THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS OF DOUBLE AND TRIPLE JACOBI POLYNOMIALS

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ABSTRACT. Formulae expressing explicitly the coefficients of an expansion of double Jacobi polynomials which has been partially differentiated an arbitrary number of times with respect to its variables in terms of the coefficients of the original expansion are stated and proved. Extension to expansion of triple Jacobi polynomials is given. The results for the special cases of double and triple ultraspherical polynomials are considered. Also the results for Chebyshev polynomials of the first, second, third and fourth kinds and of Legendre polynomials are noted. An application of how to use double Jacobi polynomials for solving Poisson's equation in two variables subject to nonhomogeneous mixed boundary conditions is described.

1. Introduction

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudospectral methods, see for instance, [3], [5], [9], [11, 12, 13], [14], [15] and [19]. In particular, Lewanowicz [22, 23, 24] has presented three different methods for obtaining recurrence relations for the expansion coefficients in Jacobi series solutions of linear ordinary differential equations with polynomial

coefficients. Solutions of such recurrence relations enables one to get spectral approximations in Jacobi series expansions for the differential equations under consideration.

The importance of Sturm-Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a differential equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm-Liouville problem.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ play important roles in mathematical analysis and its applications. It is proven that the Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm-Liouville problem, (see, [4], Sec. 9.2). This class of polynomials comprises all the polynomial solution to singular Sturm-Liouville problems on $[-1,1]$. We have therefore motivated our interest in Jacobi polynomials. Another motivation is, the expansion for the function (solution), in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)(n \geq 0, \ \alpha > -1, \ \beta > -1)$, enables one to get the sought-for Jacobi approximation for any possible values of the real parameters α and β. That is, instated of developing approximation results for each particular pair of indices (α, β), it would be very useful to carry out a study on Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with general indices which can then be directly applied to other applications. In particular, the six important special cases of ultraspherical polynomials $\alpha = \beta$ and each is replaced by $(\alpha - \frac{1}{2})$, Chebyshev polynomials of the first, second, third and fourth kinds correspond to $\left(\alpha = \beta = -\frac{1}{2}, \ \alpha = \beta = \frac{1}{2}, \alpha = -\beta = -\frac{1}{2}, \ \alpha = -\beta = \frac{1}{2}\right)$ respectively and Legendre polynomials $(\alpha = \beta = 0)$ are noted.

If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded and not by any special boundary conditions satisfied by the function itself. If the function of interest is infinitely differentiable, then the nth expansion coefficient will decrease faster than any finite power of $(1/n)$ as $n \to \infty$, cf. [17].

For the spectral method and its variants-the Galerkin and tau methods-explicit expressions for the expansion coefficients of the derivatives in terms of the original expansion coefficients of the solution are required.

Three explicit formulae expressing the Chebyshev, Legendre, ultraspherical coefficients of a general order derivative of an infinitely differentiable function in terms of its Chebyshev, Legendre, ultraspherical coefficients are given by Karageorghis [20], Phillips [28] and Doha [6] respectively. A more general formula for Jacobi coefficients is also

given-with its important special cases-for ultraspherical, Chebyshev of the first and second kinds and Legendre polynomials by Doha [10].

Formulae expressing the coefficients of expansion of double and triple Chebyshev, Legendre and ultraspherical polynomials in terms of the coefficients of the original expansions are given by Doha [7],[8] and [9].

In the present paper we state and prove the corresponding formulae expressing the coefficients of expansions of double and triple Jacobi polynomials which have been partially differentiated any number of times with respect to their variables in terms of the coefficients of the original expansions.

The paper is organized as follows. In Section 2, we give some properties of double Jacobi polynomials and in Section 3, we describe how they are used to solve Poisson's equation in two variables inside a square subject to nonhomogeneous mixed boundary conditions with the tau method as a model problem. In Section 4, we state and prove the main results of the paper which are three expressions for the coefficients of general order partial derivatives of expansion of double Jacobi polynomials in terms of the coefficients of original expansion, results for the ultraspherical polynomials, for the Chebyshev polynomials of the first, second, third, fourth kinds and for the Legendre polynomials are obtained as special cases. Extension to expansion in triple Jacobi polynomials is also given in Section 5. Numerical results are discussed in Section 6. Some concluding remarks are noted in Section 7.

2. **Some properties of double Jacobi polynomials**

The classical Jacobi polynomials associated with the real parameters $(\alpha > -1 \,, \beta > -1)$ and the weight function $w(x) = (1 - x)^\alpha (1 + x)^\beta$ (see, [1], [29], and [2]), are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$ ($n = 0, 1, 2, \ldots$), each respectively of degree n, defining by (in hypergeometric form)

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha + 1)_n}{n!} \;\; {}_2F_1\left(-n, n + \lambda; \alpha + 1 \middle| \frac{1 - x}{2}\right) \\
&= \frac{(-1)^n (\beta + 1)_n}{n!} \;\; {}_2F_1\left(-n, n + \lambda; \beta + 1 \middle| \frac{1 + x}{2}\right),
\end{aligned}
$$

where

$$\lambda = \alpha + \beta + 1, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

It is clear that

$$P_n^{(\beta,\alpha)}(-x) = (-1)^n\, P_n^{(\alpha,\beta)}(x),$$

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \quad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta+1)_n}{n!}.$$

For our present purposes it is convenient to introduce and define the normalized orthogonal Jacobi polynomials

$$R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x),$$

which gives $R_n^{(\alpha,\beta)}(1) = 1$ $(n = 0, 1, 2, \ldots)$; this is not the usual standardization, but has the desirable properties that

$$C_n^{(\alpha)}(x) = R_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x), \qquad\qquad T_n(x) = R_n^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$U_n(x) = (n+1)\, R_n^{(\frac{1}{2},\frac{1}{2})}(x), \qquad\qquad V_n(x) = R_n^{(-\frac{1}{2},\frac{1}{2})}(x),$$

$$W_n(x) = (2n+1)\, R_n^{(\frac{1}{2},-\frac{1}{2})}(x), \qquad\qquad L_n(x) = R_n^{(0,0)}(x),$$

where $C_n^{(\alpha)}(x)$, $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$ and $L_n(x)$ are the ultraspherical, the Chebyshev of the first, second, third and fourth kinds, and the Legendre polynomials respectively.

Now, it can be easily shown that the polynomials $R_n^{(\alpha,\beta)}(x)$ may be generated using the recurrence relation

$$2(n+\lambda)(n+\alpha+1)(2n+\lambda-1)R_{n+1}^{(\alpha,\beta)}(x) = (2n+\lambda-1)_3\, x\, R_n^{(\alpha,\beta)}(x)$$

$$+ (\alpha^2-\beta^2)(2n+\lambda)R_n^{(\alpha,\beta)}(x) - 2n(n+\beta)(2n+\lambda+1)R_{n-1}^{(\alpha,\beta)}(x),$$

$$n = 1, 2, \ldots,$$

starting from $R_0^{(\alpha,\beta)}(x) = 1$ and $R_1^{(\alpha,\beta)}(x) = \dfrac{1}{2(\alpha+1)}[\alpha - \beta + (\lambda+1)x]$, or obtained from Rodrigue's formula

$$R_n^{(\alpha,\beta)}(x) = \left(\frac{-1}{2}\right)^n \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}(1-x)^{-\alpha}(1+x)^{-\beta} \times$$

$$D^n\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],$$

which consequently yields, by expanding the nth-order derivative, the explicit Rodrigue's formula

$$R_n^{(\alpha,\beta)}(x) = \frac{2^{-n} n!}{(\alpha+1)_n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j}(x+1)^j,$$

with $D = \dfrac{d}{dx}$, and satisfy the orthogonality relation

$$(2.1) \quad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \, R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)\, dx = \begin{cases} 0, & m \neq n, \\ h_n^{(\alpha,\beta)}, & m = n, \end{cases}$$

where

$$h_n^{(\alpha,\beta)} = \frac{2^\lambda \, n! \, \Gamma(n+\beta+1) \, [\Gamma(\alpha+1)]^2}{(2n+\lambda)\, \Gamma(n+\lambda)\, \Gamma(n+\alpha+1)}.$$

These polynomials are eigenfunctions of the following singular Sturm-Liouville equation:

$$(1-x^2)\, \phi''(x) + [\, \beta - \alpha - (\lambda+1)\, x\,]\, \phi'(x) + n(n+\lambda)\, \phi(x) = 0.$$

A consequence of this is that spectral accuracy can be achieved for expansions in Jacobi polynomials.

The special values

$$D^q R_n^{(\alpha,\beta)}(1) = \prod_{k=0}^{q-1} \frac{(n-k)(n+k+\lambda)}{2(\alpha+k+1)},$$

$$D^q R_n^{(\alpha,\beta)}(-1) = \frac{(-1)^{n+q} \, \Gamma(\alpha+1)\, \Gamma(n+\beta+1)}{\Gamma(\beta+1)\, \Gamma(n+\alpha+1)} D^q R_n^{(\beta,\alpha)}(1),$$

will be of important use later.

Let $f(x)$ be a continuous function defined on the interval $[-1, 1]$, and let it has continuous and bounded derivatives of any order with respect to x. Then we can expand $f(x)$ and its qth derivative into a uniformly convergent series of Jacobi polynomials in the form

$$(2.2) \qquad\qquad f(x) = \sum_{n=0}^{\infty} a_n \, R_n^{(\alpha,\beta)}(x),$$

and

$$(2.3) \qquad\qquad f^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} R_n^{(\alpha,\beta)}(x),$$

where $a_n^{(q)}$ denote the Jacobi expansion coefficients of $f^{(q)}(x)$ and $a_n^{(0)} = a_n$. Multiplying both sides of (2.2) and (2.3) by $(1-x)^\alpha(1+x)^\beta R_m^{(\alpha,\beta)}(x)$ and integrate from -1 to 1. Then in view of (2.1)

$$a_n = (h_n^{(\alpha,\beta)})^{-1} \int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta R_n^{(\alpha,\beta)}(x)\,dx,$$

$$a_n^{(q)} = (h_n^{(\alpha,\beta)})^{-1} \int_{-1}^{1} f^{(q)}(x)(1-x)^\alpha(1+x)^\beta R_n^{(\alpha,\beta)}(x)\,dx,$$

$$(n = 0, 1, 2, ...).$$

The coefficients a_n(resp. $a_n^{(q)}$) are often called the Fourier coefficients associated with $f(x)$(resp. $f^{(q)}(x)$). The evaluation and estimation of these quantities for various types of functions are discussed in Luke [27], Vol. I, Section 8.4. Here we are concerned with the representation of $f(x)$ by (2.2). As in the case of Fourier series, all the conditions required to insure that the series converges and that its sum is $f(x)$ are given in (Erdélyi et al. [16], Vol. II, Section 10.19, or Luke [27], Vol. I, Section 8.3).

Now, the following two theorems are of important use hereafter.

Theorem 2.1. *The expansion coefficients $a_n^{(q)}$ and a_n are related by*

(2.4) $$a_n^{(q)} = \frac{\Gamma(n+\alpha+1)}{2^q\,n!} \sum_{i=0}^{\infty} \frac{(n+i+q+\lambda)_q\,(n+i+q)!}{\Gamma(n+i+q+\alpha+1)} \times$$
$$C_{n+i,n}(\alpha+q,\beta+q,\alpha,\beta)\,a_{n+i+q}, \quad \forall\,n \geq 0,\ q \geq 1,$$

where

$$C_{n+i,n}(\alpha+q,\beta+q,\alpha,\beta) = \frac{(n+i+2q+\lambda)_n\,(n+\alpha+q+1)_i\,\Gamma(n+\lambda)}{i!\,\Gamma(2n+\lambda)} \times$$
$${}_3F_2\left(\begin{array}{c} -i,\ 2n+i+2q+\lambda,\ n+\alpha+1 \\ n+q+\alpha+1,\ 2\,n+\lambda+1 \end{array}\middle|\,1\right).$$

Theorem 2.2. *The qth derivative of the normalized Jacobi polynomial $R_n^{(\alpha,\beta)}(x)$ is given explicitly by*

$$D^q R_n^{(\alpha,\beta)}(x) = (n+\lambda)_q\,2^{-q}\,n! \sum_{i=0}^{n-q} C_{n-q,i}(\alpha+q,\beta+q,\alpha,\beta)\,R_i^{(\alpha,\beta)}(x),$$

where

$$C_{n-q,i}(\alpha + q, \beta + q, \alpha, \beta) = \frac{(n + q + \lambda)_i \ (i + q + \alpha + 1)_{n-i-q} \ \Gamma(i + \lambda)}{(n - i - q)! \ \Gamma(2i + \lambda) \ i! \ (i + \alpha + 1)_{n-i}} \times$$

$$_3F_2 \left(\begin{array}{c} -n + q + i, \ n + i + q + \lambda, \ i + \alpha + 1 \\ i + q + \alpha + 1, \ 2\,i + \lambda + 1 \end{array} \middle| 1 \right).$$

(For the proof of Theorems 2.1 and 2.2, see Doha [10]).

Now we define the double normalized Jacobi polynomials as

$$R_{mn}^{(\alpha,\beta)}(x, y) = R_m^{(\alpha,\beta)}(x) \ R_n^{(\alpha,\beta)}(y),$$

i.e., a product of two one variable normalized Jacobi polynomials, where $R_m^{(\alpha,\beta)}(x)$ and $R_n^{(\alpha,\beta)}(y)$ are the normalized Jacobi polynomials of degrees m and n associated with the real parameters α, β in the variables x and y respectively. These polynomials are satisfying the biorthogonality relation

(2.5)
$$\int_{-1}^{1} \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta (1 - y)^\alpha (1 + y)^\beta \ R_{ij}^{(\alpha,\beta)}(x, y) \ R_{k\ell}^{(\alpha,\beta)}(x, y) \ dx \ dy$$

$$= \begin{cases} h_{ij}^{(\alpha,\beta)}, & i = k, j = \ell, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$h_{ij}^{(\alpha,\beta)} = \frac{2^{2\lambda} \ i! \ j! \ \Gamma(i + \beta + 1) \ \Gamma(j + \beta + 1) \ [\Gamma(\alpha + 1)]^4}{(2i + \lambda) \ (2j + \lambda) \ \Gamma(i + \lambda) \Gamma(j + \lambda) \ \Gamma(i + \alpha + 1) \ \Gamma(j + \alpha + 1)}.$$

It is worthy to note here that the typical orthogonal polynomials-the double ultraspherical polynomials $C_{mn}^{(\alpha)}(x, y)$, the double Chebyshev polynomials of the first kind $T_{mn}(x, y)$, the second kind $U_{mn}(x, y)$, the third kind $V_{mn}(x, y)$, the fourth kind $W_{mn}(x, y)$ and the double Legendre polynomials $L_{mn}(x, y)$-are particular forms of the double Jacobi

polynomials. These are given explicitly by

$$
\begin{aligned}
C_{mn}^{(\alpha)}(x,y) &= R_{mn}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x,y) = C_m^{(\alpha)}(x)\,C_n^{(\alpha)}(y), \\
T_{mn}(x,y) &= R_{mn}^{(-\frac{1}{2},-\frac{1}{2})}(x,y) = T_m(x)\,T_n(y), \\
U_{mn}(x,y) &= R_{mn}^{(\frac{1}{2},\frac{1}{2})}(x,y) = \frac{1}{(m+1)(n+1)}\,U_m(x)\,U_n(y), \\
V_{mn}(x,y) &= R_{mn}^{(-\frac{1}{2},\frac{1}{2})}(x,y) = V_m(x)\,V_n(y), \\
W_{mn}(x,y) &= R_{mn}^{(\frac{1}{2},-\frac{1}{2})}(x,y) = \frac{1}{(2m+1)(2n+1)}W_m(x)\,W_n(y), \\
L_{mn}(x,y) &= R_{mn}^{(0,0)}(x,y) = L_m(x)\,L_n(y).
\end{aligned}
$$

2.1. Explicit expressions for the Jacobi coefficients of the derivatives in two variables.

Let $u(x,y)$ be a continuous function defined on the square $S[-1 \le x, y \le 1]$, and let it has continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$
u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn} R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y),
$$

$$
(2.6) \quad u^{(p,q)}(x,y) = \frac{\partial^{p+q}u(x,y)}{\partial x^p \partial y^q} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y),
$$

where $a_{mn}^{(p,q)}$ denote the Jacobi expansion coefficients of $u^{(p,q)}(x,y)$ and $a_{mn}^{(0,0)} = a_{mn}$. In view of (2.5)

$$
a_{mn} = (h_{mn}^{(\alpha,\beta)})^{-1} \int_{-1}^{1}\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\alpha}(1+y)^{\beta}\, u(x,y)\times
$$

$$
R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(y)\; dx\; dy,
$$

$$
a_{mn}^{(p,q)} = (h_{mn}^{(\alpha,\beta)})^{-1} \int_{-1}^{1}\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\alpha}(1+y)^{\beta}\, u^{(p,q)}(x,y)\times
$$

$$
R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(y)\; dx\; dy.
$$

Theorem 2.3.

(2.7)

$$a_{mn}^{(p-1,q)} = \frac{2(m+\lambda-1)(m+\alpha)}{m\,(2m+\lambda-2)(2m+\lambda-1)}\,a_{m-1,n}^{(p,q)}$$
$$+ \frac{2(\alpha-\beta)}{(2m+\lambda-1)(2m+\lambda+1)}a_{mn}^{(p,q)}$$
$$- \frac{2(m+1)(m+\beta+1)}{(m+\lambda)(2m+\lambda+1)(2m+\lambda+2)}\,a_{m+1,n}^{(p,q)}, \quad m,\ p \geq 1,$$

(2.8)

$$a_{mn}^{(p,q-1)} = \frac{2(n+\lambda-1)\,(n+\alpha)}{n(2n+\lambda-2)(2n+\lambda-1)}\,a_{m,n-1}^{(p,q)}$$
$$+ \frac{2(\alpha-\beta)}{(2n+\lambda-1)(2n+\lambda+1)}a_{mn}^{(p,q)}$$
$$- \frac{2(n+1)(n+\beta+1)}{(n+\lambda)(2n+\lambda+1)(2n+\lambda+2)}\,a_{m,n+1}^{(p,q)}, \quad n,\ q \geq 1.$$

Proof. Using the expressions

$$R_m^{(\alpha,\beta)}(x) = \frac{2(m+\lambda)(m+\alpha+1)}{(m+1)\,(2m+\lambda)(2m+\lambda+1)}\,D_x R_{m+1}^{(\alpha,\beta)}(x)$$
$$+ \frac{2(\alpha-\beta)}{(2m+\lambda-1)(2m+\lambda+1)}D_x R_m^{(\alpha,\beta)}(x)$$
$$- \frac{2\,m(m+\beta)}{(m+\lambda-1)(2m+\lambda)(2m+\lambda-1)}\,D_x R_{m-1}^{(\alpha,\beta)}(x),$$

$$R_n^{(\alpha,\beta)}(y) = \frac{2(n+\lambda)(n+\alpha+1)}{(n+1)\,(2n+\lambda)(2n+\lambda+1)}\,D_y R_{n+1}^{(\alpha,\beta)}(y)$$
$$+ \frac{2(\alpha-\beta)}{(2n+\lambda-1)(2n+\lambda+1)}D_y R_n^{(\alpha,\beta)}(y)$$
$$- \frac{2\,n(n+\beta)}{(n+\lambda-1)(2n+\lambda)(2n+\lambda-1)}\,D_y R_{n-1}^{(\alpha,\beta)}(y),$$

with the assumptions that

$$D_x \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p-1,q)} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y),$$

$$D_y \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q-1)} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y),$$

it is not difficult to derive the expressions (2.7) and (2.8). Hence Theorem 2.3 is proved.   □

3. The tau method for Poisson's equation in two variables

Consider Poisson's equation in the square $S(-1 \leq x, y \leq 1)$

$$(3.1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad -1 \leq x, y \leq 1,$$

subject to the most general inhomogeneous mixed boundary conditions

$$(3.2) \qquad \left. \begin{array}{l} \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial x} = \gamma_1(y), \quad x = -1 \\[2ex] \alpha_2 u + \beta_2 \dfrac{\partial u}{\partial x} = \gamma_2(y), \quad x = 1 \end{array} \right\} \quad -1 \leq y \leq 1,$$

$$(3.3) \qquad \left. \begin{array}{l} \alpha_3 u + \beta_3 \dfrac{\partial u}{\partial y} = \gamma_3(x), \quad y = -1 \\[2ex] \alpha_4 u + \beta_4 \dfrac{\partial u}{\partial y} = \gamma_4(x), \quad y = 1, \end{array} \right\} \quad -1 \leq x \leq 1.$$

It is assumed that the solution to the above problem can be expressed in a uniformly convergent double Jacobi series expansion

$$(3.4) \qquad u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y), \quad x, y \in [-1, 1].$$

Throughout this paper we assume that the function $f(x,y)$ satisfies the boundary conditions (3.2) and (3.3) to make sure that the solution of (3.1) is free of discontinuities. We also assume that $f(x,y)$ has a series expansion of the form

$$f(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y),$$

which is uniformly convergent in $-1 \leq x, y \leq 1$. It then follows that the solution of (3.1) has a double series expansion of the form (3.4), and the solution is free of discontinuities. The case in which discontinuities are present at the vertices $(\pm 1, \pm 1)$, can often be treated by a method similar to that described by Knib and Scraton [21].

If we satisfy the Poisson's differential equation (3.1), we obtain

$$(3.5) \qquad a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn}.$$

Now application of relation (2.7) on (3.5) twice, gives

$$a_{mn} + \sum_{i=0}^{\infty} A_{mi} a_{in}^{(0,2)} = \sum_{i=0}^{\infty} A_{mi} f_{in}, \quad m \geq 2, \ n \geq 0,$$

where

$$A_{mi} = \begin{cases} \dfrac{4(m+\alpha-1)_2(m+\lambda-2)_2}{(m-1)_2(2m+\lambda-4)_4}, & i = m-2, \\[2ex] \dfrac{8(\alpha-\beta)(m+\alpha)(m+\lambda-1)}{m(2m+\lambda-2)(2m+\lambda-1)^2(2m+\lambda+1)}, & i = m-1, \\[2ex] \begin{aligned} 4\Big[&(\alpha-\beta)^2 \, (2m+\lambda-2)(2m+\lambda)(2m+\lambda+2)- \\ &(m+\alpha)(m+\beta)(2m+\lambda+1)^2(2m+\lambda+2)- \\ &(m+\alpha+1)(m+\beta+1)(2m+\lambda-2) \\ &\times (2m+\lambda-1)^2\Big] \Big/ [(2m+\lambda-2)_5 \\ &\times (2m+\lambda-1)(2m+\lambda+1)], \end{aligned} & i = m, \\[2ex] \dfrac{-8(\alpha-\beta)(m+1)(m+\beta+1)}{(2m+\lambda-1)_5}, & i = m+1, \\[2ex] \dfrac{4(m+1)_2(m+\beta+1)_2}{(2m+\lambda+1)_4(m+\lambda)_2}, & i = m+2, \\[2ex] 0, & \text{otherwise.} \end{cases}$$

Again application of relation (2.8), yields the final result:

$$(3.6) \quad \sum_{i=0}^{\infty} A_{mi}\, a_{in} + \sum_{j=0}^{\infty} a_{mj}\, B_{jn} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{mi} f_{ij} B_{jn}, \qquad m, n \geq 2,$$

where $B_{ij} = A_{ji}$.

If we assume that $\gamma_i(y), (i = 1, 2)$ and $\gamma_i(x), (i = 3, 4)$, have the following Jacobi expansions

$$\gamma_i(y) = \sum_{n=0}^{\infty} \gamma_n^{(i)} R_n^{(\alpha,\beta)}(y), \qquad i = 1, 2,$$

$$\gamma_i(x) = \sum_{m=0}^{\infty} \gamma_m^{(i)} R_m^{(\alpha,\beta)}(x), \qquad i = 3, 4,$$

then the boundary conditions (3.2) and (3.3) give

$$
(3.7) \quad
\left.
\begin{aligned}
&\sum_{m=0}^{\infty}(-1)^m \frac{\Gamma(\alpha+1)\,\Gamma(m+\beta+1)}{\Gamma(\beta+1)\,\Gamma(m+\alpha+1)} \times \\
&\qquad \left[\alpha_1 - \frac{m\,\beta_1(m+\lambda)}{2(\beta+1)}\right] a_{mn} = \gamma_n^{(1)} \\
&\sum_{m=0}^{\infty} \left[\alpha_2 + \frac{m\,\beta_2\,(m+\lambda)}{2(\alpha+1)}\right] a_{mn} = \gamma_n^{(2)}
\end{aligned}
\right\} , \quad n = 0,1,2,\ldots,
$$

$$
(3.8) \quad
\left.
\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(\beta+1)\,\Gamma(n+\alpha+1)} \times \\
&\qquad \left[\alpha_3 - \frac{n\,\beta_3(n+\lambda)}{2(\beta+1)}\right] a_{mn} = \gamma_m^{(3)} \\
&\sum_{n=0}^{\infty} \left[\alpha_4 + \frac{n\,\beta_4(n+\lambda)}{2(\alpha+1)}\right] a_{mn} = \gamma_m^{(4)}
\end{aligned}
\right\} , \quad m = 0,1,2,\ldots.
$$

It is not difficult to show that Eqs. (3.7) and (3.8), may be put in the form

$$
(3.9) \quad
\left.
\begin{aligned}
a_{0n} + \sum_{m=2}^{\infty} \mu_m\, a_{mn} = g_n \\
a_{1n} + \sum_{m=2}^{\infty} \nu_m\, a_{mn} = h_n
\end{aligned}
\right\} , \quad n = 0,1,2,\ldots,
$$

$$
(3.10) \quad
\left.
\begin{aligned}
a_{m0} + \sum_{n=2}^{\infty} U_n\, a_{mn} = k_m \\
a_{m1} + \sum_{n=2}^{\infty} V_n\, a_{mn} = \ell_m
\end{aligned}
\right\} , \quad m = 0,1,2,\ldots,
$$

$$
\begin{aligned}
\mu_m = \Bigg\{ & \frac{\beta+1}{\alpha+1}\left(\alpha_1 - \frac{\beta_1(\lambda+1)}{2(\beta+1)}\right)\left(\alpha_2 + \frac{m\,\beta_2\,(m+\lambda)}{2(\alpha+1)}\right) \\
& + (-1)^m \frac{\Gamma(\alpha+1)\,\Gamma(m+\beta+1)}{\Gamma(\beta+1)\Gamma(m+\alpha+1)}\left(\alpha_1 - \frac{m\,\beta_1\,(m+\lambda)}{2(\beta+1)}\right) \times \\
& \left(\alpha_2 + \frac{\beta_2(\lambda+1)}{2(\alpha+1)}\right)\Bigg\} \Big/ \delta_1,
\end{aligned}
$$

$$\nu_m = \left\{ \alpha_1 \alpha_2 \left[1 + (-1)^{m+1} \frac{\Gamma(\alpha+1)\,\Gamma(m+\beta+1)}{\Gamma(\beta+1)\,\Gamma(m+\alpha+1)} \right] + \frac{m(m+\lambda)}{2(\alpha+1)} \right.$$

$$\left. \times \left[(\alpha_1\beta_2 + (-1)^m \alpha_2\,\beta_1) \frac{\Gamma(\alpha+2)\,\Gamma(m+\beta+1)}{\Gamma(\beta+2)\,\Gamma(m+\alpha+1)} \right] \right\} \Big/ \delta_1,$$

$$U_n = \left\{ \frac{\beta+1}{\alpha+1} \left(\alpha_3 - \frac{\beta_3(\lambda+1)}{2(\beta+1)} \right) \left(\alpha_4 + \frac{n\,\beta_4\,(n+\lambda)}{2(\alpha+1)} \right) \right.$$

$$+ (-1)^n \frac{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+\alpha+1)} \left(\alpha_3 - \frac{n\,\beta_3\,(n+\lambda)}{2(\beta+1)} \right) \times$$

$$\left. \left(\alpha_4 + \frac{\beta_4(\lambda+1)}{2(\alpha+1)} \right) \right\} \Big/ \delta_2,$$

$$V_n = \left\{ \alpha_3\,\alpha_4 \left[1 + (-1)^{n+1} \frac{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(\beta+1)\,\Gamma(n+\alpha+1)} \right] + \frac{n(n+\lambda)}{2(\alpha+1)} \right.$$

$$\left. \times \left[(\alpha_3\,\beta_4 + (-1)^n\alpha_4\,\beta_3) \frac{\Gamma(\alpha+2)\,\Gamma(n+\beta+1)}{\Gamma(\beta+2)\,\Gamma(n+\alpha+1)} \right] \right\} \Big/ \delta_2,$$

$$g_n = \left[\left(\alpha_2 + \frac{\beta_2\,(\lambda+1)}{2(\alpha+1)} \right) \gamma_n^{(1)} + \frac{(\beta+1)}{(\alpha+1)} \left(\alpha_1 - \frac{\beta_1\,(\lambda+1)}{2(\beta+1)} \right) \gamma_n^{(2)} \right] \Big/ \delta_1,$$

$$h_n = \left[\alpha_1\gamma_n^{(2)} - \alpha_2\gamma_n^{(1)} \right] / \delta_1,$$

$$k_m = \left[\left(\alpha_4 + \frac{\beta_4\,(\lambda+1)}{2(\alpha+1)} \right) \gamma_m^{(3)} + \frac{(\beta+1)}{(\alpha+1)} \left(\alpha_3 - \frac{\beta_3\,(\lambda+1)}{2(\beta+1)} \right) \gamma_m^{(4)} \right] \Big/ \delta_2,$$

$$\ell_m = \left[\alpha_3\gamma_m^{(4)} - \alpha_4\gamma_m^{(3)} \right] / \delta_2,$$

$$\delta_1 = \frac{(\lambda+1)}{2(\alpha+1)} \,(2\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1) \neq 0,$$

$$\delta_2 = \frac{(\lambda+1)}{2(\alpha+1)} \,(2\alpha_3\alpha_4 + \alpha_3\beta_4 - \alpha_4\beta_3) \neq 0.$$

It is worthy to note here that the boundary conditions (3.9) and (3.10) are not all linearly independent. Clearly, four linear relations exist among them.

Equations (3.9) and (3.10) may be used to eliminate a_{0n}, a_{1n}, a_{m0} and a_{m1} from the L.H.S. of Eq. (3.6) to give

$$A_{m0}\,g_n + A_{m1}\,h_n + B_{0n}\,k_m + B_{1n}\,\ell_m + \sum_{i=2}^{\infty} (A_{mi} - \mu_i\,A_{m0} - \nu_i\,A_{m1})\,a_{in}$$

$$+ \sum_{j=2}^{\infty} \left(B_{jn} - U_j\, B_{0n} - V_j\, B_{1n} \right) a_{mj} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{mi} f_{ij}\, B_{jn}, \qquad m, n \geq 2,$$

which may be written in the form

$$(3.11) \qquad \sum_{i=2}^{\infty} C_{mi}\, a_{in} + \sum_{j=2}^{\infty} a_{mj}\, D_{jn} = b_{mn}, \qquad m, n \geq 2,$$

where

$$C_{mi} = A_{mi} - \mu_i\, A_{m0} - \nu_i\, A_{m1} \quad ; \quad D_{jn} = B_{jn} - U_j\, B_{0n} - V_j\, B_{1n},$$

$$b_{mn} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{mi}\, f_{ij}\, B_{jn} - \left(A_{m0}\, g_n + A_{m1}\, h_n + B_{0n}\, k_m + B_{1n}\, \ell_m \right).$$

It is now necessary to assume that a_{mn} is negligible for $m > M$ and $n > N$, and as a result, Eq. (3.11) may be written in the finite matrix form as

$$(3.12) \qquad\qquad\qquad \mathbf{CA} + \mathbf{AD} = \mathbf{B},$$

where

$$\mathbf{A} = [a_{ij}, \quad i = 2, 3, \ldots, M; j = 2, 3, \ldots, N],$$

$$\mathbf{C} = [C_{ij}, \quad i, j = 2, 3, \ldots, M],$$

$$\mathbf{D} = [D_{ij}, \quad i, j = 2, 3, \ldots, N],$$

$$\mathbf{B} = [b_{ij}, \quad i = 2, 3 \ldots, M; j = 2, 3, \ldots, N].$$

3.1. **Solution of the system of Equations (3.12).** Let $\mathbf{C} \otimes \mathbf{D} = [C_{ij}\, \mathbf{D}]$, $i, j = 2, 3, \ldots, M$ be the tensor product of the two matrices $\mathbf{C}$ and $\mathbf{D}$; $\mathbf{C} \oplus \mathbf{D} = \mathbf{C} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{M-1} \otimes \mathbf{D}$ be their tensor sum, where $\mathbf{I}_{M-1}$ and $\mathbf{I}_{N-1}$ are the identity matrices of order $M - 1$ and $N - 1$ respectively. Introducing the so called block vectors:

$$\underline{a} \equiv [\underline{a}_2, \underline{a}_3, \ldots, \underline{a}_N]^T \qquad \text{and} \quad \underline{b} \equiv [\underline{b}_2, \underline{b}_3, \ldots, \underline{b}_N]^T,$$

where

$$\underline{a}_j \equiv [a_{2j}, a_{3j}, \ldots, a_{Mj}]^T; \quad \underline{b}_j \equiv [b_{2j}, b_{3j}, \ldots, b_{Mj}]^T;$$

$$\mathbf{A} \equiv [\underline{a}_2\ \underline{a}_3\ \ldots\ \underline{a}_N]^T \ ; \quad \mathbf{B} \equiv [\underline{b}_2\ \underline{b}_3\ \ldots\ \underline{b}_N]^T,$$

and

$$vec\ \mathbf{A} = \begin{bmatrix} \underline{a}_2 \\ \underline{a}_3 \\ \vdots \\ \underline{a}_N \end{bmatrix}.$$

Utilizing the above definitions, one can reduce the system of Eqs. (3.12) to the following system

$$\mathbf{G}\underline{a} = \underline{b},$$

where

$$\mathbf{G} = [\mathbf{C} \oplus \mathbf{D}^T].$$

The interested reader is referred to Graham [18] and Loan [26], for more details about the kronecker matrix algebra.

4. Relations between the coefficients

The main result of this section is to prove the following theorem:

Theorem 4.1. *The coefficients $a_{mn}^{(p,q)}$ are related to the coefficients $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and the original coefficients a_{mn} by*

(4.1)
$$a_{mn}^{(p,q)} = \frac{\Gamma(m+\alpha+1)}{2^p\, m!} \sum_{i=0}^{\infty} \frac{(m+i+p+\lambda)_p\,(m+i+p)!}{\Gamma(m+i+p+\alpha+1)} \times$$

$$C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\, a_{m+i+p,n}^{(0,q)}, \quad p \geq 1,$$

(4.2)
$$a_{mn}^{(p,q)} = \frac{\Gamma(n+\alpha+1)}{2^q\, n!} \sum_{j=0}^{\infty} \frac{(n+j+q+\lambda)_q\,(n+j+q)!}{\Gamma(n+j+q+\alpha+1)} \times$$

$$C_{n+j,n}(\alpha+q,\beta+q,\alpha,\beta)\, a_{m,n+j+q}^{(p,0)}, \quad q \geq 1,$$

(4.3)
$$a_{mn}^{(p,q)} = \frac{\Gamma(m+\alpha+1)\,\Gamma(n+\alpha+1)}{2^{p+q}\, m!\, n!} \times$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(m+i+p+\lambda)_p\,(n+j+q+\lambda)_q \times$$

$$\frac{(m+i+p)!\,(n+j+q)!}{\Gamma(m+i+p+\alpha+1)\,\Gamma(n+j+q+\alpha+1)} \times$$

$$C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\, C_{n+j,n}(\alpha+q,\beta+q,\alpha,\beta) \times$$

$$a_{m+i+p,n+j+q}, \quad p \geq 1, q \geq 1.$$

Proof. We can write Eq. (2.6) as

$$u^{(p,q)}(x,y) = \sum_{m=0}^{\infty} b_{mn}^{(p,q)}(y)\, R_m^{(\alpha,\beta)}(x),$$

where

$$(4.4) \qquad\qquad b_{mn}^{(p,q)}(y) = \sum_{n=0}^{\infty} a_{mn}^{(p,q)}\, R_n^{(\alpha,\beta)}(y),$$

and keeping y and q fixed. In view of formula (2.4), we can deduce that

$$b_{mn}^{(p,q)}(y) \;=\; \frac{\Gamma(m+\alpha+1)}{2^p\, m!} \sum_{i=0}^{\infty} \frac{(m+i+p+\lambda)_p\,(m+i+p)!}{\Gamma(m+i+p+\alpha+1)}$$

$$(4.5) \qquad\qquad \times C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\, b_{m+i+p,n}^{(0,q)}(y).$$

Using (4.4) and (4.5), yield the formula

$$\sum_{n=0}^{\infty} a_{mn}^{(p,q)}\, R_n^{(\alpha,\beta)}(y)$$

$$= \frac{\Gamma(m+\alpha+1)}{2^p\, m!} \sum_{i=0}^{\infty} \frac{(m+i+p+\lambda)_p\,(m+i+p)!}{\Gamma(m+i+p+\alpha+1)}$$

$$\times C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\left[\sum_{n=0}^{\infty} a_{m+i+p,n}^{(0,q)}\, R_n^{(\alpha,\beta)}(y)\right]$$

$$= \frac{\Gamma(m+\alpha+1)}{2^p\, m!} \sum_{n=0}^{\infty}\left[\sum_{i=0}^{\infty} \frac{(m+i+p+\lambda)_p\,(m+i+p)!}{\Gamma(m+i+p+\alpha+1)}\right.$$

$$\left.\times C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\, a_{m+i+p,n}^{(0,q)}\right] R_n^{(\alpha,\beta)}(y),$$

which implies that

$$a_{mn}^{(p,q)} = \frac{\Gamma(m+\alpha+1)}{2^p\, m!} \sum_{i=0}^{\infty} \frac{(m+i+p+\lambda)_p\,(m+i+p)!}{\Gamma(m+i+p+\alpha+1)}$$

$$\times\, C_{m+i,m}(\alpha+p,\beta+p,\alpha,\beta)\, a_{m+i+p,n}^{(0,q)}, \quad p \geq 1,$$

and the proof of formula (4.1) is complete. $\square$

Also it can be shown that Formula (4.2) is true by following the same procedure with (4.1), keeping x and p fixed. Formula (4.3) is

obtained immediately by substituting (4.2) into (4.1). This completes the proof of Theorem 4.1.

In particular, the special case for the ultraspherical polynomials may be obtained directly by taking $\alpha = \beta$ and each is replaced by $(\alpha - \frac{1}{2})$. Moreover, the special cases for the Chebyshev polynomials of the first and second kinds may be obtained by taking $\alpha = \beta = -\frac{1}{2}$ and $\frac{1}{2}$ respectively, and for the Legendre polynomials by taking $\alpha = \beta = 0$. These are given as corollaries to the previous theorem.

Corollary 4.2. *If*

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y),$$

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y),$$

then the coefficients $a_{mn}^{(p,q)}$ are related to the coefficients $a_{mn}^{(0,q)}, a_{mn}^{(p,0)}$ and a_{mn} by:

(4.6)
$$a_{mn}^{(p,q)} = \frac{2^p (m+\alpha)\Gamma(m+2\alpha)}{(p-1)!\ m!} \times$$
$$\sum_{i=1}^{\infty} \frac{(i+p-2)!\ \Gamma(m+i+p+\alpha-1)(m+2i+p-2)!}{(i-1)!\ \Gamma(m+i+\alpha)\Gamma(m+2i+p+2\alpha-2)}\ a_{m+2i+p-2,n}^{(0,q)},$$
$$p \geq 1,$$

(4.7)
$$a_{mn}^{(p,q)} = \frac{2^q (n+\alpha)\Gamma(n+2\alpha)}{(q-1)!\ n!} \times$$
$$\sum_{j=1}^{\infty} \frac{(j+q-2)!\ \Gamma(n+j+q+\alpha-1)(n+2j+q-2)!}{(j-1)!\ \Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)}\ a_{m,n+2j+q-2}^{(p,0)},$$
$$q \geq 1,$$

$$a_{mn}^{(p,q)} = \frac{2^{p+q}(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{(p-1)!\ (q-1)!\ m!\ n!} \times$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!}{(i-1)!\ (j-1)!} \times$$

$$\frac{\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)}{\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)} \times$$

$$\frac{(m+2i+p-2)!(n+2j+q-2)!}{\Gamma(m+2i+p+2\alpha-2)\Gamma(n+2j+q+2\alpha-2)}$$

$$a_{m+2i+p-2,n+2j+q-2}, \quad p,\ q \geq 1.$$

(4.8)

Corollary 4.3. *If*

$$(4.9) \qquad\qquad u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}T_m(x)T_n(y),$$

$$(4.10) \qquad\qquad u^{(p,q)}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)}T_m(x)T_n(y),$$

then the coefficients $a_{mn}^{(p,q)}$ are related to $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and a_{mn} by:

$$(4.11) \qquad a_{mn}^{(p,q)} = \frac{2^p}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\ (m+i+p-2)!}{(i-1)!\ (m+i-1)!} \times$$
$$(m+2i+p-2)\ a_{m+2i+p-2,n}^{(0,q)},\ p \geq 1,$$

$$(4.12) \qquad a_{mn}^{(p,q)} = \frac{2^q}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\ (n+j+q-2)!}{(j-1)!\ (n+j-1)!} \times$$
$$(n+2j+q-2)\ a_{m,n+2j+q-2}^{(p,0)},\ q \geq 1,$$

(4.13)

$$a_{mn}^{(p,q)} = \frac{2^{p+q}}{(p-1)!(q-1)!} \times$$
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i+p-2)!(j+q-2)!(m+i+p-2)!(n+j+q-2)!}{(i-1)!(j-1)!(m+i-1)!(n+j-1)!} \times$$
$$(m+2i+p-2)(n+2j+q-2)\ a_{m+2i+p-2,n+2j+q-2}, \quad p,q \geq 1,$$

for all $m, n \geq 0$. Note here that the double primes in (4.9) and (4.10) indicate that the first term is $\frac{1}{4}a_{00}$; a_{m0} and a_{0n} are to be taken as $\frac{1}{2}a_{m0}$ and $\frac{1}{2}a_{0n}$ for $m, n > 0$ respectively.

Corollary 4.4. *If*

$$u(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} U_m(x) U_n(y),$$

$$u^{(p,q)}(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}^{(p,q)} U_m(x) U_n(y),$$

then the coefficients $A_{mn}^{(p,q)}$ are related to the coefficients $A_{mn}^{(0,q)}$, $A_{mn}^{(p,0)}$ and A_{mn} by:

(4.14)

$$A_{mn}^{(p,q)} = \frac{2^p(m+1)}{(p-1)!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\,(m+i+p-1)!}{(i-1)!\,(m+i)!} A_{m+2i+p-2,n}^{(0,q)}, \quad p \geq 1,$$

(4.15)

$$A_{mn}^{(p,q)} = \frac{2^q(n+1)}{(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\,(n+j+q-1)!}{(j-1)!\,(n+j)!} A_{m,n+2j+q-2}^{(p,0)}, \quad q \geq 1,$$

(4.16)

$$A_{mn}^{(p,q)} = \frac{2^{p+q}(m+1)(n+1)}{(p-1)!(q-1)!} \times$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(i+p-2)!(j+q-2)!(m+i+p-1)!(n+j+q-1)!}{(i-1)!(j-1)!(m+i)!(n+j)!} \times$$

$$\times A_{m+2i+p-2,n+2j+q-2}, \quad p, q \geq 1,$$

for all $m, n \geq 0$ and $A_{mn} = \dfrac{a_{mn}}{(m+1)(n+1)}$.

Corollary 4.5. *If*

$$u(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} L_m(x) L_n(y),$$

$$u^{(p,q)}(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)} L_m(x) L_n(y),$$

then the coefficients $a_{mn}^{(p,q)}$ are related to the coefficients $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and a_{mn} by:

(4.17)

$$a_{mn}^{(p,q)} = \frac{2^{p-1}(2m+1)}{(p-1)!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\ \Gamma(m+i+p-\frac{1}{2})}{(i-1)!\ \Gamma(m+i+\frac{1}{2})}\ a_{m+2i+p-2,n}^{(0,q)}$$

$$= \frac{(2m+1)}{2^{p-2}(p-1)!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\ (2m+2i+2p-3)!\ (m+i)!}{(i-1)!\ (2m+2i)!\ (m+i+p-2)!} \times$$

$$a_{m+2i+p-2,n}^{(0,q)}, \qquad p \geq 1,$$

(4.18)

$$a_{mn}^{(p,q)} = \frac{2^{q-1}(2n+1)}{(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\ \Gamma(n+j+q-\frac{1}{2})}{(j-1)!\ \Gamma(n+j+\frac{1}{2})}\ a_{m,n+2j+q-2}^{(p,0)}$$

$$= \frac{(2n+1)}{2^{q-2}(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\ (2n+2j+2q-3)!\ (n+j)!}{(j-1)!\ (2n+2j)!\ (n+j+q-2)!} \times$$

$$a_{m,n+2j+q-2}^{(0,q)}, \qquad q \geq 1,$$

(4.19)

$$a_{mn}^{(p,q)} = \frac{2^{p+q-2}(2m+1)(2n+1)}{(p-1)!\ (q-1)!} \times$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{(i+p-2)!\ (j+q-2)!\ \Gamma(m+i+p-\frac{1}{2})}{(i-1)!\ (j-1)!\ \Gamma(m+i+\frac{1}{2})} \times$$

$$\frac{\Gamma(n+j+q-\frac{1}{2})}{\Gamma(n+j+\frac{1}{2})}\ a_{m+2i+p-2,n+2j+q-2}$$

$$= \frac{(2m+1)(2n+1)}{2^{p+q-4}(p-1)!(q-1)!} \times$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{(i+p-2)!\ (j+q-2)!\ (2m+2i+2p-3)!}{(i-1)!\ (j-1)!\ (2m+2i)!\ (2n+2j)!} \times$$

$$\frac{(2n+2j+2q-3)!\ (m+i)!\ (n+j)!}{(m+i+p-2)!\ (n+j+q-2)!} a_{m+2i+p-2,n+2j+q-2},\ p,q \geq 1,$$

$$\textit{for all } m,n \geq 0.$$

Note 1. It is worthy to note here that Formulae (4.6-4.8), (4.11-4.13), (4.14-4.16) and (4.17-4.19) are completely consistent with those obtained in Doha [9], Doha [7], Doha [9] and Doha [8] respectively.

5. **Extension to triple Jacobi series expansions**

Let $u(x, y, z)$ be a continuous function defined on the cube $C(-1 \leq x, y, z \leq 1)$, and let it has continuous and bounded partial derivatives of any order with respect to its variables x, y and z. Then it is possible to express

$$u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} a_{\ell mn} R_\ell^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(y) R_n^{(\alpha,\beta)}(z),$$

$$
\begin{aligned}
u^{(p,q,r)}(x, y, z) &= \frac{\partial^{p+q+r} u(x, y, z)}{\partial x^p \partial y^q \partial z^r} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} a_{\ell mn}^{(p,q,r)} R_\ell^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(y) R_n^{(\alpha,\beta)}(z).
\end{aligned}
$$

Now we state without proof the following theorem which is to be considered as an extension of Theorem 4.1.

Theorem 5.1. *The expansion coefficients $a_{\ell mn}^{(p,q,r)}$ are related to the coefficients with superscripts $(0, q, r), (p, 0, r), (p, q, 0), (0, 0, r), (0, q, 0), (p, 0, 0)$ and $a_{\ell mn}$ by*

(5.1)
$$
\begin{aligned}
a_{\ell mn}^{(p,q,r)} = &\frac{\Gamma(\ell + \alpha + 1)}{2^p \, \ell!} \sum_{i=0}^{\infty} \frac{(\ell + i + p + \lambda)_p \, (\ell + i + p)!}{\Gamma(\ell + i + p + \alpha + 1)} \times \\
&C_{\ell+i,\ell}(\alpha + p, \beta + p, \alpha, \beta) \, a_{\ell+i+p,m,n}^{(0,q,r)}, \quad p \geq 1,
\end{aligned}
$$

(5.2)
$$
\begin{aligned}
a_{\ell mn}^{(p,q,r)} = &\frac{\Gamma(m + \alpha + 1)}{2^q \, m!} \sum_{j=0}^{\infty} \frac{(m + j + q + \lambda)_q \, (m + j + q)!}{\Gamma(m + j + q + \alpha + 1)} \times \\
&C_{m+j,m}(\alpha + q, \beta + q, \alpha, \beta) \, a_{\ell,m+j+q,n}^{(p,0,r)}, \quad q \geq 1,
\end{aligned}
$$

(5.3)
$$
\begin{aligned}
a_{\ell mn}^{(p,q,r)} = &\frac{\Gamma(n + \alpha + 1)}{2^r \, n!} \sum_{k=0}^{\infty} \frac{(n + k + r + \lambda)_r \, (n + k + r)!}{\Gamma(n + k + r + \alpha + 1)} \times \\
&C_{n+k,n}(\alpha + r, \beta + r, \alpha, \beta) \, a_{\ell,m,n+k+r}^{(p,q,0)}, \quad r \geq 1,
\end{aligned}
$$

(5.4)

$$a_{\ell mn}^{(p,q,r)} = \frac{\Gamma(\ell+\alpha+1)\,\Gamma(m+\alpha+1)}{2^{p+q}\,\ell!\,m!}\ \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(\ell+i+p+\lambda)_p\times$$

$$(m+j+q+\lambda)_q\ \frac{(\ell+i+p)!\,(m+j+q)!}{\Gamma(\ell+i+p+\alpha+1)\,\Gamma(m+j+q+\alpha+1)}\times$$

$$C_{\ell+i,\ell}(\alpha+p,\beta+p,\alpha,\beta)\ C_{m+j,m}(\alpha+q,\beta+q,\alpha,\beta)\ a_{\ell+i+p,m+j+q,n}^{(0,0,r)},$$

$$p\geq 1, q\geq 1,$$

(5.5)

$$a_{\ell mn}^{(p,q,r)} = \frac{\Gamma(\ell+\alpha+1)\,\Gamma(n+\alpha+1)}{2^{p+r}\,\ell!\,n!}\ \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(\ell+i+p+\lambda)_p\times$$

$$(n+k+r+\lambda)_r\frac{(\ell+i+p)!\,(n+k+r)!}{\Gamma(\ell+i+p+\alpha+1)\,\Gamma(n+k+r+\alpha+1)}\times$$

$$C_{\ell+i,\ell}(\alpha+p,\beta+p,\alpha,\beta)\ C_{n+k,n}(\alpha+r,\beta+r,\alpha,\beta)\ a_{\ell+i+p,m,n+k+r}^{(0,q,0)},$$

$$p\geq 1, r\geq 1,$$

(5.6)

$$a_{\ell mn}^{(p,q,r)} = \frac{\Gamma(m+\alpha+1)\,\Gamma(n+\alpha+1)}{2^{q+r}\,m!\,n!}\ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(m+j+q+\lambda)_q\times$$

$$(n+k+r+\lambda)_r\frac{(m+j+q)!\,(n+k+r)!}{\Gamma(m+j+q+\alpha+1)\,\Gamma(n+k+r+\alpha+1)}\times$$

$$C_{m+j,m}(\alpha+q,\beta+q,\alpha,\beta)\ C_{n+k,n}(\alpha+r,\beta+r,\alpha,\beta)\ a_{\ell,m+j+q,n+k+r}^{(p,0,0)},$$

$$q\geq 1, r\geq 1,$$

(5.7)

$$a_{\ell mn}^{(p,q,r)} = \frac{\Gamma(\ell+\alpha+1)\,\Gamma(m+\alpha+1)\,\Gamma(n+\alpha+1)}{2^{p+q+r}\,\ell!\,m!\,n!}\times$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(\ell+i+p+\lambda)_p(m+j+q+\lambda)_q\ (n+k+r+\lambda)_r\times$$

$$\frac{(\ell+i+p)!\,(m+j+q)!\,(n+k+r)!}{\Gamma(\ell+i+p+\alpha+1)\,\Gamma(m+j+q+\alpha+1)\,\Gamma(n+k+r+\alpha+1)}\times$$

$$C_{\ell+i,\ell}(\alpha+p,\beta+p,\alpha,\beta)\,C_{m+j,m}(\alpha+q,\beta+q,\alpha,\beta)\times$$

$$C_{n+k,n}(\alpha+r,\beta+r,\alpha,\beta)a_{\ell+i+p,m+j+q,n+k+r}, \quad p\geq 1, q\geq 1,\ r\geq 1.$$

Note 2. The formulae corresponding to expansions in triple ultraspherical polynomials, and in particular, in triple Chebyshev polynomials of the first, second, third and fourth kinds and in triple Legendre polynomials may be obtained from formulae (5.1)-(5.7), by specifying the proper values of the parameters α and β in each case. These would not be given here.

6. Numerical Results

Example 6.1. *Consider the problem*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -32\,\pi^2\,\sin(4\pi\,x)\,\sin(4\pi\,y),$$

subject to the boundary conditions

$$u \pm \frac{\partial u}{\partial x} = \pm\,4\pi\,\sin(4\,\pi\,y), \qquad x = \pm 1,$$

$$u \pm \frac{\partial u}{\partial y} = \pm\,4\pi\,\sin(4\,\pi\,x), \qquad y = \pm 1.$$

This problem has the analytical solution

$$
\begin{aligned}
u(x,y) &= \sin(4\pi\,x)\sin(4\pi\,y) \\[1em]
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\,R_i^{(\alpha,\beta)}(x)\,R_j^{(\alpha,\beta)}(y),
\end{aligned}
$$

where a_{ij} are given by

$$
\begin{aligned}
a_{ij} &= \frac{(-1)^{i+j}\,(4\pi)^{i+j}\,(2i+\lambda)\,(2j+\lambda)\,\Gamma(i+\lambda)\,\Gamma(j+\lambda)}{2^{i+j+2\lambda}\,i!\,j!\,\Gamma^2(\alpha+1)\,\Gamma(i+\beta+1)\,\Gamma(j+\beta+1)} \\[1em]
&\quad \times \int_{-1}^{1} \sin\left(4\,\pi x + \frac{i\,\pi}{2}\right)(1-x)^{\alpha+i}\,(1+x)^{\beta+i}\ dx \\[1em]
&\quad \times \int_{-1}^{1} \sin\left(4\,\pi y + \frac{j\,\pi}{2}\right)(1-y)^{\alpha+j}\,(1+y)^{\beta+j}\ dy, \\[1.5em]
&= \frac{2^{3(i+j)}\,(-\pi)^{i+j}\,(2i+\lambda)\,(2j+\lambda)\,\Gamma(i+\alpha+1)\,\Gamma(j+\alpha+1)}{i!\,j!\,\Gamma^2(\alpha+1)\,\Gamma(2i+\lambda+2)\,\Gamma(2j+\lambda+2)} \times \\[0.5em]
&\quad\ \Gamma(j+\lambda)\,\Gamma(i+\lambda)\,\xi_i\,\xi_j,
\end{aligned}
$$

$$\xi_i = \left[8\pi \cos\left(\frac{i\pi}{2}\right) (i + \beta + 1) \, {}_2F_3\left(\begin{array}{c} \frac{i+\beta+2}{2}, \ \frac{i+\beta+3}{2} \\ \frac{3}{2}, \ \frac{2i+\lambda+2}{2}, \ \frac{2i+\lambda+3}{2} \end{array} \middle| -16\,\pi^2 \right) \right.$$

$$\left. + \sin\left(\frac{i\pi}{2}\right) (2i + \lambda + 1) \, {}_2F_3\left(\begin{array}{c} \frac{i+\beta+1}{2}, \ \frac{i+\beta+2}{2} \\ \frac{1}{2}, \ \frac{2i+\lambda+1}{2}, \ \frac{2i+\lambda+2}{2} \end{array} \middle| -16\,\pi^2 \right) \right].$$

The maximum pointwise error of the Jacobi approximation $(u - u_{MN})$ *for various choices of* $\alpha, \beta, M = N$ *is illustrated in Table 1, where*

$$u_{MN} = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} R_i^{(\alpha,\beta)}(x) R_j^{(\alpha,\beta)}(y).$$

Table 1. Maximum pointwise error of $u - u_{MN}$
for $M = N = 16, 24, 32$

$M=N$	α	β	error
	$\frac{1}{2}$	$\frac{1}{2}$	$6.773 \cdot 10^{-2}$
	0	0	$7.758 \cdot 10^{-2}$
	$-\frac{1}{2}$	$-\frac{1}{2}$	$9.616 \cdot 10^{-2}$
	-0.95	-0.95	$1.809 \cdot 10^{-1}$
16	0	1	$7.861 \cdot 10^{-2}$
	$\frac{1}{2}$	$-\frac{1}{2}$	$9.274 \cdot 10^{-2}$
	$-\frac{1}{2}$	$\frac{1}{2}$	$8.512 \cdot 10^{-2}$
	$-\frac{1}{2}$	0	$8.417 \cdot 10^{-2}$
	0	$\frac{1}{2}$	$8.008 \cdot 10^{-2}$
	$\frac{1}{2}$	$\frac{1}{2}$	$9.590 \cdot 10^{-6}$
	0	0	$1.066 \cdot 10^{-5}$
	$-\frac{1}{2}$	$-\frac{1}{2}$	$1.495 \cdot 10^{-5}$
	-0.95	-0.95	$2.195 \cdot 10^{-5}$
24	0	1	$2.310 \cdot 10^{-5}$
	$\frac{1}{2}$	$-\frac{1}{2}$	$1.936 \cdot 10^{-5}$
	$-\frac{1}{2}$	$\frac{1}{2}$	$1.159 \cdot 10^{-5}$
	$-\frac{1}{2}$	0	$1.206 \cdot 10^{-5}$
	0	$\frac{1}{2}$	$9.811 \cdot 10^{-6}$
	$\frac{1}{2}$	$\frac{1}{2}$	$2.356 \cdot 10^{-10}$
	0	0	$5.187 \cdot 10^{-10}$
	$-\frac{1}{2}$	$-\frac{1}{2}$	$7.891 \cdot 10^{-10}$
	-0.95	-0.95	$9.657 \cdot 10^{-9}$
32	0	1	$2.955 \cdot 10^{-9}$
	$\frac{1}{2}$	$-\frac{1}{2}$	$8.274 \cdot 10^{-9}$
	$-\frac{1}{2}$	$\frac{1}{2}$	$8.407 \cdot 10^{-9}$
	$-\frac{1}{2}$	0	$2.058 \cdot 10^{-9}$
	0	$\frac{1}{2}$	$9.528 \cdot 10^{-10}$

From the numerical results presented in Table 1, we see that the results correspond to the values $\alpha = \beta = \frac{1}{2}$ give the best accuracy among all the other values of α and β. From this we conclude that the expansion based on Chebyshev polynomials of the first kind ($\alpha = \beta = -\frac{1}{2}$) is not always better than the other Jacobi series. This conclusion is ascertained in [25]. Also one should note that the results of the previous example are identical with those obtained by Doha and Abd-Elhameed [12] for all values of the parameters α and β when $\alpha = \beta$.

7. **Concluding remarks**

In this paper, we give explicit formulae expressing the coefficients of expansions of double and triple Jacobi polynomials that have been partially differentiated an arbitrary number of times with respect to their variables in terms of the coefficients of the original expansions for any possible values of the real parameters α and β. That is, instead of developing results for each particular pair of indexes (α, β), we found it would be very useful to carry out the study on the normalized Jacobi polynomials $R_n^{(\alpha,\beta)}(.)$ with general indexes which can be directly applied to other applications. In particular, the four important special cases of ultraspherical polynomials ($\alpha = \beta$) and each is replaced by ($\alpha - \frac{1}{2}$), Chebyshev polynomials of the first kind ($\alpha = \beta = -\frac{1}{2}$), Legendre polynomials ($\alpha = \beta = 0$) and Chebyshev polynomials of the second kind ($\alpha = \beta = \frac{1}{2}$) are considered. Chebyshev polynomials of the third and fourth kinds correspond to ($\alpha = -\beta = \frac{1}{2}$) and ($\alpha = -\beta = -\frac{1}{2}$) respectively, may be also obtained as direct special cases. An important application of how to use double Jacobi polynomials for solving Poisson's equation in two variables subject to the most general inhomogenous mixed boundary conditions is discussed. Numerical results ascertain that the expansion based on Chebyshev polynomials of the first kind is not always better than the other Jacobi series.

Acknowledgments

References

[1] M. Abramowitz and I. A. Stegun, (Eds.), Handbook of Mathematical Functions, Applied Mathematical Series, Vol. **55**, National Bureau of Standards, New York, 1970.

[2] G. E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.

[3] A. H. Bhrawy and S. I. El-Soubhy, Jacobi spectral Galerkin method for the integrated forms of second-order differential equations, *Appl. Math. Comput.* **217** (2010), no. 6, 2684–2697.

[4] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, New York, 1988.

[5] E. A. Coutsias, T. Hagstrom and D. Torres, An efficient spectral method for ordinary differential equations with rational function coefficients, *Math. Comp.* **65** (1996), no. 214, 611–635.

[6] E. H. Doha, The coefficients of differentiated expansions and derivatives of ultraspherical polynomials, *Comput. Math. Appl.* **21** (1991), no. 2-3, 115–122.

[7] E. H. Doha, The Chebyshev coefficients of general-order derivatives of an infinitely differentiable function in two or three variables, *Ann. Univ. Sci. Budapest. Sect. Comput.* **13** (1992) 83–91.

[8] E. H. Doha, On the coefficients of differentiated expansions of double and triple Legendre polynomials, *Ann. Univ. Sci. Budapest. Sect. Comput.* **15** (1995) 23–35.

[9] E. H. Doha, The coefficients of differentiated expansions of double and triple ultraspherical polynomials, *Ann. Univ. Sci. Budapest. Sect. Comput.* **19** (2000) 57–73.

[10] E. H. Doha, On the coefficients of differentiated expansions and derivatives of Jacobi polynomials, *J. Phys. A* **35** (2002), no. 15, 3467–3478.

[11] E. H. Doha and W. M. Abd-Elhameed, Efficient spectral-Galerkin algorithms for direct solution of second-order equations using ultraspherical polynomials, *SIAM J. Sci. Comput.* **24** (2002), no. 2, 548–571.

[12] E. H. Doha and W. M. Abd-Elhameed, Accurate spectral solutions for the parabolic and elliptic partial differential equations by the ultraspherical tau method, *J. Comput. Appl. Math.* **181** (2005), no. 1, 24–45.

[13] E. H. Doha and W. M. Abd-Elhameed, Efficient spectral ultraspherical-dual-Petrov-Galerkin algorithms for the direct solution of $(2n+1)$th-order linear differential equations, *Math. Comput. Simulation* **79** (2009), no. 11, 3221–3242.

[14] E. H. Doha and A. H. Bhrawy, Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials, *Appl. Numer. Math.* **58** (2008), no. 8, 1224–1244.

[15] E. H. Doha, W. M. Abd-Elhameed and A. H. Bhrawy, Efficient spectral ultraspherical-Galerkin algorithms for the direct solution of $2n$th-order linear differential equations, *Appl. Math. Model.* **33** (2009), no. 4, 1982–1996.

[16] A. Erdélyi, W. Magnus and F. Oberhettinger, Higher Transcedental Functions, **II**, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[17] D. Gottlieb and S. A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, **26**, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1977.

[18] A. Graham, Kronecker Product and Matrix Calculus With Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1981.

[19] T. Zang and D. B. Haidvogel, The accurate solution of Poisson's equation by expansion in Chebyshev polynomials, *J. Comput. Phys.* **30** (1979), no. 2, 167–180.

[20] A. Karageorghis, A note on the Chebyshev coefficients of the general-order derivative of an infinitely differentiable function, *J. Comput. Appl. Math.* **21** (1988), no. 1, 129–132.

[21] D. knibb and R. E. Scraton, On the solution of parabolic partial differential equation in Chebyshev series, *Comput. J.* **14** (1971) 428–432.

[22] S. Lewanowicz, Recurrence relations for the coefficients in Jacobi series solutions of linear differential equations, *SIAM J. Math. Anal.* **17** (1986), no. 5, 1037–1052.

[23] S. Lewanowicz, A new approach to the problem of constructing recurrence relations for the Jacobi coefficients, *Appl. Math.* **21** (1991), no. 2, 303–326.

[24] S. Lewanowicz, Quick construction of recurrence relations for the Jacobi coefficients, *J. Comput. Appl. Math.* **43** (1992), no. 3, 355–372.

[25] W. A. Light, Are Chebyshev expansions really better, *Bull. Inst. Math. Appl.* **22** (1986) 180–181.

[26] C. V. Loan, Computational Frameworks for Fast Fourier Transform, SIAM, Philadelphia, 1992.

[27] Y. L. Luke, The Special Functions and Their Approximations, II, Academic Press, New York-London, 1969.

[28] T. N. Phillips, On the Legendre coefficients of a general-order derivative of an infinitely differentiable function, *IMA J. Numer. Anal.* **8** (1988), no. 4, 455–459.

[29] G. Szegö, Orthogonal Polynomials, AMS. Colloquium Publications, **23**, 1985.

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