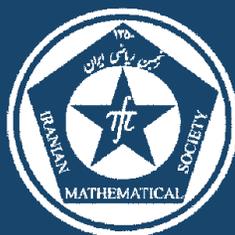


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ON THE NORM OF THE DERIVED SUBGROUPS OF ALL SUBGROUPS OF A FINITE GROUP

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ABSTRACT. In this paper, we give a complete proof of Theorem 4.1(ii) and a new elementary proof of Theorem 4.1(i) in [Li and Shen, On the intersection of the normalizers of the derived subgroups of all subgroups of a finite group, *J. Algebra*, 323 (2010) 1349–1357]. In addition, we also give a generalization of Baer’s Theorem.

Keywords: Derived subgroup, Solvable group, Nilpotency class, Fitting length.

MSC(2010): Primary: 20D10; Secondary: 20D20.

1. Introduction

Let G be a finite group (all groups considered in this paper are finite); the notation and terminology used in this paper are standard, as in [10-11]. By $N(G)$ denote the intersection of the normalizers of all subgroups of G and by $\omega(G)$ denote the intersection of the normalizers of all subnormal subgroups of G . Those concepts were introduced by R. Baer and H. Wielandt in 1934 and 1958, respectively, and were investigated by many authors, for example, see [1-2, 4-5, 7-9, 14 and 16-19]. Li and Shen [13] investigated the following concept:

Definition 1.1. *Let G be a finite group. By $D(G)$ denote the intersection of the normalizers of the derived subgroups of all subgroups of G .*

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That is

$$D(G) = \bigcap_{H \leq G} N_G(H').$$

Obviously, $D(G)$ is a characteristic subgroup of G . Let $Z_2(G)$ be the second term of the ascending central series of G . In the light of a theorem of P. Hall [12, III, Hauptsatz 2.11], $[G', Z_2(G)] = 1$, so $Z_2(G)$ centralizes G' and hence centralizes the derived subgroups of all the subgroups of G , thereby

$$Z_2(G) \leq D(G).$$

Definition 1.2. For a finite group G , there exists a series of normal subgroups:

$$1 = D_0(G) \leq D_1(G) \leq D_2(G) \leq \cdots \leq D_n(G) \leq \cdots$$

satisfying $D_{i+1}(G)/D_i(G) = D(G/D_i(G))$ for $i = 0, 1, 2, \dots$ and $D_n(G) = D_{n+1}(G)$ for some integer $n \geq 1$. Write $D_\infty(G)$ for the terminal term of the ascending series.

Throughout the paper, we denote by \mathcal{F}_{dn} the class of finite groups G with G' nilpotent. It is well-known that \mathcal{F}_{dn} is a saturated formation containing all supersolvable groups.

2. A generalization of Baer's Theorem

Theorem 2.1. (R. Baer, [3, Corollary 2, p.159]) The following properties of the group G are equivalent:

- (i) $G \in \mathcal{F}_{dn}$;
- (ii) Every homomorphic image of G induces in each of its minimal normal subgroups a cyclic group of automorphisms;
- (iii) If M is a maximal subgroup of G , then M/M_G is cyclic;
- (iv) If M is a maximal subgroup of G , then M/M_G is abelian;
- (v) $(G/\Phi(G))'$ is nilpotent.

The following basic properties of the subgroup $D(G)$ are required in this paper.

Proposition 2.2. ([13]) If $M \leq G$, then $M \cap D(G) \leq D(M)$.

Proposition 2.3. ([13]) Let $N \leq D(G)$ and $N \trianglelefteq G$. Then $D(G)/N \leq D(G/N)$.

Theorem 2.4. ([13]) *Let G be a finite group. Then the following statements are equivalent:*

- (i) $G \in \mathcal{F}_{dn}$;
- (ii) $(G/D_\infty(G)) \in \mathcal{F}_{dn}$;
- (iii) $G = D_\infty(G)$.

The next theorem is a generalization of Theorem 2.1.

Theorem 2.5. *The following properties of the group G are equivalent:*

- (i) G' is nilpotent;
- (ii) Every homomorphic image of $G/D_\infty(G)$ induces in each of its minimal normal subgroups a cyclic group of automorphisms;
- (iii) If M is a maximal subgroup of G of composite index, then M/M_G is abelian;
- (iv) If M is a maximal subgroup of G of composite index containing $D_\infty(G)$, then M/M_G is cyclic;
- (v) If M is a maximal subgroup of G of composite index containing $D_\infty(G)$, then M/M_G are abelian.

Proof. (i) implies (ii), (iii), (iv) and (v) by Theorem 2.1. (iv) \Rightarrow (v); clear. (ii) \Rightarrow (i): By Theorem 2.1 (ii), $G/D_\infty(G)$ belongs to \mathcal{F}_{dn} , so G' is nilpotent by applying Theorem 2.4.

(iii) \Rightarrow (i): Suppose that the group G satisfies (iii). First of all, we show that G is soluble. If every maximal subgroup of G is of index a prime, then G is supersolvable and hence G' is nilpotent, as desired. So we may assume that there exists a maximal subgroup M of G such that $|G : M|$ is a composite index. By hypothesis, M/M_G is abelian. As a group with an abelian maximal subgroup is solvable, we know that G/M_G is solvable. Thus

$$G / \bigcap_M M_G$$

is solvable. Moreover, the intersection of all maximal subgroups of a group of composite index is supersolvable [6, Theorem 3], so the intersection of all M_G is supersolvable. Consequently, G is solvable.

Now, if every maximal subgroup M of G satisfies that M/M_G is abelian, then G satisfies (iv) of Theorem 2.1, and hence G' is nilpotent, as desired. Thus we assume that there exists a maximal subgroup M of G such that M/M_G is non-abelian. By hypothesis, every maximal subgroup L of composite index of G satisfies that L/L_G is

abelian, so the subgroup M possesses prime index. It follows that we have $G/M_G = [N/M_G]M/M_G$ where $|N/M_G| = |G : M| = p$ is prime, so M/M_G would be cyclic, a contradiction.

(v) \Rightarrow (i): Clearly, $G/D_\infty(G)$ satisfies (iii). Therefore $G/D_\infty(G)$ is an \mathcal{F}_{dn} -group. Thus apply Theorem 2.4 to conclude that G' is nilpotent and (i) holds. \square

Lemma 2.6. *Let G be a finite group and suppose that M is a maximal subgroup of G . Then, either $M' \leq M_G$ or $D_\infty(G) \leq M$.*

Proof. Assuming that $D_k(G) \leq M$, we get that $M/D_k(G)$ is a maximal subgroup of $G/D_k(G)$. Now either $N_{G/D_k(G)}((M/D_k(G))') = M/D_k(G)$ or $(M/D_k(G))' = M'D_k(G)/D_k(G) \leq G/D_k(G)$. Thus, either $M'D_k(G) \leq G$, so that $M' \leq M'D_k(G) \leq M_G$ or $D_{k+1}(G)/D_k(G) \leq M/D_k(G)$ and $D_{k+1}(G) \leq M$. It follows that either $M' \leq M_G$ or $D_\infty(G) \leq M$. \square

The next consequence follows from Theorem 2.5(v) and Lemma 2.6.

Corollary 2.7. *If every maximal subgroup M of G of composite index satisfies $M' \leq D_\infty(G)$, then G' is nilpotent.*

Proof. If M is a maximal subgroup of G of composite index, then either $M' \leq M_G$ or $D_\infty(G) \leq M$. But, in the latter case we have by assumption that $M' \leq D_\infty(G) \leq M_G$, as required. \square

Remark 2.8. *If $H \leq K$, then, it is clear that $D(K) \leq D(H)$. And thus, it follows that $D(D(G)) = D(G)$. Hence, we must have $D_\infty(D(G)) = D(G)$, and thus, by theorem 2.4, $D(G)'$ is nilpotent.*

3. Some new results of D -groups

Definition 3.1. *A finite group G is called a D -group if $G = D(G)$, that is, the derived subgroups of all subgroups of G are normal.*

Theorem 3.2. *If G is a supersolvable D -group, then the nilpotent residual $G^{\mathcal{N}}$ is abelian.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Since G is supersolvable, by hypothesis, G has unique minimal normal subgroup N of order p , where p is the largest prime divisor of $|G|$. If N_1, N_2 are distinct minimal normal subgroups of G and $M_1/N_1, M_2/N_2$ are the nilpotent residuals of $G/N_1, G/N_1$, resp.,

then, by induction M_1/N_1 , M_2/N_2 are both abelian and the nilpotent residual of G is contained in $M_1 \cap M_2$. But, then, there is a 1-1 mapping $M_1 \cap M_2$ to $M_1/N_1 \cap M_2/N_2$. In this case it would follow that the nilpotent residual of G is abelian, as required.

By the induction hypothesis and Proposition 2.3, $G^{\mathcal{N}}/N$ is abelian and $O_{p'}(G) = 1$. Let P be Sylow p -subgroup of G . Then $G = [P]H$, where H is a p' -Hall subgroup. Since $[P, H] \trianglelefteq \langle P, H \rangle = G$ and $G/[P, H]$ is nilpotent, $G^{\mathcal{N}} = [P, H]$. Moreover, $[P, H, H] = [P, H]$ and the minimality of G imply $G^{\mathcal{N}} = P$. Hence $G^{\mathcal{N}} = F(G)$.

Since G is a supersolvable, all chief factors of G contained in $G^{\mathcal{N}}$ are cyclic. Assume $G^{\mathcal{N}}$ is not cyclic. Then there is a chief factor J/I of G contained in $G^{\mathcal{N}}$ which is a cyclic group of order p such that I is cyclic and J is non-cyclic. We have $J = I\langle x \rangle$, $x^p \in I$ and this is an abelian group because $I \leq Z(G^{\mathcal{N}})$ by N/C -Theorem. By the theory of abelian groups, J possesses (p^f, p) -type, so $\Omega_1(J)$ is abelian with (p, p) -type and normal in G .

Let $U = \Omega_1(J)$. Then $U = \langle z \rangle \times \langle y \rangle$, $z^p = y^p = 1$, where $\langle z \rangle = N$ is normal in G . Set $V = \langle y \rangle$. We consider UH . Since G is supersolvable, V is also H -invariant. If H acts nontrivially on V , then V is the derived subgroup of VH . By hypothesis, V is normal in G , which contradicts the uniqueness of N . Therefore, H acts trivially on V and so H acts trivially on U/N . Hence $C_{G^{\mathcal{N}}/N}(H) > 1$. This is a contradiction as the nilpotent residual $G^{\mathcal{N}}/N$ of G/N which is abelian and HN/N is a p' -action on it so that $[P/N, HN/N] = P/N$, the nilpotent residual of G/N . \square

D.J.S. Robinson had proved the following: If N is a nilpotent normal subgroup of a group G and G/N is supersolvable, then G is supersolvable [16]. Theorem 3.3 is immediate from the Robinson theorem. But, Theorem 3.4 is not obvious.

Theorem 3.3. *A D -group G is supersolvable if and only if G/G'' is supersolvable.*

Theorem 3.4. *The D -group G is nilpotent if and only if the nilpotent residual $G^{\mathcal{N}} \subseteq G''$.*

Proof. The necessity of the theorem is clear. Assume the converse is not true and let G be a counterexample of minimal order. Then $G^{\mathcal{N}} > 1$. Let $G^{\mathcal{N}}/K$ be a chief factor of G and consider the quotient group G/K . By Theorem 3.3, $G^{\mathcal{N}}/K$ is cyclic of order a prime p , and we know that G/K is a D -group by definition. By the choice of G we see that K must be 1, namely $G^{\mathcal{N}}$ is cyclic of order p . Obviously, $G^{\mathcal{N}} \not\subseteq \Phi(G)$ (otherwise G is nilpotent, a contradiction). Thus there exists a maximal subgroup M of G such that $G = G^{\mathcal{N}}M$ and $G^{\mathcal{N}} \cap M = 1$. If M is abelian, then $G' = G^{\mathcal{N}}$, but $G^{\mathcal{N}} \leq G''$ by hypothesis, it follows that $G' = G''$, which gives $G' = 1$, a contradiction. Therefore we can let M be non-abelian. Then we can find a minimal non-abelian subgroup $Q \leq M$, and hence Q' is of order a prime q . By hypothesis Q' is normal in G and so Q' is in the center $Z(M)$ because M is nilpotent. Also, $[G^{\mathcal{N}}, Q'] \leq G^{\mathcal{N}} \cap Q' = 1$, so $G^{\mathcal{N}}$ centralizes Q' too. Consequently, $Z(G) \geq Q' > 1$. Now, the quotient group $G/Z(G)$ satisfies the condition obviously, it follows that $G/Z(G)$ is nilpotent by the choice of G . But then, G would be nilpotent, a final contradiction. \square

4. A complete proof

Li and Shen [13], gave the following: For a finite group G , if all elements of prime order of G are in $D(G)$, then G is solvable and the Fitting length of G is bounded by 3. However, in the course of proof we omitted the following theorem. In fact, the following theorem has its own interest.

Theorem 4.1. *Let G be a p -solvable group. Suppose that all elements of G of order p are in $D(G)$. If $p = 2$, in addition, all elements of G of order 4 are in $D(G)$, then $l_p(G) \leq 1$.*

Proof. We use induction on $|G|$. Clearly, $G/O_{p'}(G)$ satisfies the hypothesis and $l_p(G/O_{p'}(G)) = l_p(G)$. We may assume that $O_{p'}(G) = 1$.

Let P be a Sylow p -subgroup of $D(G)$. By Theorem 2.4, $D(G)'$ is nilpotent. Thus $O_{p'}(G) = 1$ implies $D(G)' \leq P$, it follows that P is normal in $D(G)$ and so P is normal in G . Also, $F_p(G) = O_{p',p}(G) = O_p(G)$. As G is p -solvable, by [15, p.269, Theorem 9.3.1], we know

$$C_G(O_p(G)) \leq O_p(G).$$

We now claim that G is q -nilpotent for any prime $q \neq p$. Otherwise, there exists a prime q such that G is non- q -nilpotent. Then there exists

a subgroup K with the following properties: K is non- q -nilpotent but all proper subgroups of K are q -nilpotent. By a theorem of Itô [15, p.296, Theorem 10.3.3], $K = [Q]R$, where Q is a normal q -subgroup, $\exp(Q) = p$ or 4 , and R is a cyclic r -subgroup, the prime $r \neq q$. We know that $K' = Q$. Consider the subgroup

$$M = O_p(G)Q.$$

Let $p > 2$. By above, $\Omega_1(G_p) \leq P \leq O_p(G)$, so $\Omega_1(G_p) = \Omega_1(O_p(G))$. Then $\Omega_1(O_p(G)) \trianglelefteq G$. By hypothesis, $\Omega_1(O_p(G))$ normalizes $K' = Q$, it follows that $[Q, \Omega_1(O_p(G))] = 1$. By [12, p.437, 5.12], we get $[Q, O_p(G)] = 1$. Thus $Q \leq C_G(O_p(G))$. As $C_G(O_p(G)) \leq O_p(G)$ and Q is a p' -group, Q must be 1, a contradiction. Similar for the case when $p = 2$.

Now let $G_{q'}$ denote the normal q -complement of G for every prime $q \neq p$. Then $G_p \leq G_{q'}$ and G_p is the intersection of all $G_{q'}$, hence $G_p \trianglelefteq G$, and of course, $l_p(G) = 1$. The proof is now complete. \square

Next, we give a new elementary proof of [13, Theorem 4.1(i)] by Burnside theorem and a complete proof of [13, Theorem 4.1(ii)].

Theorem 4.2. *Let G be a finite group. If all elements of odd prime order of G are in $D(G)$, then:*

- (i) G is solvable;
- (ii) The Fitting length of G is bounded by 3.

Proof. First we show (i). Assume that the theorem is false and let G be a counterexample of minimal order. If M is a proper subgroup of G , by Proposition 2.2 we have $M \cap D(G) \leq D(M)$. Thus all cyclic subgroups of M of odd prime order are in $D(M)$. So M satisfies the condition. By the choice of G , M is solvable. Consequently, G is a non-solvable group in which all proper subgroups are solvable, so that $G/\Phi(G)$ is a minimal simple group. As $D(G)$ is normal in G and solvable, it follows that $D(G) \leq \Phi(G)$, the Frattini subgroup of G .

Let p be an odd prime dividing the order of G and let G_p be a Sylow p -subgroup of G . We firstly claim the following two conclusions:

- (i) $\Omega_1(G_p) \trianglelefteq G$ and
- (ii) $C_G(\Omega_1(G_p)) \leq \Phi(G)$.

It is well known that $\Phi(G)$ is nilpotent, so all Sylow subgroups of $\Phi(G)$ are normal in G . Let P be a Sylow p -subgroup of $\Phi(G)$. By hypothesis, all subgroups of G of order p are in $D(G)$ and hence in P , so $\Omega_1(G_p) = \Omega_1(P)$. Thus $\Omega_1(G_p) \text{ char } P \trianglelefteq G$, (i) follows. Let us show (ii). By (i), $\Omega_1(G_p)$ is normal in G , it follows that $C_G(\Omega_1(G_p))$ is normal in G . Thus $G/\Phi(G)$ contains a normal subgroup $C_G(\Omega_1(G_p))\Phi(G)/\Phi(G)$. As $G/\Phi(G)$ has no non-trivial normal subgroups, we have $C_G(\Omega_1(G_p))\Phi(G) = \Phi(G)$ or $C_G(\Omega_1(G_p))\Phi(G) = G$. Suppose that the second case happens. Then we have $C_G(\Omega_1(G_p)) = G$, i.e. $\Omega_1(G_p) \leq Z(G)$. Thus all elements of G of order p are in $Z(G)$. Noting that p is an odd prime, we can apply the Itô lemma [12, p.435, Satz 5.5] to see that G is p -nilpotent. Because the quotient groups of a p -nilpotent group is also p -nilpotent, we see that $G/\Phi(G)$ would be p -nilpotent. But $G/\Phi(G)$ has no non-trivial normal subgroup, which implies that $G/\Phi(G)$ is a p' -group. However, by [12, III, Satz 3.8], $p \mid |G/\Phi(G)|$ holds whenever $p \mid |\Phi(G)|$. This is a contradiction. We thus conclude that only the first case is true, which implies (ii).

Fix an odd prime p as above. Consider the subgroup

$$N = N_G(G_p).$$

By Schur-Zassenhaus theorem [15, p.253, Theorem 9.12], N possesses a Hall p' -subgroup H such that $N = [G_p]H$. By condition, $\Omega_1(G_p) \leq D(G)$, namely $\Omega_1(G_p)$ normalizes the derived subgroup of every subgroup of G , so $\Omega_1(G_p)$ normalizes H' . On the other hand, by (i), we have $\Omega_1(G_p) \trianglelefteq N$. Thus $[\Omega_1(G_p), H'] \leq \Omega_1(G_p) \cap H' = 1$, hence H' acts trivially on $\Omega_1(G_p)$ by conjugation. By [12, p.437, Satz 5.12], H' acts trivially on G_p . That is, the subgroup $G_p H' = G_p \times H'$. Now, $N/G_p H' = G_p H/G_p H' \cong H/(G_p H' \cap H) = H/H'$, so $N' \leq G_p \times H'$. Because G_p and H are subgroups of N , we have $G'_p \leq N'$ and $H' \leq N'$, so we can write for some $P \leq G_p$,

$$N' = P \times H', G'_p \leq P \leq G_p.$$

As G is non-solvable, by the Burnside $\{p, q\}$ -theorem [15, p.247, Theorem 8.5.3], the order of G contains at least three distinct primes. Therefore there exists another odd prime q dividing the order G such that $q \neq p$. Let G_q be a Sylow q -subgroup of G . By (i), we have $\Omega_1(G_q) \trianglelefteq G$. Also, by hypothesis, $\Omega_1(G_q) \leq D(G)$, so $\Omega_1(G_q)$ normalizes N' . As $P \text{ char } N'$, it follows that $\Omega_1(G_q)$ normalizes P too. Thus the subgroup $\Omega_1(G_q)P = \Omega_1(G_q) \times P$, and hence $C_G(\Omega_1(G_q)) \geq P$. Applying (ii), we see that $P \leq \Phi(G)$. Recall that H' acts trivially on $\Omega_1(G_p)$, applying

(ii) again, we have $H' \leq \Phi(G)$. Thus

$$N' = P \times H' \leq \Phi(G).$$

Study the quotient group $\overline{G} = G/\Phi(G)$. Then $\overline{G}_p = G_p\Phi(G)/\Phi(G)$ is a Sylow p -subgroup of $G/\Phi(G)$. Write $N_{\overline{G}}(\overline{G}_p) = M/\Phi(G)$. Then $G_p\Phi(G) \trianglelefteq M$ and G_p is a Sylow p -subgroup of $G_p\Phi(G)$. By Frattini argument $M = N_M(G_p)\Phi(G)$. Therefore $N_{\overline{G}}(\overline{G}_p) = N_G(G_p)\Phi(G)/\Phi(G)$. Now, $N_{\overline{G}}(\overline{G}_p) \cong N_G(G_p)/(N_G(G_p) \cap \Phi(G))$, and, by the above, $N_G(G_p)' = N' \leq \Phi(G)$, so $N_G(G_p)/(N_G(G_p) \cap \Phi(G))$ is abelian. Consequently, $N_{\overline{G}}(\overline{G}_p)$ is abelian. By a theorem of Burnside [12, IV, Hauptsatz 2.6], \overline{G} is p -nilpotent. This is not possible because \overline{G} is a minimal simple group. The proof now is complete.

The proof of (ii): Let p be any odd prime dividing $|G|$ and let P be a Sylow p -subgroup of G . As G is solvable, it is p -solvable. According to Theorem 4.1, we have $F_p(G) = O_{p',p}(G) = O_{p'}(G)P$, the maximal normal p -nilpotent subgroup of G . Then $C_G(P) \leq F_p(G)$ by [15, p.269, Theorem 9.3.1]. Next, by Frattini argument $G = N_G(P)O_{p'}(G)$. On the other hand, by Schur-Zassenhaus's theorem [15, p.253, Theorem 9.1.2], $N_G(P) = [P]M$, where M is a Hall p' -subgroup of $N_G(P)$. By hypothesis, $\Omega_1(P)$ normalizes M' . Hence M' centralizes $\Omega_1(P)$, and thus centralizes P . Consequently

$$M' \leq F_p(G).$$

Now $G = F_p(G)M$, it follows that $G/F_p(G) \cong M/F_p(G) \cap M$. As $M' \leq F_p(G) \cap M$, $G/F_p(G)$ is an abelian group. Let T be the intersection of all $F_p(G)$. Then T is p -nilpotent for every odd prime p , and hence T is an extension of an abelian 2-group by a nilpotent group of odd order. Thus we get a series of normal subgroups of G :

$$1 \leq T_2 \leq T \leq G,$$

where T_2 is the Sylow 2-subgroup of T . In this series all the factor groups are nilpotent, which indicates the Fitting length of G is at most 3, completing the proof. \square

5. Some relation conjectures

By hypercenter results [20], we give the following conjectures:

Conjecture 5.1. *Let G be a p -solvable group. Suppose that all elements of G of order p are in $D_\infty(G)$. If $p = 2$, in addition, all elements of G of order 4 are in $D_\infty(G)$, then the $l_p(G) \leq 1$.*

Conjecture 5.2. *Let G be a finite group. If all elements of prime order of G are in $D_\infty(G)$, then:*

- (i) G is solvable;
- (ii) The Fitting length of G is bounded by 3.

On generation results, Thompson, Baer and Flavell [3, 10, 21-22] gave the following:

Theorem 5.3. *Let G be a group. Then G is a solvable group if and only if $\langle x, y \rangle$ is a solvable group, $\forall x, y \in G$.*

Theorem 5.4. *Let G be a group. Then G is a supersolvable group if and only if $\langle x, y \rangle$ is a supersolvable group, $\forall x, y \in G$.*

Theorem 5.5. *Let G be a group. Then G is a \mathcal{F}_{dn} -group if and only if $\langle x, y, z \rangle$ is a \mathcal{F}_{dn} -group, $\forall x, y, z \in G$.*

We observe that D -groups are closely related to supersolvable groups and \mathcal{F}_{dn} -groups. So we give the following conjecture:

Conjecture 5.6. *Let G be a group. Then G is a D -group if and only if $\langle x, y, z \rangle$ is a D -group, for $\forall x, y, z \in G$.*

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REFERENCES

- [1] A. R. Ashrafi, B. Taeri, On finite groups with exactly seven element centralizers, *J. Appl. Math. Comput.* **22** (2006), no. 1-2, 403-410.
- [2] R. Baer, Norm and hypernorm, *Publ. Math. Debrecen*, **4** (1956) 347-356.
- [3] R. Baer, Classes of finite groups and their properties, *Illinois J. Math.* **1** (1957) 115-187.
- [4] R. Baer, Zentrum und kern von gruppen mit elementen unendlicher ordnung, *Compositio Math.* **2** (1935) 247-249.

- [5] J. C. Beidleman, H. Heineken and M. Newell, Center and norm, *Bull. Austral. Math. Soc.* **69** (2004), no. 3, 457–464.
- [6] P. Bhattacharya and N. P. Mukherjee, On the intersection of a class of maximal subgroups of a finite group, *J. Pure and Appl. Algebra* **42** (1986), no. 2, 117–124.
- [7] R. A. Bryce and J. Cossey, The Wielandt subgroup of a finite soluble group, *J. London Math. Soc. (2)* **40** (1989), no. 2, 244–256.
- [8] J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970) 15–17.
- [9] A. R. Camina, The Wielandt length of finite groups, *J. Algebra* **15** (1970) 142–148.
- [10] P. Flavell, Finite groups in which every two elements generate a soluble subgroup, *Invent. Math.* **121**(1995), no. 2, 279–285.
- [11] D. Gorenstein, *Finite Groups*, Chelsea, New York, 1980.
- [12] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
- [13] S. Li and Z. Shen, On the intersection of the normalizers of derived subgroups of all subgroups of a finite group, *J. Algebra*, **323** (2010), no. 5, 1349–1357.
- [14] G. A. Miller and H. C. Moreno, Non-abelian groups in which every subgroup is abelian. *Trans. Amer. Math. Soc.* **4** (1903), no. 4, 398–404.
- [15] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York-Berlin, 1982.
- [16] D. J. S. Robinson, A property of the lower central series of a group, *Math. Z.* **107** (1968) 225–231.
- [17] E. Schenkman, On the norm of a group, *Illinois J. Math.* **4** (1960) 150–152.
- [18] J. G. Thompson, Nonsolvable groups all of whose local subgroups are solvable, *I. Bull. Amer. Math. Soc.* **74** (1968) 383–437.
- [19] J. Wang and X. Guo, On the norm of finite groups, *Algebra Colloquium* **14** (2007), no. 4, 605–612.
- [20] H. Wielandt, Über der normalisator der subnormalen untergruppen, *Math. Z.* **69** (1958) 463–465.
- [21] A. Yokoyama, Finite soluble groups whose \mathcal{F} -hypercenter contains all minimal subgroups, *Arch. Math.* **26** (1975) 123–130.
- [22] J. Zhang, *Algebra in the 21th*, Beijing University press, 2003.

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