A FIXED POINT THEOREM IN GENERALIZED D–METRIC SPACES

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ABSTRACT. In this paper, we consider the concept of Δ–distance on a complete D–metric space and prove a fixed point theorem.

1. Introduction and preliminaries

Recently, Dhage [1] introduced the concept of D-metric. Afterwards, many authors [4, 5, 6] proved some fixed point theorems in these spaces.

In this paper, using the concept of D-metric, we define a Δ–distance on a complete D-metric space which is a generalization of the concept of ω–distance due to Kada, Suzuki and Takahashi [2]. This generalization is non trivial because a D-metric does not always define a topology, and even when it does, this topology is not necessarily Hausdorff (see [3] and [4, Ch.1]). Using the concept of Δ-distance, we prove a fixed point theorem, which is the main result of this paper.

We state the definition of D–metric, Δ–distance and prove a lemma. For more information on D–metrics, we refer the reader to [1, 4, 6].

Definition 1.1. ([1]) Let X be a non-empty set. A function $D : X \times X \times X \longrightarrow [0, \infty)$ is called a D–metric if the following conditions are satisfied:

   (a) (coincidence) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$,
   (b) (symmetry) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation of x, y, z,
   (c) (tetrahedral inequality) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$.

Definition 1.2. ([1]) (a) A sequence $\{x_n\}$ in X is called a D-Cauchy sequence if for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m > n$, $p \geq n_0$,

$$D(x_m, x_n, x_p) < \varepsilon.$$

   (b) A sequence $\{x_n\}$ in X is said to be D-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$,

$$D(x_m, x_n, x) < \varepsilon.$$

Definition 1.3. Let X be a metric space with metric D. Then a function $\Delta : X \times X \times X \longrightarrow [0, \infty)$ is called a Δ–distance on X if the following conditions are satisfied:

   (a) $\Delta(x, y, z) \leq \Delta(x, y, a) + \Delta(x, a, z) + \Delta(a, y, z)$ for all $x, y, z, a \in X$,
   (b) for any $x, y \in X$, $\Delta(x, y, .) : X \longrightarrow [0, \infty)$ is lower semi-continuous,
   (c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\Delta(a, x, y) \leq \delta$, $\Delta(a, x, z) \leq \delta$ and $\Delta(a, y, z) \leq \delta$ imply $D(x, y, z) \leq \varepsilon$.

Example 1.4. Let X be a metric space with the metric D defined by

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},$$

for all $x, y, z \in X$ (see [4, Corollary 1.20]). Then $\Delta = D$ is a Δ–distance on X.

Proof. (a) and (b) are obvious. We show (c). Let $\varepsilon > 0$ be given and put $\delta = \varepsilon$. If $D(a, x, y) \leq \delta$, $D(a, x, z) \leq \delta$ and $D(a, y, z) \leq \delta$, we have $d(x, y) \leq \delta$, $d(x, z) \leq \delta$, and $d(y, z) \leq \delta$, which implies that $D(x, y, z) \leq \delta = \varepsilon$.   □

Example 1.5. Let X be a metric space with the metric D defined by

$$D(x, y, z) = d(x, y) + d(y, z) + d(x, z),$$

for all $x, y, z \in X$ (see [4, Theorem 1.13 and Corollary 1.17]). Then the function $\Delta : X^3 \to [0, \infty)$ defined by

$$\Delta(x, y, z) = t,$$

for all $x, y, z \in X$, is a Δ–distance on X, where t is a positive real number.

Proof. The proofs of (a) and (b) are immediate. To show (c), for any $\varepsilon > 0$, put $\delta = \frac{\varepsilon}{3}$. Then $\Delta(a, x, y) \leq \delta$, $\Delta(a, x, z) \leq \delta$ and $\Delta(a, y, z) \leq \delta$ imply $D(x, y, z) \leq \varepsilon$. $\qquad\square$

Example 1.6. Let $X = \mathbb{R}$ be a metric space with the metric D defined by

$$D(x, y, z) = |x - y| + |y - z| + |x - z|,$$

for all $x, y, z \in \mathbb{R}$ (see [4, Theorem 1.13 and Corollary 1.17]). Then a function $\Delta : \mathbb{R}^3 \to [0, \infty)$ defined by

$$\Delta(x, y, z) = |y - z|,$$

for all $x, y, z \in \mathbb{R}$ is a Δ–distance on $\mathbb{R}$.

Proof. The proofs of (a) and (b) are immediate. We show (c). Let $\varepsilon > 0$ be given and put $\delta = \frac{\varepsilon}{3}$. If $\Delta(a, x, y) \leq \delta$, $\Delta(a, x, z) \leq \delta$ and $\Delta(a, y, z) \leq \delta$, then $|x - y| \leq \delta$, $|x - z| \leq \delta$ and $|y - z| \leq \delta$, which imply that $D(x, y, z) \leq \delta + \delta + \delta = \varepsilon$. $\qquad\square$

Lemma 1.7. *Let X be a metric space with metric D and let Δ be a Δ–distance on X. Let $\{x_n\}$, $\{y_n\}$ be sequences in X and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, \infty)$ converging to zero and assume that $x, y, z, a \in X$. Then we have the following implications:*

    (a) *If $\Delta(x_n, a_n, y_n) \leq \alpha_n$, $\Delta(x_n, a_n, z) \leq \beta_n$ and $\Delta(x_n, y_n, z) \leq \gamma_n$, for any $n \in \mathbb{N}$, then $D(a_n, y_n, z) \to 0$.*

    (b) *If $\Delta(x_n, x_m, x_p) \leq \alpha_n$, for any $p, n, m \in \mathbb{N}$ with $m < n < p$, then $\{x_n\}$ is a D-Cauchy sequence.*

Proof. (a) Let $\varepsilon > 0$ be given. From the definition of Δ–distance, there exists $\delta > 0$ such that $\Delta(a, u, v) \leq \delta$, $\Delta(a, u, z) \leq \delta$ and $\Delta(a, v, z) \leq \delta$

imply $D(u, v, z) \leq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$, $\beta_n \leq \delta$ and $\gamma_n \leq \delta$ for every $n \geq n_0$. Then for any $n \geq n_0$ we have

$$\Delta(x_n, a_n, y_n) \leq \alpha_n \leq \delta, \ \Delta(x_n, a_n, z) \leq \beta_n \leq \delta, \ \Delta(x_n, y_n, z) \leq \gamma_n \leq \delta,$$

and hence

$$D(a_n, y_n, z) \leq \varepsilon.$$

If we replace $\{a_n\}$ with $\{y_n\}$, then $\{y_n\}$ converges to z.

Let us now prove (b). Let $\varepsilon > 0$ be given. As in the proof of (a), choose $\delta > 0$ and then $n_0 \in \mathbb{N}$. Then, for any $p > n > m \geq n_0 + 1$,

$$\Delta(x_{n_0}, x_n, x_m) \leq \alpha_{n_0} \leq \delta, \ \Delta(x_{n_0}, x_n, x_p) \leq \beta_{n_0} \leq \delta,$$

$$\Delta(x_{n_0}, x_m, x_p) \leq \gamma_{n_0} \leq \delta,$$

and hence

$$D(x_n, x_m, x_p) \leq \varepsilon.$$

This implies that $\{x_n\}$ is a D-Cauchy sequence. $\square$

2. The main result

In [4], the author showed that there are some D–metrics which are not continuous. In this paper, we assume that D–metrics lie in $\mathcal{D}$, where $\mathcal{D}$ is the class of continuous D–metrics. Also, X is said to be Δ–*bounded* if there is a constant $M > 0$ such that $\Delta(x, y, z) \leq M$ for all $x, y, z \in X$.

Now, we give the main result of this paper.

Theorem 2.1. *Let X be a complete metric space with metric D, Δ a Δ–distance on X and T a mapping from X into itself. Let X be Δ–bounded. Suppose that there exists $r \in [0, 1)$ such that*

$$\Delta(Tx, T^2x, Tw) \leq r\Delta(x, Tx, w),$$

for all $x, w \in X$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $\Delta(v, v, v) = 0$.

Proof. We claim that

$$\inf\{\Delta(x, Tx, y) + \Delta(x, Tx, T^2x) + \Delta(x, T^2x, y) \ : \ x \in X \ \} > 0,$$

for all $y \in X$ with $y \neq Ty$. For the moment, suppose that the claim is true. Let $u \in X$ and define a sequence $\{u_n\}$ in X by

$$u_n = T^n u,$$

for all $n \in \mathbb{N}$. Then, for all $n, t \in \mathbb{N}$, we have

$$\Delta(u_n, u_{n+1}, u_{n+t}) \le r\Delta(u_{n-1}, u_n, u_{n+t-1}) \le \cdots \le r^n \Delta(u, u_1, u_t).$$

Thus, for any $p > m > n$ for which $m = n + k$ and $p = m + t$ $(k, t \in \mathbb{N})$, we have

$$\begin{aligned}
\Delta(u_n, u_m, u_p) &\le \Delta(u_n, u_{n+1}, u_{n+2}) + \cdots + \Delta(u_{p-2}, u_{p-1}, u_p) \\
&\le \sum_{j=n}^{p} 2Mr^j \\
&\le \frac{r^n}{1-r} 2M.
\end{aligned}$$

(For more details see [4, page 71]).By part (b) of Lemma 1.7, $\{u_n\}$ is a D-Cauchy sequence. Since X is complete, $\{u_n\}$ converges to a point $z \in X$. Let $n \in \mathbb{N}$ be fixed. Then, by lower semi-continuity of Δ, we have

$$\Delta(u_n, u_m, z) \le \liminf_{p \to \infty} \Delta(u_n, u_m, u_p) \le \frac{r^n}{1-r} 2M.$$

Assume that $z \ne Tz$. Then, by hypothesis, we have

$$\begin{aligned}
0 &< \inf\{\Delta(x, Tx, z) + \Delta(x, Tx, T^2x) + \Delta(x, T^2x, z)\} \\
&\le \inf\{\Delta(u_n, u_{n+1}, z) + \Delta(u_n, u_{n+1}, u_{n+2}) + \Delta(u_n, u_{n+2}, z) : n \in \mathbb{N}\} \\
&\le \inf\left\{\frac{r^n}{1-r} 2M + r^n M + \frac{r^{n+1}}{1-r} 2M : n \in \mathbb{N}\right\} = 0.
\end{aligned}$$

This is a contradiction. Therefore, we have $z = Tz$.

Now, if $v = Tv$, we have

$$\Delta(v, v, v) = \Delta(Tv, T^2v, T^3v) \le r\Delta(v, Tv, T^2v) = r\Delta(v, v, v),$$

and so $\Delta(v, v, v) = 0$.

Now, it remains to prove the claim. Assume that there exists $y \in X$ with $y \ne Ty$ and

$$\inf\{\Delta(x, Tx, y) + \Delta(x, Tx, T^2x) + \Delta(x, T^2x, y)\} = 0.$$

Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty}\{\Delta(x_n, Tx_n, y) + \Delta(x_n, Tx_n, T^2x_n) + \Delta(x_n, T^2x_n, y)\} = 0.$$

Thus we have

$$\lim_{n \to \infty} \Delta(x_n, Tx_n, y) = 0, \quad \lim_{n \to \infty} \Delta(x_n, Tx_n, T^2x_n) = 0,$$

$$\lim_{n \to \infty} \Delta(x_n, T^2x_n, y) = 0,$$

and hence, by part (a) of Lemma 1.7, we have $\lim_{n\to\infty} D(Tx_n, T^2x_n, y) = 0$, and by continuity of D–metric,

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T^2x_n = y.$$

We also have

$$\lim_{n\to\infty} \Delta(Tx_n, T^2x_n, Ty) \leq r \lim_{n\to\infty} \Delta(x_n, Tx_n, y) = 0,$$

$$\lim_{n\to\infty} \Delta(Tx_n, y, Ty) \leq \liminf_{n\to\infty} \Delta(Tx_n, T^2x_n, Ty)$$

$$\leq r \liminf_{n\to\infty} \Delta(x_n, Tx_n, y) = 0,$$

and

$$\begin{aligned}
\lim_{n\to\infty} \Delta(Tx_n, T^2x_n, y) &\leq \liminf_{n\to\infty} \Delta(Tx_n, T^2x_n, T^2x_n)\\
&\leq r \liminf_{n\to\infty} \Delta(x_n, Tx_n, Tx_n)\\
&\leq r \liminf_{n\to\infty} \Delta(x_n, Tx_n, T^2x_n) = 0.
\end{aligned}$$

By part (a) of Lemma 1.7, we have $\lim_{n\to\infty} D(T^2x_n, y, Ty) = 0$ and hence $y = Ty$. This is a contradiction. This completes the proof. $\square$

Now, to justify that the set of functions satisfying the conditions of Theorem 2.1 is not void, we give an example.

Example 2.2. Consider Example 1.6 Define a function $T : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$Tx = \frac{x}{2}$$

for all $x \in \mathbb{R}$. Then we have

$$\Delta(Tx, T^2x, Tw) = |T^2x - Tw| = \left|\frac{x}{4} - \frac{w}{2}\right| = \frac{1}{2}\left|\frac{x}{2} - w\right| \leq \frac{1}{2}\Delta(x, Tx, w)$$

for all $x, w \in \mathbb{R}$. Thus all the conditions of Theorem 2.1 are satisfied and 0 is a fixed point of T.

References

[1] B. C. Dhage, Generalized metric spaces and mappings with fixed point, *Bull. Calcutta Math. Soc.* **84** (1992), 329–336.

[2] O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.* **44** (1996), 381–391.

[3] S. V. R. Naidu, K. P. R. Rao and N. S. Rao, On the topology of D-metric spaces and generation of D-metric spaces from metric spaces, *Int. J. Math. Math. Sci.* **51** (2004), 2719-2740.

[4] N. S. Rao, A study of the topological properties of D–metric spaces and fixed point theorems in metric and D–metric spaces, Ph. D. Thesis, Acharia Nagarjuna University, 2004.

[5] B. E. Rhoades, A fixed point theorem for generalized metric spaces, *Int. J. Math. Math. Sci.* **19** (1996), 457-460.

[6] J. S. Ume and J. K. Kim, Common fixed point theorems in D-metric spaces with local boundedness, *Indian J. Pure Appl. Math.* **31** (2000), 865–871.

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