x = \pm t$, that is, $u(x, x) = sin^2 x$, $u(-t, t) = -sin^2 t$. The exact solution is $u(x, t) = sin(x)sin(t)$. By changing of variables: $x' = x + t$ and $t' = x - t$, Eq.(6.3) can be rewritten in the form

$$
\begin{aligned}
4U(X, T) &= \int_0^T \int_0^X \left\{ sinU(x', t')cos(\frac{1}{2}(x' - t'))sin(\frac{1}{2}(x' + t')) \right. \\
&\quad \left. + cosU(x', t')sin(\frac{1}{2}(x' - t'))cos(\frac{1}{2}(x' + t')) \right\} dx'dt' \\
&\quad - \int_0^X (cosU(x', T) - sinU(x', T))dx' + \int_0^T (cosU(X, t') \\
&\quad - sinU(X, t'))dt' + \int_0^X \left\{ cos(sin^2\frac{x'}{2}) + sin(sin^2(\frac{x'}{2})) \right\} dx' \\
&\quad + \int_0^T \left\{ sin(-sin^2\frac{t'}{2}) - cos(-sin^2(\frac{t'}{2})) \right\} dt' \\
&\quad + 4sin^2\frac{X}{2} - 4sin^2\frac{T}{2}.
\end{aligned}
$$

Now, we solve this two-dimensional second kind Volterra integral equation by using the described method in section 3.

The numerical results corresponding to the problems A, B, C and D are shown respectively in tables 1, 2,...,7, in which the absolute and relative errors reported at some mesh points. In tables 1-3 we have presented a comparison between the errors of the collocation method that reported in [3], and the block by block method for $a = b = 1$ and $c = -1$, in which **Bl. M.** and **Co. M.** denote respectively the block by block and collocation methods. It is clear from these results that the block by block method is more accurate than the collocation method. In the last rows of tables, we saved computing times for the

given method which show that the method is very fast. Tables 5-7 compare the errors at selected grid points for the Euler and trapezoidal methods that obtained in [10], and the block by block method. For these cases, the error of the block by block method for larger step sizes is less than the error of the other methods. Also, the collocation, Euler and trapezoidal methods have been proposed only for the linear case of the (1.1), while as it is clear from the results of table 4, the block by block method is also an efficient method for the nonlinear case.

Table 1: *Numerical results of example A; M = N = 4.*

(t, s)	Bl. M.		Co. M.
	$e(relative)$	$e(absolute)$	$e(absolute)$
$(1, 1)$	$2.07e{-}6$	$4.13e{-}7$	$1.64e{-}4$
$(2, 1)$	$2.94e{-}4$	$2.15e{-}5$	$1.82e{-}4$
$(1, 2)$	$4.05e{-}3$	$6.21e{-}4$	$1.66e{-}3$
$(2, 2)$	$7.33e{-}3$	$4.13e{-}4$	$5.56e{-}4$
$time$	$1.703''$		$-$

Table 2:   *Numerical results of example A; M = N = 8.*

(t, s)	Bl. M.		Co. M.
	$e(relative)$	$e(absolute)$	$e(absolute)$
$(1, 1)$	$3.03e{-}5$	$6.04e{-}6$	$2.14e{-}5$
$(2, 1)$	$5.82e{-}5$	$4.26e{-}6$	$2.67e{-}5$
$(1, 2)$	$5.32e{-}5$	$5.09e{-}6$	$1.93e{-}4$
$(2, 2)$	$5.13e{-}5$	$2.89e{-}6$	$6.12e{-}5$
$time$	$4.077''$		$-$

Table 3: *Numerical results of example A; M = N = 16.*

(t, s)	Bl. M.		Co. M.
	$e(relative)$	$e(absolute)$	$e(absolute)$
$(1, 1)$	$4.01e{-}6$	$7.97e{-}8$	$3.67e{-}6$
$(2, 1)$	$7.22e{-}6$	$5.28e{-}8$	$3.53e{-}6$
$(1, 2)$	$4.19e{-}4$	$6.42e{-}5$	$2.30e{-}8$
$(2, 2)$	$6.21e{-}4$	$3.50e{-}5$	$7.10e{-}5$
$time$	$13.984''$		$-$

Table 4: *Numerical results of example B.*

(t,s)	$M = N = 4$	$M = N = 8$	$M = N = 12$
	$e(relative)$	$e(relative)$	$e(relative)$
$(0.25, 0.25)$	0	$2.00e{-}10$	$5.74e{-}11$
$(0.25, 0.5)$	$4.40e{-}09$	$2.54e{-}09$	$1.87e{-}10$
$(0.5, 0.5)$	$3.66e{-}07$	$2.20e{-}08$	$1.14e{-}09$
$(0.5, 0.75)$	$9.61e{-}08$	$3.93e{-}09$	$3.11e{-}08$
$(0.75, 0.75)$	$6.86e{-}07$	$3.69e{-}08$	$2.91e{-}07$
$(0.75, 1)$	$1.47e{-}06$	$5.83e{-}08$	$4.43e{-}07$
$(1, 1)$	$7.33e{-}06$	$1.11e{-}07$	$4.01e{-}07$
time	$1.397''$	$5.187''$	$11.486''$

Table 5: *Numerical results of example C; $h = k = 0.1$.*

(t,s)	Euler's M.	trapezoidal M.	Block M.	
	$e(absolute)$	$e(absolute)$	$e(absolute)$	$e(relative)$
$(1,1)$	$9.45e{-}02$	$3.93e{-}03$	$3.18e{-}10$	$7.64e{-}10$
$(1,2)$	$1.97e{-}02$	$2.20e{-}03$	$1.48e{-}06$	$1.49e{-}6$
$(2,1)$	$1.97e{-}02$	$2.20e{-}03$	$8.94e{-}07$	$9.03e{-}7$
$(2,2)$	$8.76e{-}02$	$8.17e{-}03$	$1.77e{-}05$	$2.71e{-}5$
time	$-$	$-$	$12.797''$	
$\|e(h)\|_\infty$	$1.09e{-}01$	$8.17e{-}03$	$1.77e{-}05$	

$\|e(h)\|_\infty := max\{|f(t_i, s_j) - f_{i,j}|, \qquad 0 \le i \le N, \quad 0 \le j \le M\}$

Table 6: *Numerical results of example C; $h = k$.*

(t,s)	Euler's M.	trapezoidal M.	Block M.	
	$h = 0.0125$	$h = 0.0125$	$h = 0.16$	
	$e(absolute)$	$e(absolute)$	$e(absolute)$	$e(relative)$
$(1,1)$	$1.13e{-}02$	$6.12e{-}05$	$2.41e{-}06$	$5.79e{-}06$
$(1,2)$	$3.53e{-}03$	$3.41e{-}05$	$3.62e{-}06$	$3.65e{-}06$
$(2,1)$	$3.53e{-}03$	$3.42e{-}05$	$3.62e{-}06$	$3.65e{-}06$
$(2,2)$	$9.46e{-}03$	$1.27e{-}04$	$4.98e{-}06$	$7.61e{-}06$
time	$-$	$-$	$4.500''$	
$\|e(h)\|_\infty$	$1.34e{-}02$	$1.27e{-}04$	$4.98e{-}06$	

$\|e(h)\|_\infty := max\{|f(t_i, s_j) - f_{i,j}|, \qquad 0 \le i \le N, \quad 0 \le j \le M\}$

Table 7: *Numerical results of example D; h = k = 0.1.*

(t,s)	*Euler's M.*	*Block M.*	
	e(absolute)	*e(absolute)*	*e(relative)*
$(0.1, 0.1)$	–	$5.40e{-}22$	$5.41e{-}20$
$(1, 0)$	$7.03e{-}04$	$5.07e{-}08$	–
$(3/2, 1/2)$	$3.01e{-}03$	$1.65e{-}06$	$3.45e{-}6$
$(2, 0)$	$9.32e{-}04$	$3.03e{-}06$	–
time	–	$267.405''$	
$\|e(h)\|_\infty$	$3.72e{-}03$	$4.03e{-}05$	

7. **Conclusion**

In this paper, we propose a method with at least 6 order of convergence for the numerical solution of two-dimensional linear and nonlinear VIE. The method can be improved to be more accurate by using other varieties of Romberg rule or other suitable integration methods.

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