COMPACT ENDOMORPHISMS OF CERTAIN SUBALGEBRAS OF THE DISC ALGEBRA

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ABSTRACT. In this paper we study endomorphisms of the following subalgebras of the disc algebra $A(\mathbb{D})$: The natural uniform subalgebras of $A(\mathbb{D})$, the analytic Lipschitz algebras $Lip_A(\mathbb{D}, \alpha)$ of order α $(0 < \alpha \leq 1)$ and the n-times differentiable Lipschitz algebras $Lip^n(\mathbb{D}, \alpha)$ of order α $(0 < \alpha \leq 1)$. Every nonzero endomorphism T of many commutative semisimple Banach algebras including these subalgebras of $A(\mathbb{D})$ has the form $Tf = f \circ \varphi$ for some $\varphi : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ in them. We show that a sufficient condition for φ to induce a compact endomorphism of these algebras is that either φ is constant or $\|\varphi\|_{\mathbb{D}} < 1$. We then show that these conditions are also necessary when $\alpha = 1$.

1. Introduction

In this note we consider endomorphisms of three types of Banach algebras of analytic functions on the open unit disc $\mathbb{D}$. We recall that a compact endomorphism of a Banach algebra B is a compact linear map from B into B which preserves multiplication. Let B be a Banach function algebra on a compact Hausdorff space X,

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i.e., an algebra of complex-valued continuous functions on X which separates the points of X, contains the constants and is complete under an algebra norm. A Banach function algebra B on X is called natural, if its maximal ideal space is X, i.e., each complex homomorphism on B has the form $f \mapsto f(x)$ for some $x \in X$. It is well known that, if B is a natural Banach function algebra on X and T is a nonzero bounded endomorphism of B, then there exists a self-map φ on X such that $Tf = f \circ \varphi$ for all $f \in B$. Conversely, if φ is a self-map on X such that for every $f \in B$, $f \circ \varphi \in B$, then $T : f \to f \circ \varphi$ is an endomorphism of B. In each case, we say that T is induced by φ. If X is a compact plane set and B contains the coordinate map z, then obviously $\varphi \in B$. It is interesting to see that under what conditions such φ induces compact endomorphisms. For the disc algebra $A(\mathbb{D})$, the uniform algebra of complex-valued functions analytic on the open unit disc $\mathbb{D}$ and continuous on its closure $\overline{\mathbb{D}}$, H. Kamowitz [5] showed that if T is a nonzero endomorphism of the disc algebra $A(\mathbb{D})$ induced by a map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, then T is compact if and only if φ is either constant or $\|\varphi\|_{\mathbb{D}} = \sup\{|\varphi(z)| : |z| \leq 1\} < 1$. H. Kamowitz and S. Scheinberg [6] also showed that an endomorphism T of the Lipschitz algebra $Lip(X, d)$ induced by a map $\varphi : X \longrightarrow X$ is compact if and only if φ is supercontraction, that is $\frac{d(\varphi(x), \varphi(y))}{d(x,y)} \longrightarrow 0$ as $d(x, y) \longrightarrow 0$ where (X, d) is a metric space. In [1] it was shown that an endomorphism of D^n, the algebra of functions on the closed unit disc $\overline{\mathbb{D}}$ with continuous nth derivatives, is compact if and only if φ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$. Now we consider the following three types of subalgebras of $A(\mathbb{D})$: The natural uniform subalgebras of $A(\mathbb{D})$, the analytic Lipschitz algebras $Lip_A(\mathbb{D}, \alpha)$ of order α $(0 < \alpha \leq 1)$, and the n-times differentiable Lipschitz algebras $Lip^n(\mathbb{D}, \alpha)$ of order α $(0 < \alpha \leq 1)$. We show that a self-map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ in any of these subalgebras induces a compact endomorphism of the subalgebra, if φ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$, and in the case that $\alpha = 1$ these conditions are also necessary. In general we have the following:

Proposition 1.1. *Let B be a subalgebra of $A(\mathbb{D})$ which is a natural Banach function algebra on $\overline{\mathbb{D}}$ under a norm. If $\varphi \in B$ and $\|\varphi\|_{\mathbb{D}} < 1$, then φ induces an endomorphism of B.*

Proof. By the naturality of B, the spectrum $\sigma(\varphi) = \varphi(\overline{\mathbb{D}})$. So $\sigma(\varphi) \subset \mathbb{D}$, since $\|\varphi\|_{\mathbb{D}} < 1$. Therefore, every $f \in B$ is analytic on a neighborhood of $\sigma(\varphi)$, so $f \circ \varphi \in B$, by the Functional Calculus Theorem. That is, φ induces an endomorphism of B. $\square$

2. Natural uniform subalgebras of $\mathbf{A}(\mathbb{D})$

Let B be a natural uniform subalgebra of $A(\mathbb{D})$ with the uniform norm $\|f\|_{\mathbb{D}} = \sup\{|f(z)| : |z| \leq 1\}$, $(f \in B)$. Then we have

Theorem 2.1. *A nonzero endomorphism T of B induced by φ is compact if and only if, φ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$.*

Proof. Obviously, the constant map φ induces a compact endomorphism of B. Let φ be non-constant and $\|\varphi\|_{\mathbb{D}} < 1$. For compactness of T, suppose $\{f_n\}$ is a bounded sequence in B with $\|f_n\|_{\mathbb{D}} \leq 1$. Then by Montel Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is uniformly convergent on every compact subset of $\mathbb{D}$, in particular on $\varphi(\overline{\mathbb{D}})$. So $\|Tf_{n_k} - Tf_{n_j}\|_{\mathbb{D}} = \|f_{n_k} \circ \varphi - f_{n_j} \circ \varphi\|_{\mathbb{D}} = \|f_{n_k} - f_{n_j}\|_{\varphi(\overline{\mathbb{D}})} \to 0$ as $k, j \to \infty$. By completeness of B, $Tf_{n_k} = f_{n_k} \circ \varphi$ is convergent in B. That is, T is compact.

Conversely, let T be a nonzero compact endomorphism of B induced by φ. Suppose for some c, $|c| = 1$ and $|\varphi(c)| = 1$. Define $f_n(z) = (\frac{z}{\varphi(c)})^n$ on $\overline{\mathbb{D}}$. By the compactness of T, the bounded sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $Tf_{n_k} = f_{n_k} \circ \varphi = (\frac{\varphi}{\varphi(c)})^{n_k}$ converges to a function f in B. Since $f_n(\varphi(c)) = 1$ for each n, $f(c) = 1$. If φ is not constant, the maximum modulus principle implies that $|\varphi(z)| < 1$ whenever $|z| < 1$. So $(\varphi(z))^{n_k} \to 0$ when $|z| < 1$. Therefore, $f = 0$ on $\mathbb{D}$ and hence on $\overline{\mathbb{D}}$, by the continuity of f, which is a contradiction. $\square$

3. The analytic Lipschitz algebras of order α $(0 < \alpha \leq 1)$

Let the analytic Lipschitz algebra $Lip_A(\mathbb{D}, \alpha)$ of order α $(0 < \alpha \leq 1)$, be the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$ and satisfy the Lipschitz condition $p_\alpha(f) = \sup\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z, w \in \overline{\mathbb{D}}, z \neq w\} < \infty$. The algebra $Lip_A(\mathbb{D}, \alpha)$ is a Banach function algebra on $\overline{\mathbb{D}}$, if it is equipped with the norm $\|f\| = \|f\|_{\mathbb{D}} + p_\alpha(f)$. As it is shown in [1], the maximal ideal space of $Lip_A(\mathbb{D}, \alpha)$ is $\overline{\mathbb{D}}$, i.e., $Lip_A(\mathbb{D}, \alpha)$ is natural.

To give necessary conditions under which φ induces a compact endomorphism of $Lip_A(\mathbb{D}) = Lip_A(\mathbb{D}, 1)$ we need the following lemma [2; Chapter I of Part Six, p 32].

Lemma 3.1. *Let $f(z)$ be a non-constant analytic function of bound one in the disc $|z| < 1$, and consider any triangle in this disc that has one of its vertices at $z = 1$. Then for any sequence $\{z_n\}$ of points from the interior of the triangle that converges to $z = 1$, the limit*

$$\lim_{n \to \infty} \frac{1 - f(z_n)}{1 - z_n}$$

exists and either always (that is, for every such sequence) equals infinity or always equals a positive number α_0.

In the second case, we refer to the number α_0 as "the angular derivative" of the function f at $z = 1$.

Corollary 3.2. *If $\varphi \in Lip_A(\mathbb{D})$ is non-constant and $\|\varphi\|_{\mathbb{D}} = |\varphi(c)| = 1$ for some c with $|c| = 1$, then φ has a nonzero angular derivative at c.*

Theorem 3.3. *Let T be a nonzero endomorphism of $Lip_A(\mathbb{D}, \alpha)$, $0 < \alpha \leq 1$, induced by a map $\varphi : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$. Then T is compact, if φ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$. For $\alpha = 1$, these conditions are necessary.*

Proof. If φ is constant, clearly T is compact. Let $\|\varphi\|_{\mathbb{D}} < 1$. For the compactness of T, we assume that $\{f_n\}$ is a bounded sequence

in $Lip_A(\mathbb{D}, \alpha)$ with $\|f_n\| = \|f_n\|_{\mathbb{D}} + p_\alpha(f_n) \le 1$. Then $\{f_n\}$ is a bounded sequence of analytic functions on $\mathbb{D}$. By Montel Theorem $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ and its derivative $\{f'_{n_k}\}$ are uniformly convergent on every compact subset of $\mathbb{D}$. We claim that $\{f_{n_k} \circ \varphi\}$ is convergent in $Lip_A(\mathbb{D}, \alpha)$. Let $\Delta_r = \{z : |z| \le r\}$ where $\|\varphi\|_{\mathbb{D}} < r < 1$. Then Δ_r is a compact subset of $\mathbb{D}$ containing the compact set $\varphi(\overline{\mathbb{D}})$. Hence

$$\|f_{n_k} \circ \varphi - f_{n_j} \circ \varphi\|_{\mathbb{D}} = \|f_{n_k} - f_{n_j}\|_{\varphi(\overline{\mathbb{D}})} \to 0 \;\; as \; k, j \to \infty,$$

and for all $z, w \in \overline{\mathbb{D}}$, with $z \ne w$ and $k, j \in \mathbb{Z}_+$ we have

$$\frac{|(f_{n_k} \circ \varphi - f_{n_j} \circ \varphi)(w) - (f_{n_k} \circ \varphi - f_{n_j} \circ \varphi)(z)|}{|w - z|^\alpha}$$
$$= \frac{|(f_{n_k} - f_{n_j})(\varphi(w)) - (f_{n_k} - f_{n_j})(\varphi(z))|}{|w - z|^\alpha}$$
$$\le \frac{\|f'_{n_k} - f'_{n_j}\|_{\Delta_r} |\varphi(w) - \varphi(z)|}{|w - z|^\alpha} \le p_\alpha(\varphi) \|f'_{n_k} - f'_{n_j}\|_{\Delta_r}.$$

Hence

$$p_\alpha(f_{n_k} \circ \varphi - f_{n_j} \circ \varphi) \le p_\alpha(\varphi) \|f'_{n_k} - f'_{n_j}\|_{\Delta_r} \to 0 \;\; as \; k, j \to \infty.$$

Therefore, $\{f_{n_k} \circ \varphi\}$ is a Cauchy sequence in $Lip_A(\mathbb{D}, \alpha)$ and hence T is compact.

Conversely, let $\alpha = 1$ and let $0 \ne T$ be a compact endomorphism of $Lip_A(\mathbb{D})$ induced by φ. Suppose for some c, $|c| = 1$ and $|\varphi(c)| = 1$. So $\|\varphi\|_{\mathbb{D}} = 1$. Define $f_n(z) = \dfrac{z^n}{n}$. Then $\|f_n\|_{\mathbb{D}} = \dfrac{1}{n}$ and $p_1(f_n) \le 1$. Therefore, $\{f_n\}$ is a bounded sequence in $Lip_A(\mathbb{D})$. By the compactness of T, there exists a subsequence $\{f_{n_k}\}$ such that $Tf_{n_k} = f_{n_k} \circ \varphi$ converges in $Lip_A(\mathbb{D})$. Since $f_{n_k} \longrightarrow 0$ uniformly on $\overline{\mathbb{D}}$, $f_{n_k} \circ \varphi \longrightarrow 0$ in $Lip_A(\mathbb{D})$. Thus

$$p_1(f_{n_k} \circ \varphi) = \sup_{\substack{z,w \in \overline{\mathbb{D}} \\ z \ne w}} \left| \frac{\varphi^{n_k}(w) - \varphi^{n_k}(z)}{n_k(w - z)} \right| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

Fix $\epsilon > 0$. Then

$$\sup_{\substack{z,w\in\overline{\mathbb{D}}\\z\neq w}}\left|\frac{\varphi^{n_k}(w)-\varphi^{n_k}(z)}{n_k(w-z)}\right|<\epsilon,$$

for some n_k. In particular,

$$\frac{1}{n_k}\sup_{\substack{z\in\overline{\mathbb{D}}\\z\neq c}}\left|\frac{\varphi^{n_k}(z)-\varphi^{n_k}(c)}{z-c}\right|<\epsilon,\qquad(1)$$

for some n_k. We note that, the self-map φ is in $Lip_A(\mathbb{D})$, since $Lip_A(\mathbb{D})$ contains the identity map z on $\overline{\mathbb{D}}$. If φ is non-constant, then $\varphi \in Lip_A(\mathbb{D})$ satisfies the hypotheses of Corollary 3.2, so φ has a nonzero angular derivative at c. However, by (1)

$$\frac{1}{n_k}\lim_{\substack{z\to c\\z\in\Gamma}}\left|\frac{\varphi^{n_k}(z)-\varphi^{n_k}(c)}{z-c}\right|<\epsilon,$$

where Γ is a triangle in $\mathbb{D}$ that has one of its vertices at c. This means that the angular derivative of φ at c is zero and this is a contradiction. $\square$

We conjecture that the same conditions are necessary for the compactness of an endomorphism T of $Lip_A(\mathbb{D},\alpha)$, $0<\alpha<1$.

4. The differentiable Lipschitz algebras of order α ($0 < \alpha \leq 1$)

A complex-valued function f on $\overline{\mathbb{D}}$ is differentiable on $\overline{\mathbb{D}}$ if at each point $a \in \overline{\mathbb{D}}$,

$$f'(a)=\lim_{\substack{z\to a\\z\in\overline{\mathbb{D}}}}\frac{f(z)-f(a)}{z-a}$$

exists. Note that, every differentiable function on $\overline{\mathbb{D}}$ is analytic on $\mathbb{D}$.

Let the n-times differentiable Lipschitz algebra $Lip^n(\mathbb{D},\alpha)$ of order α ($0 < \alpha \leq 1$), be the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ whose derivatives up to order n exist and for each k ($0 \leq k \leq n$), $f^{(k)}$ satisfy the Lipschitz condition $p_\alpha(f^{(k)}) =$

$\sup\{\frac{|f^{(k)}(z) - f^{(k)}(w)|}{|z-w|^\alpha} : z, w \in \overline{\mathbb{D}}, z \neq w\} < \infty$. These algebras are Banach function algebras on $\overline{\mathbb{D}}$, if they are equipped with the norm

$$\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_{\mathbb{D}} + p_\alpha(f^{(k)})}{k!}, \quad (f \in Lip^n(\mathbb{D}, \alpha)).$$

These algebras were introduced in [3], and it was shown that they are natural Banach function algebras. By Proposition 1.1, every self-map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ in $Lip^n(\mathbb{D}, \alpha)$ with $\|\varphi\|_{\mathbb{D}} < 1$ induces an endomorphism of $Lip^n(\mathbb{D}, \alpha)$. However, for these algebras this is true without extra condition on φ, that is

Theorem 4.1. *Every map* $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *in* $Lip^n(\mathbb{D}, \alpha)$ *induces an endomorphism of* $Lip^n(\mathbb{D}, \alpha)$.

Proof. We carry out the proof for the case $n = 1$; the case $n > 1$ is similar. It is enough to show that $f \circ \varphi \in Lip^1(\mathbb{D}, \alpha)$ for every $f \in Lip^1(\mathbb{D}, \alpha)$. For every $z, w \in \overline{\mathbb{D}}$ with $\varphi(z) \neq \varphi(w)$ (so $z \neq w$) and for every $f \in Lip^1(\mathbb{D}, \alpha)$ we have

$$\frac{|f \circ \varphi(z) - f \circ \varphi(w)|}{|z - w|^\alpha} \leq \|f'\|_{\mathbb{D}} \frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha}$$
$$\leq \|f'\|_{\mathbb{D}} \, p_\alpha(\varphi)$$

and

$$\frac{|(f \circ \varphi)'(z) - (f \circ \varphi)'(w)|}{|z - w|^\alpha} = \frac{|f'(\varphi(z))\varphi'(z) - f'(\varphi(w))\varphi'(w)|}{|z - w|^\alpha}$$
$$\leq \frac{|f'(\varphi(z)) - f'(\varphi(w))|}{|\varphi(z) - \varphi(w)|^\alpha} |\varphi'(z)| |\frac{\varphi(z) - \varphi(w)}{z - w}|^\alpha$$
$$+ \frac{|\varphi'(z) - \varphi'(w)|}{|z - w|^\alpha} |f'(\varphi(w))|$$
$$\leq p_\alpha(f')\|\varphi'\|_{\mathbb{D}}^{1+\alpha} + p_\alpha(\varphi')\|f'\|_{\mathbb{D}}.$$

This implies that $f \circ \varphi \in Lip^1(\mathbb{D}, \alpha)$. $\square$

We now consider sufficient conditions that such φ induces compact endomorphisms of $Lip^n(X, \alpha)$, and we show that in the case $\alpha = 1$, these conditions are also necessary. For this we need the following lemma due to Julia [2, Part Six].

Lemma 4.2. *Let f be a continuously differentiable complex-valued function on the closed unit disc $\overline{\mathbb{D}}$. If f is non-constant and $f(1) = 1 = \|f\|_{\mathbb{D}}$, then $f'(1)$ is a strictly positive number.*

Theorem 4.3. *Let T be a nonzero endomorphism of $Lip^1(\mathbb{D}, \alpha)$, $0 < \alpha \leq 1$, induced by a map $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Then T is compact, if φ is either constant or $\|\varphi\|_{\mathbb{D}} < 1$. In the case that $\alpha = 1$, these conditions are necessary.*

Proof. When φ is constant, it is clear. Let $\|\varphi\|_{\mathbb{D}} < 1$. For the compactness of T, we assume that $\{f_n\}$ is a bounded sequence in $Lip^1(\mathbb{D}, \alpha)$ with $\|f_n\| = \|f_n\|_{\mathbb{D}} + \|f_n'\|_{\mathbb{D}} + p_\alpha(f_n) + p_\alpha(f_n') \leq 1$. Then $\{f_n\}$ is a bounded sequence of analytic functions on $\mathbb{D}$. So by Montel Theorem $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ and its derivatives $\{f_{n_k}^{(r)}\}$ of each order are uniformly convergent in every compact subset of $\mathbb{D}$. We denote this subsequence by $\{f_n\}$ again. We want to show that $\{f_n \circ \varphi\}$ is convergent in $Lip^1(\mathbb{D}, \alpha)$. Set $\Delta_r = \{z : |z| \leq r\}$ where $\|\varphi\|_{\mathbb{D}} < r < 1$. So Δ_r is a compact subset of $\mathbb{D}$ and $\varphi(\overline{\mathbb{D}}) \subseteq \Delta_r$. Clearly, $\|f_n \circ \varphi\|_{\mathbb{D}} = \|f_n\|_{\varphi(\overline{\mathbb{D}})}$, $\|(f_n \circ \varphi)'\|_{\mathbb{D}} \leq \|f_n'\|_{\varphi(\overline{\mathbb{D}})} \|\varphi'\|_{\mathbb{D}}$ and for all $z, w \in \overline{\mathbb{D}}$ with $\varphi(z) \neq \varphi(w)$ (so $z \neq w$) we have

$$\frac{|f_n \circ \varphi(z) - f \circ \varphi(w)|}{|z - w|^\alpha} \leq \frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} \|f_n'\|_{\Delta_r} \leq p_\alpha(\varphi)\|f_n'\|_{\Delta_r},$$

so that $p_\alpha(f_n \circ \varphi) \leq p_\alpha(\varphi)\|f_n'\|_{\Delta_r}$, and

$$\frac{|(f_n \circ \varphi)'(z) - (f_n \circ \varphi)'(w)|}{|z - w|^\alpha} = \frac{|f_n'(\varphi(z))\varphi'(z) - f_n'(\varphi(w))\varphi'(w)|}{|z - w|^\alpha}$$

$$\leq \frac{|f_n'(\varphi(z)) - f_n'(\varphi(w))||\varphi'(z)|}{|z - w|^\alpha} + \frac{|\varphi'(z) - \varphi'(w)||f_n'(\varphi(w))|}{|z - w|^\alpha}$$

$$\leq \|f_n''\|_{\Delta_r} \frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} \|\varphi'\|_{\mathbb{D}} + \|f_n'\|_{\Delta_r} p_\alpha(\varphi')$$

$$\leq \|f_n''\|_{\Delta_r} p_\alpha(\varphi)\|\varphi'\|_{\mathbb{D}} + \|f_n'\|_{\Delta_r} p_\alpha(\varphi'),$$

so that $p_\alpha((f_n \circ \varphi)') \leq \|f_n''\|_{\Delta_r} p_\alpha(\varphi)\|\varphi'\|_{\mathbb{D}} + \|f_n'\|_{\Delta_r} p_\alpha(\varphi')$.

Hence all sequences $\{\|f_n \circ \varphi\|_{\mathbb{D}}\}$, $\{\|(f_n \circ \varphi)'\|_{\mathbb{D}}\}$, $\{p_\alpha(f_n \circ \varphi)\}$ and $\{p_\alpha((f_n \circ \varphi)')\}$ are Cauchy sequences. Thus $\{f_n \circ \varphi\}$ is a Cauchy

sequence in $Lip^1(\mathbb{D}, \alpha)$, and hence it is convergent, by the completeness of $Lip^1(\mathbb{D}, \alpha)$.

Conversely, let $\alpha = 1$, $0 \neq T$ be compact and $|c| = 1, |\varphi(c)| = 1$ for some c. Define $f_n(z) = \dfrac{z^n}{n(n-1)}$. Then

$$\|f_n\| = \|f_n\|_{\mathbb{D}} + \|f_n'\|_{\mathbb{D}} + p_1(f_n) + p_1(f_n')$$
$$\leq \frac{1}{n(n-1)} + \frac{2}{n-1} + 1.$$

Therefore, $\{f_n\}$ is a bounded sequence in $Lip^1(\mathbb{D}) = Lip^1(\mathbb{D},1)$. By the compactness of T, there exists a subsequence $\{f_{n_k}\}$ such that $Tf_{n_k} = f_{n_k} \circ \varphi$ is convergent in $Lip^1(\mathbb{D})$. Since $f_{n_k} \to 0$ uniformly on $\overline{\mathbb{D}}$, $f_{n_k} \circ \varphi \to 0$ in $Lip^1(\mathbb{D})$. Thus

$$p_1((f_{n_k} \circ \varphi)') = \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} \frac{|\varphi^{n_k-1}(z)\varphi'(z) - \varphi^{n_k-1}(w)\varphi'(w)|}{(n_k-1)|z-w|}$$
$$= \frac{1}{n_k-1} \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} |\frac{\varphi^{n_k-1}(z) - \varphi^{n_k-1}(w)}{z-w}\varphi'(z)$$
$$+ \frac{\varphi'(z) - \varphi'(w)}{z-w}\varphi^{n_k-1}(w)| \to 0 \quad \text{as} \quad k \to \infty.$$

Considering

$$\frac{1}{n_k-1} \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} |\frac{\varphi'(z) - \varphi'(w)}{z-w}\varphi^{n_k-1}(w)| \leq \frac{1}{n_k-1}p_1(\varphi') \to 0$$

$$\text{as} \quad k \to \infty,$$

we have

$$\frac{1}{n_k-1} \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} |\frac{\varphi^{n_k-1}(z) - \varphi^{n_k-1}(w)}{z-w}\varphi'(z)| \to 0 \quad \text{as} \quad k \to \infty.$$

In particular,

$$\frac{1}{n_k-1} \sup_{\substack{z\in\overline{\mathbb{D}} \\ z\neq c}} |\frac{\varphi^{n_k-1}(z) - \varphi^{n_k-1}(c)}{z-c}\varphi'(z)| \to 0 \quad \text{as} \quad k \to \infty,$$

and hence

$$\frac{1}{n_k - 1} \sup_{\substack{z \in \overline{\mathbb{D}} \\ z \neq c}} |\frac{\varphi^{n_k - 1}(z) - \varphi^{n_k - 1}(c)}{z - c} \varphi'(z)| < \epsilon,$$

for arbitrary $\epsilon > 0$ and some n_k. Then

$$\frac{1}{n_k - 1} \lim_{\substack{z \to c \\ z \in \mathbb{D}}} |\frac{\varphi^{n_k - 1}(z) - \varphi^{n_k - 1}(c)}{z - c} \varphi'(z)| \leq \epsilon,$$

so $|\varphi'(c)|^2 = |(\varphi'(c))^2 \varphi^{n_k - 2}(c)| \leq \epsilon$, for any $\epsilon > 0$. Hence $\varphi'(c) = 0$.

On the other hand, $g = \frac{\varphi}{\varphi(c)} \in Lip^1(\mathbb{D})$, has continuous complex derivative on $\overline{\mathbb{D}}$, and $g(c) = 1 = \|g\|_{\mathbb{D}}$. Then by Lemma 4.2, the function g is constant on $\overline{\mathbb{D}}$, so φ must be constant. $\square$

Remark 4.4. *Using a similar method, one can conclude Theorem 4.3 for $Lip^n(\mathbb{D}, \alpha)$ when $n \geq 1, 0 < \alpha \leq 1$.*

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