A NOTE ON STRONGLY QUOTIENT GRAPHS

R. K. ZAFERANI

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ABSTRACT. The notion of strongly quotient graph was introduced by Adiga et al. [3]. Here, we show that some well known families of graphs are strongly quotient graphs. We also establish an upper bound for the energy of a strongly quotient graph with respect to the distance matrix.

1. **Introduction**

During the past forty years or so an enormous amount of research has been done on graph labeling, where the vertices are assigned values subject to certain conditions. These interesting problems have been motivated by practical problems. Applications of graph labeling have been found in x-ray, crystallography, coding theory, radar, circuit design, astronomy and communication design. Particularly interesting applications of graph labeling can be found in Bloom and Golomb [4, 5]. Recently, Adiga et al. [3] have introduced the notion of strongly quotient graphs and studied them. Throughout this paper, by a labeling f of a graph G of order n we mean an injective mapping,

$$f : V(G) \longrightarrow \{1, 2, \ldots, n\}.$$

We define the quotient function,

$$f_q : E(G) \to \boldsymbol{Q}$$

by

$$f_q(e) = min \ \{ \ \frac{f(v)}{f(w)}, \ \frac{f(w)}{f(v)} \ \}$$

if e joins v and w. Note that for any $e \in E(G), 0 < f_q(e) < 1$.

A graph with n vertices is called a strongly quotient graph if its vertices can be labeled with $1, 2, \ldots, n$, such that the quotient function f_q is injective; i.e., the values $f_q(e)$ on the edges are all distinct. Throughout this paper, SQG stands for strongly quotient graph of order n with maximum number of edges.

In [3], Adiga et al. showed that only a few complete graphs and complete bipartite graphs are strongly quotient graphs, and they also established that all cycles, wheels and grids are strongly quotient graphs. They derived an explicit formula for $\mu(n)$, the maximum number of edges in a strongly quotient graph of order n. In [1], Adiga and Zaferani have established that the clique number $\omega(G)$ and chromatic number $\chi(G)$ are both equal to $1 + \pi(n)$, where $\pi(n)$ is the number of primes not exceeding n. They also determined the size of a maximal independent set $\alpha(G)$ and the minimum defining set $d(G, \chi)$ of a strongly quotient graph of order n. Adiga and Zaferani [2] obtained two eigenvalues of SQG and an upper bound for the energy of SQG with respect to the adjacency matrix.

The remainder of this paper organized as follows. In Section 2, we show that some families of graphs like $P_2 \times C_n, P_3 \times C_n, P_2 \odot C_n, C_n \odot P_m$, ladder, triangular ladder, star, double star, S_{n_1, n_2, n_3}, fan and $K_{\alpha, n-\alpha}$ are strongly quotient graphs. In Section 3, we obtain two eigenvalues of SQG and an upper bound for the energy of SQG with respect to the distance matrix.

2. Some new families of strongly quotient graphs

Definition. A *cartesian product* G_1 and G_2, denoted by $G = G_1 \times G_2$, is a graph G such that $V(G) = V(G_1) \times V(G_2)$ and $(x_1 x_2)(y_1 y_2) \in E(G)$ if and only if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1 y_1 \in E(G_1)$.

Definition. A *path* of length $n - 1$, denoted by P_n, is a sequence of distinct edges $v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n$ with $v_i v_{i+1} \in E(P_n)$. A closed path, with $v_1 = v_n$, is called a *cycle* or a *circuit* and denoted by C_{n-1}.

Theorem 2.1. $P_2 \times C_n$ *is a strongly quotient graph.*

Proof. For $n = 3$, we label the vertices of $P_2 \times C_3$ as shown in Fig. 2.1.

Figure 2.1

We can arrange the values of edges of $P_2 \times C_3$ in an increasing sequence,

$$\{ \; \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \; \}.$$

Hence, $P_2 \times C_3$ is a strongly quotient graph.
For $n \geq 4$, we label the vertices of $P_2 \times C_n$ as shown in Fig. 2.2.

Figure 2.2

We can arrange the values of edges of $P_2 \times C_n$ in an increasing sequence,

$$\{ \frac{1}{2n}, \frac{1}{n+2}, \frac{1}{n+1}, \frac{2}{n+2}, \frac{2}{n+1}, \frac{3}{n+3},$$

$$\frac{4}{n+4}, \ldots, \frac{n}{2n}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{n+2}{n+3}, \frac{n+3}{n+4}, \ldots, \frac{2n-1}{2n} \}.$$

Hence, $P_2 \times C_n$ is a strongly quotient graph.

Theorem 2.2. *$P_3 \times C_n$ is a strongly quotient graph.*

Proof. For $n = 3$, we label the vertices of $P_3 \times C_3$ as shown in Fig. 2.3.

Figure 2.3

It is clear that the values of edges of $P_3 \times C_3$ are all distinct, and hence $P_3 \times C_3$ is a strongly quotient graph.

For $n > 3$ (n odd), we label the vertices of $P_3 \times C_n$ as shown in Fig. 2.4.

Figure 2.4

Using the fact that $x/y < (x+1)/(y+1)$ whenever $x < y$, we can arrange the values of edges of $P_3 \times C_n$ in an increasing sequence,

$$\{ \frac{1}{3n-1}, \frac{1}{2n+1}, \frac{1}{2n}, \frac{2}{3n}, \frac{2}{n+2}, \frac{3}{n+3}, \frac{4}{n+4}, \ldots, \frac{\frac{n-1}{2}}{n+\frac{n-1}{2}}, \frac{n}{3n},$$

$$\frac{\frac{n+1}{2}}{n+\frac{n+1}{2}}, \frac{n+1}{3n}, \frac{1+\frac{n+1}{2}}{n+1+\frac{n+1}{2}}, \ldots, \frac{n-1}{2n-1}, \frac{n}{2n}, \frac{n+1}{2n+1}, \frac{n+2}{2n+2}, \ldots, \frac{2n-1}{3n-1},$$

$$\frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n}, \frac{n+1}{n+2}, \ldots, \frac{2n-1}{2n}, \frac{2n+1}{2n+2}, \ldots, \frac{3n-2}{3n-1} \}.$$

From Fig. 2.5, it is clear that $P_3 \times C_4$ is a strongly quotient graph.

Figure 2.5

For $n > 4$ (n even), we label the vertices of $P_3 \times C_n$ as shown in Fig. 2.6.

Figure 2.6

We can arrange the values of edges of $P_3 \times C_n$ in an increasing sequence,

$$\{ \frac{1}{3n}, \frac{1}{3n-1}, \frac{1}{2n}, \frac{2}{3n}, \frac{2}{2n+3}, \frac{2}{2n+2}, \frac{3}{n+3}, \frac{3}{n+2}, \frac{4}{n+4}, \frac{5}{n+5}, \ldots,$$

$$\frac{n+2}{2n+2}, \frac{n+3}{2n+3}, \frac{n+4}{2n+4}, \frac{n+3}{2n+2}, \frac{n+5}{2n+5}, \ldots, \frac{2n-1}{3n-1}, \frac{2n+1}{3n}, \frac{3}{4},$$

$$\ldots, \frac{n+1}{n+2}, \frac{n+3}{n+4}, \ldots, \frac{2n}{2n+1}, \frac{2n+1}{2n+2}, \frac{2n+3}{2n+4}, \ldots, \frac{3n-2}{3n-1} \}.$$

Hence, $P_3 \times C_n$ is a strongly quotient graph.   □

Definition. A *crown product* of G_1 and G_2, denoted by $G = G_1 \odot G_2$, is defined as follows: Fix a vertex v in G_2. Take $\mid V(G_1) \mid$ copies of G_2 and attach the i-th copy of G_2 to the i-th vertex of G_1 by identifying the vertex v in the i-th copy of G_2 with the i-th vertex of G_1.

Theorem 2.3. $P_2 \odot C_n, n \geq 3$, *is a strongly quotient graph.*

Proof. We label the vertices of $P_2 \odot C_n$ as shown in Fig. 2.7.

Figure 2.7

In Fig. 2.7, p is a prime number with $n < p < 2n$. Such a prime exists by Bertrand's Postulate [10]. We can arrange the values of edges of ladder in an increasing sequence,

$$\{ \frac{1}{p}, \frac{1}{n}, \frac{1}{2}, \frac{n+1}{2n}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \frac{n+1}{n+2},$$

$$\frac{n+2}{n+3}, \ldots, \frac{p-1}{p}, \frac{p}{p+1}, \ldots, \frac{2n-1}{2n} \}.$$

Hence, $P_2 \odot C_n$ is a strongly quotient graph.   □

Theorem 2.4. $C_n \odot P_m$ *is a strongly quotient graph.*

Proof. We label the vertices of $C_n \odot P_m$ as shown in Fig. 2.8.

Figure 2.8

Then values of edges are of the form,

$$\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{m}{m+1}, \frac{m+1}{m+2}, \ldots, \frac{2m-1}{2m},$$

$$\frac{2m+1}{2m+2}, \ldots, \frac{3m-1}{3m}, \frac{3m+1}{3m+2}, \ldots, \frac{nm-1}{nm} \},$$

$$\{ \frac{m+1}{2m+1}, \frac{2m+1}{3m+1}, \frac{3m+1}{4m+1}, \ldots, \frac{(n-2)m+1}{(n-1)m+1} \}.$$

Since $x/y < (x+1)/(y+1)$ whenever $x < y$, it follows that the members of the first set are increasing. Similarly, members in the second set are also increasing for

$$\frac{km+1}{(k+1)m+1} < \frac{(k+1)m+1}{(k+2)m+1} , \qquad k = 1, 2, ..., (n-3).$$

Moreover, these two sets are disjoint. Hence, $C_n \odot P_m$ is a strongly quotient graph. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition. A *ladder* L_n is a graph $K_2 \times P_n$ with
$$V(L_n) = \{\ u_i, v_i : 1 \leq i \leq n\ \},$$
and
$$E(L_n) = \{\ u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\ \} \cup \{\ u_i v_i : 1 \leq i \leq n\ \}.$$

Theorem 2.5. *Ladder is a strongly quotient graph.*

Proof. We label the vertices of ladder as shown in Fig. 2.9.

Figure 2.9

We can arrange the values of edges of ladder in an increasing sequence,
$$\{\ \frac{1}{2n}, \frac{1}{n+1}, \frac{2}{n+2}, \frac{3}{n+3}, \frac{4}{n+4}, \ldots, \frac{n}{2n}, \frac{2}{3},$$
$$\frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{n+2}{n+3}, \ldots, \frac{2n-1}{2n}\ \},$$
so that all edges have different values. Hence, ladder is a strongly quotient graph.

Definition. A *triangular ladder* $\mathbb{L}_n, n \geq 2$, is a graph obtained by completing the ladder $L_n \simeq P_n \times P_2$ by adding the edges $u_i v_{i+1}$, for $1 \leq i \leq n-1$.

Theorem 2.6. *Triangular ladder is a strongly quotient graph.*

Proof. For $n=2$ and $n=3$, respectively, we label the vertices of triangular ladder as shown in Figures 2.10 and 2.11.

3 2

1 4

Figure 2.10

3 5 4

2 6 1

Figure 2.11

We can arrange the values of edges of triangular ladder in an increasing sequence,

$$\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \},$$

$$\{ \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}, \frac{5}{6} \},$$

and for $n \geq 4$, we label the vertices of triangular ladder as shown in Fig. 2.12.

2 3 4 5   n n+1

. . .

n+2 n+3 n+4 n+5   2n 1

. . .

Figure 2.12

Using the fact that $\frac{k}{n+k} < \frac{k+1}{n+k+2}$ for $k < n$, we can arrange the values of edges of triangular ladder in an increasing sequence,

$$\{ \frac{1}{2n}, \frac{1}{n+1}, \frac{1}{n}, \frac{2}{n+3}, \frac{2}{n+2}, \frac{3}{n+4}, \frac{3}{n+3}, \ldots, \frac{n-1}{2n},$$

$$\frac{n}{2n}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{n+2}{n+3}, \ldots, \frac{2n-1}{2n} \}.$$

Hence, triangular ladder is a strongly quotient graph. $\square$

Definition. A *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be seen as the union of a sequence of stars $S_1 \cup S_2 \cup \ldots \cup S_r$, where each S_i is a star with center c_i and n_i leaves, $i = 1, 2, 3, \ldots, r$, and the leaves of S_i include c_{i-1} and c_{i+1}, $i = 2, 3, \ldots, r-1$. We denote the caterpillar as $S_{n_1, n_2, \ldots, n_r}$, where the vertex set is:

$$V(S_{n_1, n_2, \ldots, n_r}) = \{c_i : 1 \le i \le r\} \cup \bigcup_{i=2}^{r-1} \{x_i^j : 2 \le j \le n_i - 1\}$$

$$\cup \{x_1^j : 1 \le j \le n_1 - 1\} \cup \{x_r^j : 2 \le j \le n_r\},$$

and the edge set is:

$$E(S_{n_1, n_2, \ldots, n_r}) = \{c_i c_{i+1} : 1 \le i \le r-1\} \cup \bigcup_{i=2}^{r-1} \{c_i x_i^j : 2 \le j \le n_i - 1\}$$

$$\cup \{c_1 x_1^j : 1 \le j \le n_1 - 1\} \cup \{c_r x_r^j : 2 \le j \le n_r\}.$$

If $r = 2$ then the graph is called a double star.

Theorem 2.7. *Star is a strongly quotient graph.*

Proof. Let p be the largest prime less than or equal to n. In Fig. 2.13, we give a labeling showing that star is a strongly quotient graph.

Figure 2.13

Observe that $(p, m) = 1$, for $m = 1, 2, \ldots, p-1, p+1, \ldots, n$. This implies that $k/p \ne p/l$ for $k = 1, 2, \ldots, p-1$ and $l = p+1, \ldots, n$, and hence the edge values are all distinct. $\square$

Theorem 2.8. *Double star is a strongly quotient graph.*

Proof. We suppose that $n_1 \geq n_2$. Then, we label $c_1 = 1, c_2 = 2$, we assign odd numbers $3, 5, 7, \ldots, 2n_2 - 1$, on $n_2 - 1$ leaves and even numbers greater than 2 and the remaining odd numbers on $n_1 - 1$ leaves (see Fig 2.14).

Figure 2.14

Then, values of edges are of the form:

$$\{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots, \frac{2}{2n_2 - 1}\},$$

$$\{\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots, \frac{1}{2n_2}, \frac{1}{2n_2 + 1}, \frac{1}{2n_2 + 2}, \ldots, \frac{1}{n_1 + n_2}\}.$$

It is easy to check that all these values are distinct. Hence, double star is a strongly quotient graph. $\square$

Theorem 2.9. S_{n_1, n_2, n_3} *is a strongly quotient graph.*

Proof. Suppose that $n_1 \geq n_2 \geq n_3$. Then, we label $c_1 = 1$, $c_2 = p$ and $c_3 = 2$, where p is the largest prime less than or equal to the number of vertices of S_{n_1, n_2, n_3}, namely $n_1 + n_2 + n_3 - 1$. If $p > 2n_3 - 1$, then we label $n_3 - 1$ leaves with odd numbers $3, 5, 7, \ldots, 2n_3 - 1$. If $p \leq 2n_3 - 1$, then we label $n_3 - 1$ leaves with odd numbers $3, 5, \ldots, p-2, p+2, \ldots, 2n_3 - 1, 2n_3 + 1$. We assign the remaining $n_1 + n_2 - 3$ numbers on the other leaves. $\square$

Definition. A *fan* graph F_n can be constructed from a wheel W_{n+1} with n spokes by deleting one edge on the $n-$cycle.

Theorem 2.10. *Fan is a strongly quotient graph.*

Proof. For $n = 3$, we label the vertices of a fan as shown in Fig. 2.16.

Figure 2.16

We can arrange the values of edges of a fan in an increasing sequence,

$$\{ \ \frac{1}{4}, \frac{1}{3}, \frac{2}{4}, \frac{2}{3}, \frac{3}{4} \ \},$$

and for $n > 3$, we label the vertices of the fan as shown in Fig. 2.17.

Figure 2.17

Here, p is the largest prime less than or equal to $n + 1$. Then, values of edges are of the forms,

$$\{\frac{1}{p+1}, \frac{1}{p}, \frac{1}{p-1}, \frac{2}{p}, \frac{3}{p}, \dots, \frac{p-1}{p} \ \},$$

$$\{ \ \frac{p}{n}, \frac{p}{n-1}, \frac{p}{n-2}, \dots, \frac{p}{p+1} \ \},$$

and

$$\{ \ \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{p-2}{p-1}, \frac{p+1}{p+2}, \frac{p+2}{p+3}, \dots, \frac{n}{n+1}\}.$$

It is easy to check that all these values are distinct. Hence, a fan is a strongly quotient graph. $\square$

Theorem 2.11. *The bipartite graph $K_{\alpha,n-\alpha}$ is a strongly quotient graph, where,*

$$P = \{ \, p \, \mid \, p \text{ is prime and } \tfrac{n}{2} < p < n \, \}, \, \alpha = |P| + 1.$$

Proof. Let $V_1 = \{ \, p_1, p_2, \ldots, p_{\alpha-1}, n \, \}$, where the p_i are distinct members of P and $V_2 = \{k : 1 \leq k < n, k \notin V_1\}$.

Define,

$$A_i = \{m : \; 1 \leq m < p_i, \; p_i \in P\},$$

$$B_i = \{l : \; l < n \, \& \, l > p_i, \; p_i \in P\}.$$

Then, values of edges are of the forms $\frac{m}{p_i}$, $\frac{p_i}{l}$ and $\frac{k}{n}$, where $m \in A_i$, $l \in B_i$, $i = 1, 2, \ldots, \alpha - 1$, $k < n$ and $k \notin P$. It is easy to check that all these values are distinct. $\qquad\square$

3. An upper bound for the energy of strongly quotient graph with respect to the distance matrix

Let G be a connected graph with n vertices and m edges. The vertices of G are labeled as $v_1, v_2, \ldots, v_n$. The distance between the vertices v_i and v_j is the length of the shortest path between v_i and v_j in G and is denoted by $d(v_i, v_j)$.

Throughout this section, by distance between j and k we mean the distance between vertices u and v having labels j and k. The distance matrix $D(G) = [d_{ij}]$ of a graph G is a square matrix of order n in which $d_{ij} = d(v_i, v_j)$. The characteristic polynomial of the distance matrix of G is $\psi(G; \gamma) = \det(\gamma I - D(G))$, where I is the unit matrix of order n. The roots of $\psi(G; \gamma) = 0$ are the eigenvalues of $D(G)$ and they are denoted by $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$, [6], [7], [8], [9]. The energy of a graph G with respect to the distance matrix is defined as the sum of the absolute values of the eigenvalues of G and denoted by $E(G)$. In this section, we obtain two eigenvalues of SQG and obtain an upper bound for the energy of SQG with respect to the distance matrix. We need the following result due to Ramane and Revankar [11].

Theorem 3.1. *Let G be a connected graph. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $D(G)$ then*

$$\sum_{i=1}^{n} \gamma_i = 0,$$

and

$$\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{1 \le i \le j \le n} d(v_i, v_j)^2. \tag{3.1}$$

Theorem 3.2. *If G is an SQG, then -1 is an eigenvalue of G with multiplicity greater than or equal to $\mid P \mid$, where,*

$$P = \{ \ p \ \mid \ p \ \text{is prime and} \ \tfrac{n}{2} < p \le n \ \}.$$

Proof. Let p be any prime number such that $\frac{n}{2} < p \le n$. In an SQG, vertices with labels p and 1 are adjacent to every other vertex. We recall that an SQG is a strongly quotient graph with the maximum number of edges for a fixed order. Thus, if $j \ne p$ and $j \ne 1$ then the distance between p and j is 1, as well as the distance between 1 and j. Hence,

$$D(G) = \begin{array}{c} \\ \\ \\ \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_p \\ \vdots \\ R_n \end{array} \begin{array}{c} c_1 \quad c_2 \quad c_3 \quad c_4 \quad \ldots \quad c_p \quad \ldots \quad c_n \\ \left(\begin{array}{cccccccc} 0 & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 \\ 1 & 0 & 1 & 2 & \ldots & 1 & \ldots & a_2 \\ 1 & 1 & 0 & 1 & \ldots & 1 & \ldots & a_3 \\ 1 & 2 & 1 & 0 & \ldots & 1 & \ldots & a_2 \\ \vdots & & & & & & & \\ 1 & 1 & 1 & 1 & \ldots & 0 & \ldots & 1 \\ \vdots & & & & & & & \\ 1 & a_2 & a_3 & a_2 & \ldots & 1 & \ldots & 0 \end{array} \right) \end{array}$$

where,

$$a_2 = \begin{cases} 2 & \text{if } \ n \equiv 0 \ (\text{mod } 2), \\ 1 & \text{otherwise,} \end{cases}$$

$$a_3 = \begin{cases} 2 & \text{if } \ n \equiv 0 \ (\text{mod } 3), \\ 1 & \text{otherwise,} \end{cases}$$

and so on.

So, the characteristic polynomial $\psi(G; \gamma)$ of D is given by:

$$\psi(G; \gamma) = |\gamma I - D(G)| =$$

$$
\begin{array}{c|ccccccc}
 & c_1 & c_2 & c_3 & c_4 & \ldots & c_p & \ldots & c_n \\
\hline
R_1 & \gamma & -1 & -1 & -1 & \ldots & -1 & \ldots & -1 \\
R_2 & -1 & \gamma & -1 & -2 & \ldots & -1 & \ldots & -a_2 \\
R_3 & -1 & -1 & \gamma & -1 & \ldots & -1 & \ldots & -a_3 \\
R_4 & -1 & -2 & -1 & \gamma & \ldots & -1 & \ldots & -a_2 \\
\vdots & \vdots & & & & & & & \\
R_p & -1 & -1 & -1 & -1 & \ldots & \gamma & \ldots & -1 \\
\vdots & \vdots & & & & & & & \\
R_n & -1 & -a_2 & -a_3 & -a_2 & \ldots & -1 & \ldots & \gamma
\end{array}.
$$

Replacing R_p by $R_p - R_1$, we see that $(\gamma + 1)$ is a factor of $\psi(G; \gamma)$. As this is true for every $p \in P$, we see that $(\gamma + 1)^{|P|}$ is a factor of $\psi(G; \gamma)$. This completes the proof. $\qquad \square$

Theorem 3.3. *If G is an SQG, then -2 is an eigenvalue of G with multiplicity greater than or equal to k where,*

$$k = \sum_{\substack{p \ prime \\ p \leq [\frac{n}{2}]}} [\, \log_p n \,].$$

Proof. If p is any prime less than or equal to $[\frac{n}{2}]$, then the distance between the vertices p and p^c $(c = 2, 3, \ldots, [\, \log_p n \,])$ is 2. Note that if $j \neq p$ and $j \neq p^c$, then the distance between p and j is the same as the distance between p^c and j. Let $[d_{ij}]$ be the $(i, j)-$th entry of the distance matrix D of G. Then,

$$
d_{ij} = \begin{cases} 1 & \text{if } (i, j) = 1, \\ 2 & \text{if } (i, j) \neq 1, \\ 0 & \text{if } i = j. \end{cases}
$$

Thus, the distance matrix of SQG is:

$$
D(G) = \begin{array}{c}
 \\
R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_p \\ \vdots \\ R_{p^c} \\ \vdots \\ R_n
\end{array}
\begin{array}{cccccccccc}
c_1 & c_2 & c_3 & \ldots & c_p & \ldots & c_{p^c} & \ldots & c_n \\
\left(\begin{array}{ccccccccc}
0 & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
1 & 0 & 1 & \ldots & 1 & \ldots & 1 & \ldots & a_2 \\
1 & 1 & 0 & \ldots & 1 & \ldots & 1 & \ldots & a_3 \\
\vdots & & & & & & & & \\
1 & 1 & 1 & \ldots & 0 & \ldots & 2 & \ldots & a_p \\
\vdots & & & & & & & & \\
1 & 1 & 1 & \ldots & 2 & \ldots & 0 & \ldots & a_p \\
\vdots & & & & & & & & \\
1 & a_2 & a_3 & \ldots & a_p & \ldots & a_p & \ldots & 0
\end{array}\right)
\end{array}
$$

where,

$$
a_2 = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 2, \\ 1 & \text{otherwise,} \end{cases}
$$

$$
a_3 = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{otherwise,} \end{cases}
$$

$$
a_p = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{\text{p}}, \\ 1 & \text{otherwise,} \end{cases}
$$

and so on.

If $c = [\, \log_p n \,]$, then using the same argument as above, we see that $(\gamma + 2)^{[log_p n]}$ is a factor of $\psi(G; \gamma)$. If we apply this method for every prime p, $1 < p < [\frac{n}{2}]$, we get the required result. $\qquad \square$

Theorem 3.4. *Let G be an SQG with n $(n > 3)$ vertices and maximum edges m. Let $P = \{\, p \mid p$ is prime and $\frac{n}{2} < p \leq n \,\}$ and $l =\mid P \mid$. Then,*

$$
E_D(G) \leq l + 2k + \sqrt{(n - l - k)\Big(2 \sum_{1 \leq i \leq j \leq n} d(v_i, v_j)^2 - l - 4k\Big)},
$$

where,

$$k = \sum_{\substack{p \ prime \\ p \leq [\frac{n}{2}]}} [\ \log_p n\ ].$$

Proof. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, the Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right). \tag{3.2}$$

Setting $a_i = 1, b_i = |\gamma_i|$ and replacing n by $n - l - k$, in (3.2), we obtain

$$\left(\sum_{i=1}^{n-l-k} |\gamma_i| \right)^2 \leq (n - l - k) \left(\sum_{i=1}^{n-l-k} |\gamma_i|^2 \right). \tag{3.3}$$

By Theorems 3.2 and 3.3, we have that -1 and -2 are eigenvalues of G with multiplicity greater than or equal to l and k, respectively. Hence,

$$E_D(G) = \sum_{i=1}^{n} |\gamma_i| = \sum_{i=1}^{n-l-k} |\gamma_i| + k|-2| + l|-1|,$$

$$E(G) - l - 2k = \sum_{i=1}^{n-l-k} |\gamma_i|. \tag{3.4}$$

From (3.1), we have that

$$2 \sum_{1 \leq i \leq j \leq n} d(v_i, v_j)^2 = \sum_{i=1}^{n-l-k} \gamma_i^2 + k(-2)^2 + l(-1)^2,$$

i.e.,

$$\sum_{i=1}^{n-l-k} \gamma_i^2 = 2 \sum_{1 \leq i \leq j \leq n} d(v_i, v_j)^2 - l - 4k. \tag{3.5}$$

Employing (3.4) and (3.5) in (3.3), we deduce that

$$(E_D(G) - l - 2k)^2 \leq (n - l - k)(2 \sum_{1 \leq i \leq j \leq n} d(v_i, v_j)^2 - l - 4k),$$

or equivalently,

$$E_D(G) \leq l + 2k + \sqrt{(n - l - k)(2 \sum_{1 \leq i \leq j \leq n} d(v_i, v_j)^2 - l - 4k)}.$$

□

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R. K. ZAFERANI

Department of Studies in Mathematics, University of Mysore, Manasagangothri, MYSORE - 570 006, INDIA.

Email: rkzaferan@yahoo.com