MODIFIED NOOR ITERATIONS FOR INFINITE FAMILY OF STRICT PSEUDO-CONTRACTION MAPPINGS

L.-P. YANG*

Communicated by Mohammad Sal Moslehian

ABSTRACT. We introduce a modified Noor iteration scheme generated by an infinite family of strict pseudo-contractive mappings and prove the strong convergence theorems of the scheme in the framework of $q-$uniformly smooth and strictly convex Banach space. Results shown here are extensions and refinements of previously known results.

1. Introduction

Let E be a real Banach space, and K be a nonempty closed convex subset of E. Recall that a mapping $f : K \to K$ is said to be a *contraction* on K if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in K$. We use Π_K to denote the collection of all contractions on K; that is, $\Pi_K = \{f | f : K \to K$ is a contraction with constant $\alpha\}$. A mapping $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. In the sequel, $F(T) = \{x \in K : Tx = x\}$ denotes the fixed point set of T. For fixed

43

$x_1 \in K$, one classical way to study nonexpansive mappings is to use the following *Mann iteration process* [1]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1,$$

where, $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$. For the approximation of a fixed point of a nonexpansive mapping, the Mann iteration process is always applicable and has been studied extensively by several authors.

Another iteration process found to be successful for the approximation of a fixed point of a nonexpansive map is the Halpern-type process. Let K be a nonempty closed convex subset of a Hilbert space and $T : K \to K$ be a nonexpansive mapping. For an arbitrary $u \in K$ and any initial value $x_0 \in K$, define a sequence $\{x_n\} \subset K$ in an explicit iterative way by

(1.1) $$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0.$$

In 1967, Halpern [2] proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, or equivalently, $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$. However, it is unclear whether the conditions (C1) and (C2) are sufficient. Consequently, several authors have concentrated to study the convergence of Halpern iteration under a different restriction on the parameter $\{\alpha_n\}$. For example, Lions [3] proved strong convergence of Halpern iteration $\{x_n\}$ defined by (1.1) to a fixed point of T in Hilbert space if $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions:

$(C1) \lim_{n\to\infty} \alpha_n = 0; \quad (C2) \sum_{n=0}^{\infty} \alpha_n = \infty; \quad (C3) \lim_{n\to\infty} \dfrac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$

It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was rectified in 1992 by Wittmann [4], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the conditions (C1), (C2) and the condition, (C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Xu [5] (see also [6]) improved Lions' result in two respects. First, he suggested the following control condition (C5) instead of the conditions (C3) or (C4):

$(C5) \lim_{n\to\infty} \dfrac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$,or equivalently, $\lim_{n\to\infty} \dfrac{\alpha_{n-1}}{\alpha_n} = 1$

so that the canonical choice of $\alpha_n = 1/(n+1)$ is made possible. Second, he proved the strong convergence of Halpern-type process in the framework of real uniformly smooth Banach space. And Xu [6] showed that condition (C3) and condition (C5) are not comparable.

In 2005, Kim and Xu [7] introduced the following iterative algorithm:

$$(1.2) \quad \begin{cases} x_0 = x \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where T is a nonexpansive mapping of K into itself and $u \in K$ is a given point. They proved that the sequence $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T provided that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1), (C2), (C4) and

$$(B1) \; \lim_{n \to \infty} \beta_n = 0; \quad (B2) \; \sum_{n=0}^{\infty} \beta_n = \infty; \quad (B3) \; \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Recently, Yao et al. [8] modified the recursion formula (1.2) to have strong convergence by using the viscosity approximation method. They introduced the following iteration scheme:

$$(1.3) \quad \begin{cases} x_0 = x \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad f \in \Pi_K, \quad n \geq 0. \end{cases}$$

They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T, where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1), (C2) and (B4): $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Let E^* denote the dual space of a Banach space E. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \; \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a real number. In particular, $J = J_2$ is said to be normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$ (the identity mapping). It is well-known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

A mapping T is said to be a pseudo-contraction, if there exists some $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q, \quad \forall x, y \in K.$$

A mapping T is said to be a $\lambda-$strict pseudo-contraction in the terminology of Browder and Petryshyn [9], if there exists a constant $\lambda > 0$

such that

$$\begin{aligned}
\langle Tx - Ty, j_q(x - y)\rangle &\leq \|x - y\|^q \\
&- \lambda\|(I - T)x - (I - T)y\|^q
\end{aligned}$$

(1.4)

for every $x, y \in K$ and for some $j_q(x - y) \in J_q(x - y)$.

A mapping T is said to be a strong pseudo-contraction if there exists $k \in (0, 1)$ such that $\langle Tx - Ty, j_q(x - y)\rangle \leq k\|x - y\|^q, \quad \forall x, y \in K$.

It can be proved that if T is $\lambda-$strict pseudo-contraction in the terminology of Browder and Petryshyn, then T is Lipschitz continuous with the Lipschitz constant $L = (1 + \lambda)/\lambda$. The class of strong pseudo-contractive mappings is independent of the class of $\lambda-$strict pseudo-contractions.

Iterative methods for nonexpansive mapping have been extensively studied. Iterative methods for the $\lambda-$strict pseudo-contractive mapping, introduced by Browder and Petryshyn [9] in 1967, are far less developed than those for nonexpansive mapping; the reason is probably that the second term appearing in the right-hand side of (1.4) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping. On the other hand, $\lambda-$strict pseudo-contractive mapping has more powerful applications than nonexpansive mapping do in solving inverse problems (see, for example, Scherzer [10]). Therefore, it is more interesting to study the theory of iterative methods for $\lambda-$strict pseudo-contractive mappings.

Zhou and Su [11] proved the relation between the $\lambda-$strict pseudo-contraction and the nonexpansive mappings using the following lemma.

Lemma 1.1. *(See* [11, *Lemma 2.2]) Let K be a nonempty convex subset of a real $q-$uniformly smooth Banach space E and $T : K \to K$ be a $\lambda-$strict pseudo-contraction. For $\alpha \in (0, 1)$, define $T_\alpha x = (1 - \alpha)x + \alpha Tx$. Then, for $\alpha \in (0, \mu]$, $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, $T_\alpha : K \to K$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Here, for any $n \in \mathbb{N}$ (the set of positive integers), we consider the mapping W_n to be defined by

$$(1.5) \quad \begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I, \\ U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\ \vdots \\ U_{n,k} = \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I, \\ U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\ \vdots \\ U_{n,2} = \gamma_2 S_2 U_{n,3} + (1 - \gamma_2)I, \\ W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I, \end{cases}$$

where I is the identity operator on E, and $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$, for every $i \in \mathbb{N}$, $S_i = t_i T_i + (1 - t_i)I$, where T_i is λ_i-strict pseudo-contractive mapping of K into itself and $t_i \in (0, \mu]$, $\mu \in \min\{1, (\frac{q\lambda}{C_q})^{q-1}\}$. Such a mapping W_n is called the $W - mapping$, generated by $T_n, T_{n-1}, \ldots, T_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1, t_n, t_{n-1}, \ldots, t_1$. It follows from Lemma 1.1 that non-expansivity of each S_i ensures the non-expansivity of W_n.

Cho et al. [12] proposed the following iterative scheme:

$$(1.6) \quad \begin{cases} x_0 = x \in K \text{ chosen arbitrarily,} \\ y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad f \in \Pi_K, \quad n \ge 0. \end{cases}$$

Under the conditions (C1), (C2) and (B4), they also proved the strong convergence of the sequence $\{x_n\}$, defined by (1.6), and extended the results of [8].

Recently, Yao et al. [13] considered the following iterative algorithm:

$$(1.7) \quad \begin{cases} x_0 = x \in K \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)W_n x_n, \end{cases}$$

where $f \in \Pi_K$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\beta_n\}$ is a sequence in $[0,1)$ and $\alpha_n + \beta_n < 1$, for all $n \in N$. They proved that the iterative algorithm (1.7) converges strongly to a common fixed points of an infinite countable family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$.

Shimoji and Takahashi [14] first introduced an iterative algorithm given by an infinite family of nonexpansive mappings. Furthermore, they considered the feasibility problem of finding a solution of infinite convex inequalities and the problem of finding a common fixed point of infinite nonexpansive mappings. Bauschke and Borwein [15] pointed out that

the well-known convex feasibility problem reduced to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest having practical importance (see [16]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [16,17]).

Noor [18] first introduced a three-step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [19] applied a three-step iterative sequence for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory. In [19], they showed that the three-step iterative schemes performed better than the Ishikawa type and Mann type iterative methods. Haubruge et al. [20] studied the convergence analysis of the three-step iterations to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus, three-step iteration process and multistep iteration process have been investigated extensively by some authors (see [21–24] and the references therein).

Motivated and inspired by these facts, as the viscosity approximation method, we consider a new modified Noor iteration scheme for an infinite family of λ_i-strict pseudo-contractive mappings $\{T_i\}_{i=1}^{\infty}$:

$$(1.8) \quad \begin{cases} x_0 = x \in K, \\ z_n = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n, \\ y_n = (1 - b_n) z_n + b_n W_n z_n, \\ x_{n+1} = (1 - c_n) y_n + c_n W_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{b_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\alpha_n + \beta_n\} \subset (0,1)$, $f \in \Pi_K$, and W_n is a mapping defined by (1.5). If $b_n = c_n = 0$, for all $n \geq 0$, then (1.8) reduces to (1.7). By using viscosity approximation methods, our purpose here is to study some sufficient and necessary conditions of the three-step iterative algorithm (1.8) for finding approximate common fixed points of an infinite countable family of λ_i-strict pseudo-contractive mappings $\{T_i\}_{i=1}^{\infty}$. The results presented here extend and improve some recent results.

2. **Preliminaries**

Let E be a Banach space with dimension $E \geq 2$ and E^* be its dual. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$, defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \right\}.$$

A Banach space E is *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, x \neq y \text{ implies } \frac{\|x + y\|}{2} < 1.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$. The space E is said to be *Gâteaux differentiable* (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists, for all $x, y \in S(E)$. For any $x, y \in E(x \neq 0)$, we denote this limit by (x, y). The norm is said to be *uniformly Gâteaux differentiable* if for $y \in S(E)$, the limit is attained uniformly for $x \in S(E)$. The norm $\|\cdot\|$ of E is said to be *Fréchet differentiable* if for all $x \in S(E)$, the limit (x, y) exists uniformly, for all $y \in S(E)$. It is known that E is smooth if and only if each normalized duality mapping J is single-valued.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$. A Banach space E is said to be $q-$unifromly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. a typical example of both uniformly convex and uniformly smooth Banach space is L^p $(p > 1)$.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the *inward set* of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \to E$ is said to be *weakly inward* if $Tx \in cl[I_K(x)]$, for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. It is obvious that every self-map is trivially weakly inward.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$. Then a mapping $P : C \to D$ is said to be retraction if $Px = x$, for all $x \in C$. A retraction $P : C \to D$ is said to be *sunny* [25] if $P(Px + t(x - Px)) = Px$, for all $x \in C$ and $t \geq 0$, with $Px + t(x - Px) \in C$.

Suppose that $\{x_n\}$ is a sequence in E. In the sequel, $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

Concerning W_n, the following two lemmas play crucial roles in proving our main results.

Lemma 2.1. *(see [14]) Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 < \gamma_n \leq b < 1$, for any $n \geq 1$. Then, for any $x \in K$ and $k \in N$, $\lim_{n \to \infty} U_{n,k} x$ exists.*

Using Lemma 2.1, we can define the mapping W of K into itself as follows:

$$Wx = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x, \quad \forall x \in K.$$

Such a mapping W is said to be the W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots, t_1, t_2, \ldots, t_n$. Throughout this paper, we will assume that $0 < \gamma_n \leq b < 1$, for all $n \geq \mathbb{N}$, $t_i \in (0, \mu)$ and $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, where $\lambda = \inf \lambda_i > 0$, $\forall i \in \mathbb{N}$.

Lemma 2.2. *(see [14]) Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^{\infty} F(S_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 < \gamma_n \leq b < 1$, for any $n \geq 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.*

It follows from Lemma 1.1 and Lemma 2.2 that $F(W) = \bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n)$.

We also need the following lemmas for the proof of our main results.

Lemma 2.3. *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad j(x + y) \in J(x + y).$$

Lemma 2.4. *(see [6, Lemma 2.5]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \sigma_n + \mu_n, \quad n \geq 0,$$

where, (i) $\{\lambda_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \lambda_n = \infty$; (ii) $\limsup_{n\to\infty} \sigma_n \leq 0$; and (iii) $\mu_n \geq 0$, $\sum_{n=1}^{\infty} \mu_n < \infty$. Then, $\{a_n\}$ converges to zero, as $n \to \infty$.

Lemma 2.5. *(see [26]) Let $\{x_n\}, \{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$, for all integers $n \geq 0$, and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.6. *(see [27]) Let E be a uniformly smooth Banach space and K be a closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_K$. Then, the sequence $\{x_t\}$, defined by*

$$x_t = tf(x_t) + (1 - t)Tx_t,$$

converges strongly to a point in $F(T)$. If we define a mapping $P : \Pi_K \to F(T)$ by

$$P(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_K,$$

then $P(f)$ solves the following variational inequality:

$$\langle (I - f)P(f), J(P(f) - p)\rangle \leq 0, \quad \forall f \in \Pi_K, \quad p \in F(T).$$

3. **Main Results**

Theorem 3.1. *Let K be a closed convex subset of a real $q-$uniformly smooth and strictly convex Banach space E. Let T_i be a λ_i-strict pseudo-contractive mapping from K into itself, for $i \in \mathbb{N}$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f \in \Pi_K$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\alpha_n + \beta_n\}$ in $(0, 1)$ satisfy the following conditions:*
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$;
(3) $\limsup_{n\to\infty} \beta_n < 1$.
Then, the modified Noor iterative scheme defined by (1.8) converges

strongly to $P(f) \in F$, where $P(f)$ is the unique solution of the following variational inequality:

$$\langle (I - f)P(f), J(P(f) - p) \rangle \leq 0, \quad \forall f \in \Pi_K, \quad p \in F.$$

Proof. We proceed with the following steps.

Step 1. We should prove that $\|x_n - p\| \leq \max\{\|x_0 - p\|, (1/(1-\alpha))\|f(p) - p\|\}$, for all $n \geq 0$ and all $p \in F$. So, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{W_n x_n\}$, $\{W_n y_n\}$ and $\{W_n z_n\}$ are bounded. Indeed, take a point $p \in F$. It follows from (1.8) that

$$
\begin{aligned}
\|z_n - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) \\
&\quad + (1 - \alpha_n - \beta_n)(W_n x_n - p)\| \\
&\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)\|W_n x_n - p\| \\
&\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= (1 - (1 - \alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\},
\end{aligned}
$$

(3.1)

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - b_n)(z_n - p) + b_n(W_n z_n - p)\| \\
&\leq (1 - b_n)\|z_n - p\| + b_n\|W_n z_n - p\| \\
&\leq (1 - b_n)\|z_n - p\| + b_n\|z_n - p\| = \|z_n - p\|.
\end{aligned}
$$

(3.2)

It follows from (1.8), (3.1) and (3.2) that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - c_n)(y_n - p) + c_n(W_n y_n - p)\| \\
&\leq (1 - c_n)\|y_n - p\| + c_n\|y_n - p\| = \|y_n - p\| \\
&\leq \|z_n - p\| \leq \max\{\|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\}.
\end{aligned}
$$

(3.3)

Using mathematical induction, we obtain:

$$
(3.4) \qquad \|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\},
$$

for all $n \geq 0$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{W_n x_n\}$, $\{W_n y_n\}$ and $\{W_n z_n\}$.

Step 2. We prove:

$$(3.5) \qquad \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Indeed, putting $l_n = \frac{x_{n+1} - \rho_n x_n}{1 - \rho_n}$, we have

$$(3.6) \qquad x_{n+1} = \rho_n x_n + (1 - \rho_n)l_n, \quad \forall n \geq 0.$$

It follows from (1.5) and (1.8) that

$$
\begin{aligned}
z_n &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)W_n x_n \\
&= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)[\gamma_1 S_1 U_{n,2} x_n + (1 - \gamma_1)x_n] \\
&= \alpha_n f(x_n) + [1 - (1 - \gamma_1)\alpha_n - (1 - \beta_n)\gamma_1]x_n \\
&\quad + (1 - \alpha_n - \beta_n)\gamma_1 S_1 U_{n,2} x_n \\
(3.7) \quad &= \alpha_n f(x_n) + \rho_n x_n + (1 - \alpha_n - \rho_n)S_1 U_{n,2} x_n,
\end{aligned}
$$

where $\rho_n = 1 - (1 - \gamma_1)\alpha_n - (1 - \beta_n)\gamma_1$. It follows from conditions (1) and (3) that $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 1$. It follows from (3.6), (3.7) and (1.8) that

$$
\begin{aligned}
l_{n+1} - l_n &= \frac{(1 - c_{n+1})y_{n+1} + c_{n+1}W_{n+1}y_{n+1} - \rho_{n+1}x_{n+1}}{1 - \rho_{n+1}} \\
&\quad - \frac{(1 - c_n)y_n + c_n W_n y_n - \rho_n x_n}{1 - \rho_n} \\
&= \frac{c_{n+1}(W_{n+1}y_{n+1} - y_{n+1}) + y_{n+1} - \rho_{n+1}x_{n+1}}{1 - \rho_{n+1}} \\
&\quad - \frac{c_n(W_n y_n - y_n) + y_n - \rho_n x_n}{1 - \rho_n} \\
&= \frac{c_{n+1}}{1 - \rho_{n+1}}(W_{n+1}y_{n+1} - y_{n+1}) - \frac{c_n}{1 - \rho_n}(W_n y_n - y_n) \\
&\quad + \frac{b_{n+1}}{1 - \rho_{n+1}}(W_{n+1}z_{n+1} - z_{n+1}) - \frac{b_n}{1 - \rho_n}(W_n z_n - z_n) \\
&\quad + \frac{\alpha_{n+1}}{1 - \rho_{n+1}}(f(x_{n+1}) - S_1 U_{n+1,2} x_{n+1}) \\
&\quad - \frac{\alpha_n}{1 - \rho_n}(f(x_n) - S_1 U_{n,2} x_n) + (S_1 U_{n+1,2} x_{n+1} \\
&\quad - S_1 U_{n+1,2} x_n) + (S_1 U_{n+1,2} x_n - S_1 U_{n,2} x_n).
\end{aligned}
$$

We have

$$
\begin{aligned}
\|l_{n+1} - l_n\| \leq\; & \frac{c_{n+1}}{1 - \rho_{n+1}} \|W_{n+1}y_{n+1} - y_{n+1}\| + \frac{c_n}{1 - \rho_n} \|W_n y_n - y_n\| \\
& + \frac{b_{n+1}}{1 - \rho_{n+1}} \|W_{n+1}z_{n+1} - z_{n+1}\| + \frac{b_n}{1 - \rho_n} \|W_n z_n - z_n\| \\
& + \frac{\alpha_{n+1}}{1 - \rho_{n+1}} \|f(x_{n+1}) - S_1 U_{n+1,2}x_{n+1}\| \\
& + \frac{\alpha_n}{1 - \rho_n} \|f(x_n) - S_1 U_{n,2}x_n\| \\
& + \|x_{n+1} - x_n\| + \|S_1 U_{n+1,2}x_n - S_1 U_{n,2}x_n\|.
\end{aligned}
$$
(3.8)

Since S_i and $U_{n,i}$ are nonexpansive, from (1.5), we obtain:

$$
\begin{aligned}
\|S_1 U_{n+1,2}x_n - S_1 U_{n,2}x_n\| \leq\; & \|U_{n+1,2}x_n - U_{n,2}x_n\| \\
=\; & \|\gamma_2 S_2 U_{n+1,3}x_n - \gamma_2 S_2 U_{n,3}x_n\| \\
\leq\; & \gamma_2 \|U_{n+1,3}x_n - U_{n,3}x_n\| \\
=\; & \gamma_2 \|\gamma_3 S_3 U_{n+1,4}x_n - \gamma_3 S_3 U_{n,4}x_n\| \\
\leq\; & \gamma_2 \gamma_3 \|U_{n+1,4}x_n - U_{n,4}x_n\| \\
\leq\; & \cdots \\
\leq\; & \gamma_2 \gamma_3 \cdots \gamma_n \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
\leq\; & M \prod_{i=2}^{n} \gamma_i,
\end{aligned}
$$
(3.9)

where $M \geq 0$ is a constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M$, for all $n \geq 0$. Substituting (3.9) into (3.8), we have

$$
\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq\; & \frac{c_{n+1}}{1 - \rho_{n+1}} \|W_{n+1}y_{n+1} - y_{n+1}\| \\
& + \frac{c_n}{1 - \rho_n} \|W_n y_n - y_n\| \\
& + \frac{b_{n+1}}{1 - \rho_{n+1}} \|W_{n+1}z_{n+1} - z_{n+1}\| \\
& + \frac{b_n}{1 - \rho_n} \|W_n z_n - z_n\| \\
& + \frac{\alpha_{n+1}}{1 - \rho_{n+1}} \|f(x_{n+1}) - S_1 U_{n+1,2}x_{n+1}\| \\
& + \frac{\alpha_n}{1 - \rho_n} \|f(x_n) - S_1 U_{n,2}x_n\| + M \prod_{i=2}^{n} \gamma_i.
\end{aligned}
$$

From the conditions (1), (2), (3) and $0 < \gamma_n \leq b < 1$, we get

$$\limsup_{n \to \infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 2.5 that $\lim_{n \to \infty} \|l_n - x_n\| = 0$. Notice that using (3.6), we obtain:

$$x_{n+1} - x_n = (1 - \rho_n)(l_n - x_n).$$

Thus, we get $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Step 3. We will show $\lim_{n \to \infty} \|Wz_n - z_n\| = 0$. Observing that $x_{n+1} - y_n = c_n(W_n y_n - y_n)$, $y_n - z_n = b_n(W_n z_n - z_n)$ and condition (2), we have

$$(3.10) \qquad \lim_{n \to \infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n \to \infty} \|y_n - z_n\| = 0.$$

On the other hand, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

This, together with (3.5) and (3.10), imply:

$$(3.11) \qquad\qquad \lim_{n \to \infty} \|y_n - x_n\| = 0.$$

It follows from (1.8) that

$$\begin{aligned}
\|x_n - W_n x_n\| &\leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - W_n x_n\| \\
&\leq \|x_n - y_n\| + \|y_n - z_n\| + \beta_n \|x_n - W_n x_n\| \\
&\quad + \alpha_n \|f(x_n) - W_n x_n\|.
\end{aligned}$$

This implies:

$$(1 - \beta_n)\|x_n - W_n x_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \alpha_n \|f(x_n) - W_n x_n\|.$$

From condition (2), (3), (3.10) and (3.11), we have

$$(3.12) \qquad\qquad \lim_{n \to \infty} \|x_n - W_n x_n\| = 0.$$

It follows from (1.8) that $z_n - x_n = (1 - \beta_n)(W_n x_n - x_n) + \alpha_n(f(x_n) - W_n x_n)$. Therefore, we have

$$\begin{aligned}
\|z_n - x_n\| &\leq (1 - \beta_n)\|W_n x_n - x_n\| + \alpha_n \|f(x_n) - W_n x_n\| \\
&\leq \|W_n x_n - x_n\| + \alpha_n(\|f(x_n)\| + \|W_n x_n\|).
\end{aligned}$$

This, together with (3.12) and $\lim_{n \to \infty} \alpha_n = 0$, imply $\lim_{n \to \infty} \|z_n - x_n\| = 0$. Noticing that

$$\begin{aligned}
\|z_n - W_n z_n\| &\leq \|z_n - x_n\| + \|x_n - W_n x_n\| + \|W_n x_n - W_n z_n\| \\
&\leq 2\|z_n - x_n\| + \|x_n - W_n x_n\|,
\end{aligned}$$

we have $\lim_{n\to\infty}\|z_n - W_n z_n\| = 0$. On the other hand, we have

(3.13) $\qquad \|Wz_n - z_n\| \leq \|Wz_n - W_n z_n\| + \|W_n z_n - z_n\|.$

From [28, Remark 2.2], we have

$$\|Wz_n - W_n z_n\| \to 0 \quad (n \to \infty).$$

This together with (3.13) imply:

(3.14) $\qquad\qquad\qquad \lim_{n\to\infty}\|Wz_n - z_n\| = 0.$

Step 4. We show that $\limsup_{n\to\infty}\langle P(f) - f(P(f)), J(P(f) - z_n)\rangle \leq 0$, where $P(f) = \lim_{t\to 0^+} x_t$, with x_t being the fixed point of the contraction,

$$x \mapsto tf(x) + (1-t)Wx.$$

Then, we can write

(3.15) $\qquad x_t - z_{n_j} = t(f(x_t) - z_{n_j}) + (1-t)(Wx_t - z_{n_j}).$

Suppose that a subsequence $\{z_{n_j}\} \subset \{z_n\}$ is such that

$$\limsup_{n\to\infty}\langle P(f) - f(P(f)), J(P(f) - z_n)\rangle$$

(3.16)
$$= \lim_{j\to\infty}\langle P(f) - f(P(f)), J(P(f) - z_{n_j})\rangle$$

and $z_{n_j} \rightharpoonup p$, for some $p \in E$. It follows from (3.14) that $\lim_{j\to\infty}\|z_{n_j} - Wz_{n_j}\| = 0$. Putting

$$
\begin{aligned}
f_j(t) &= (1-t)^2 \|z_{n_j} - Wz_{n_j}\|(2\|x_t - z_{n_j}\| \\
&\quad + \|z_{n_j} - Wz_{n_j}\|) \to 0 \quad (j \to \infty),
\end{aligned}
$$

(3.17)

it follows from (3.15), Lemma 2.3 and Step 3 that

$$
\begin{aligned}
\|x_t - z_{n_j}\|^2 &\leq (1-t)^2\|Wx_t - z_{n_j}\|^2 + 2t\langle f(x_t) - z_{n_j}, J(x_t - z_{n_j})\rangle \\
&\leq (1-t)^2(\|Wx_t - Wz_{n_j}\| + \|Wz_{n_j} - z_{n_j}\|)^2 \\
&\quad + 2t\langle f(x_t) - x_t, J(x_t - z_{n_j})\rangle \\
&\quad + 2t\langle x_t - z_{n_j}, J(x_t - z_{n_j})\rangle \\
&= (1-t)^2\|x_t - z_{n_j}\|^2 + f_j(t) + 2t\langle f(x_t) - x_t, J(x_t - z_{n_j})\rangle \\
&\quad + 2t\|x_t - z_{n_j}\|^2.
\end{aligned}
$$

(3.18)

The last inequality implies:

$$\langle x_t - f(x_t), J(x_t - z_{n_j})\rangle \leq \frac{t}{2}\|x_t - z_{n_j}\|^2 + \frac{1}{2t}f_j(t).$$

Letting $j \to \infty$ and noting (3.17) yield:

$$(3.19) \qquad \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq \frac{t}{2} M_1,$$

where $M_1 > 0$ is a constant such that $M_1 \geq \|x_t - z_{n_j}\|^2$, for all $n \geq 0$ and $t \in (0,1)$. Taking $t \to 0$ in (3.19) and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$(3.20) \qquad \limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle \leq 0.$$

Indeed, letting $t \to 0$, from (3.19) we have

$$(3.21) \qquad \limsup_{t \to 0} \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq 0.$$

Thus, for arbitrary $\epsilon > 0$, there exists a positive number δ_1 such that for any $t \in (0, \delta_1)$, we have

$$(3.22) \qquad \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq \frac{\epsilon}{2}.$$

Since $x_t \to P(f)$, as $t \to 0$, the set $\{x_t - z_{n_j}\}$ is bounded and the duality mapping J is norm-to-norm uniformly continuous on bounded subset of E, then there exists $\delta_2 > 0$ such that, for any $t \in (0, \delta_2)$,

$$
\begin{aligned}
&|\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle - \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle| \\
&= |\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) - J(x_t - z_{n_j}) \rangle \\
&\quad + \langle P(f) - f(P(f)) - (x_t - f(x_t)), J(x_t - z_{n_j}) \rangle| \\
&\leq |\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) - J(x_t - z_{n_j}) \rangle| \\
&\quad + \|P(f) - f(P(f)) - (x_t - f(x_t))\| \|x_t - z_{n_j}\| < \epsilon/2.
\end{aligned}
$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, for all $t \in (0, \delta)$ and $j \in N$, we have

$$\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle < \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle + \frac{\epsilon}{2},$$

which implies:

$$\limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle$$

$$\leq \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle + \frac{\epsilon}{2}.$$

This together with (3.22) imply:

$$\limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle \leq \epsilon.$$

Since ϵ is arbitrary, we have $\limsup_{j\to\infty}\langle P(f) - f(P(f)), J(P(f) - z_{n_j})\rangle \leq 0$.

Step 5. We claim that $\lim_{n\to\infty}\|x_n - P(f)\| = 0$. Indeed, it follows from (3.3) and (1.8) that

$$
\begin{aligned}
\|x_{n+1} - P(f)\| &\leq \|z_n - P(f)\| = \|(1 - \alpha_n - \beta_n)(W_n x_n - P(f)) \\
&\quad + \beta_n(x_n - P(f)) + \alpha_n(f(x_n) - P(f))\|.
\end{aligned}
$$

Thus, it follows from Lemma 2.3 and (3.1) that

$$
\begin{aligned}
\|x_{n+1} - P(f)\|^2 &\leq \|z_n - P(f)\|^2 \\
&\leq \|(1 - \alpha_n - \beta_n)(W_n x_n - P(f)) + \beta_n(x_n - P(f))\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - P(f), J(z_n - P(f))\rangle \\
&\leq ((1 - \alpha_n - \beta_n)\|W_n x_n - P(f)\| + \beta_n\|x_n - P(f)\|)^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(P(f)), J(z_n - P(f))\rangle \\
&\quad + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f))\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - P(f)\|^2 \\
&\quad + 2\alpha\alpha_n\|x_n - P(f)\|\|z_n - P(f)\| \\
&\quad + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f))\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - P(f)\|^2 + 2\alpha\alpha_n\|x_n - P(f)\|^2 \\
&\quad + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f))\rangle.
\end{aligned}
$$

Therefore, we obtain:

$$
\begin{aligned}
\|x_{n+1} - P(f)\|^2 &\leq (1 - 2(1 - \alpha)\alpha_n + \alpha_n^2)\|x_n - P(f)\|^2 \\
&\quad + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f))\rangle \\
&\leq (1 - 2(1 - \alpha)\alpha_n\|x_n - P(f)\|^2 \\
\text{(3.23)} &\quad + \alpha_n^2 M_2^2 + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f))\rangle,
\end{aligned}
$$

where $M_2 = \sup_{n\geq 0}\|x_n - P(f)\|$. Set

$$
\begin{aligned}
\lambda_n &= 2(1 - \alpha)\alpha_n, \\
\sigma_n &= \frac{\alpha_n}{2(1-\alpha)}M_2^2 + \frac{1}{1-\alpha}\langle f(P(f)) - P(f), J(z_n - P(f))\rangle.
\end{aligned}
$$

It follows from condition (1) and Step 4 that $\lambda_n \to 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $\limsup_{n\to\infty}\sigma_n \leq 0$. Then, (3.23) reduces to

$$
\|x_{n+1} - P(f)\|^2 \leq (1 - \lambda_n)\|x_n - P(f)\|^2 + \alpha_n\sigma_n.
$$

From Lemma 2.4 with $\mu_n = 0$, we see that $\lim_{n\to\infty}\|x_n - P(f)\| = 0$. This completes the proof. $\qquad\square$

Remark 3.2. *If $\{T_i\}_{i=1}^{\infty}$ is composed of nonexpansive mappings, then the $S_i = t_iT_i + (1 - t_i)I$ are also nonexpansive mappings. Therefore, $q-$uniformly smoothness cprresponding to E in Theorem 3.1 can be extended to uniformly smooth. If we take $b_n = c_n = 0$ in Theorem 3.1, then Theorem 3.1 becomes Theorem 2.1 of Cho et al. [12] and Theorem 3.1 of Yao [13].*

Remark 3.3. *Theorem 3.1 partially improves main results of [11] from a finite family of λ_i-strict pseudo-contractions to an infinite family of λ_i-strict pseudo-contractions.*

If $f(x) = u \in K$, for all $x \in K$, in Theorem 3.1, then we have the following result.

Theorem 3.4. *Let K be a closed convex subset of a real $q-$uniformly smooth and strictly convex Banach space E. Let T_i be a λ_i-strict pseudo-contractive mapping from K into itself, for $i \in \mathbb{N}$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\alpha_n + \beta_n\}$ in $(0,1)$ satisfy the following conditions:*
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$;
(3) $\limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}$ be the three-step iterative scheme defined by

$$(3.24) \quad \begin{cases} x_0 = x \in K, \\ z_n = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)W_n x_n, \\ y_n = (1 - b_n)z_n + b_n W_n z_n, \\ x_{n+1} = (1 - c_n)y_n + c_n W_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{b_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\alpha_n + \beta_n\} \subset (0,1)$, and W_n is a mapping defined by (1.5). Then, $\{x_n\}$ converges strongly to $z \in F$, where $z = P_F(u)$, and $P_F : K \to F$ is the unique sunny nonexpansive retraction from K onto F.

Remark 3.5. *Theorem 3.4 mainly improves Theorem 2.3 of Zhou [29] from a single $\lambda-$strict pseudo-contractive mapping to an infinite family of λ_i-strict pseudo-contractive mappings and from one-step iteration scheme to three-step iteration scheme if K is a closed convex subset of a $2-$uniformly smooth and strictly convex Banach space E.*

Acknowledgments

The author thanks the referees for many valuable suggestions. This work was supported by the National Natural Science Foundation of China (60974143) and the Natural Science Foundation of Guangdong Province (8351009001000002).

References

[1] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953) 506-510.

[2] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* **73** (1967) 957-961.

[3] P. L. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Paris Ser. A-B* **284** (1977) A1357-A1359.

[4] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **59** (1992) 486-491.

[5] H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.* **65** (2002) 109-113.

[6] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* **66** (2002) 240-256.

[7] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.* **61** (2005) 51-60.

[8] Y. Yao, R. Chen and J. C. Yao, Strong convergence and certain control conditions for modified Mann iteration, *Nonlinear Anal.* **68** (2008) 1687-1693.

[9] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.* **20** (1967) 197-228.

[10] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonliner problems, *J. Math. Anal. Appl.* **194** (1991) 911-933.

[11] H. Zhang and Y. Su, Strong convergence theorems for strict pseudo-contractions in $q-$uniformly smooth Banach spaces, *Nonlinear Anal.* **70** (2009) 3236-3242.

[12] Y. J. Cho, S. M. Kang and X. Qin, Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces, *Comput. Math. Appl.* **56** (2008) 2058-2064.

[13] Y. Yao, Y. C. Liou and S. M. Kang, Strong convergence of an iterative algorithm on an infinite countable family of nonexpansive mappings, *Appl. Math. Comput.* **208** (2009) 211-218.

[14] K. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.* **5** (2001) 387-404.

[15] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.* **38** (1996) 367-426.

[16] D. C. Youla, Mathematical theory of image restoration by the method of convex projections, in: H. Stark (Ed.), *Image Recovery: Theory and Applications*, Academic Press, Orlando, 1987, pp. 29-77.

[17] A. N. Iusem and A. R. De Pierro, On the convergence of Han's method for convex programming with quadratic objective, *Math. Programming* **52** (1991) 265-284.

[18] M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* **251** (2000) 217–229.

[19] R. Glowinski and P. Le Tallec, *Augemented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, 1989.

[20] S. Haubruge, V.H. Nguyen and J.J. Strodiot, Convergence analysis and applications of the Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators, *J. Optim. Theory Appl.* **97**(3) (1998) 645–673.

[21] K. Nammanee and S. Suantai, The modified Noor iterations with errors for non-Lipschitzian mappings in Banach spaces, *Appl. Math. Comput.* **187** (2007) 669–679.

[22] K. Nammanee, M. A. Noor and S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **314** (2006) 320–334.

[23] M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* **251** (2004) 199-277.

[24] L. Yang, Modified multistep iterative process for some common fixed point of a finite family of nonself asymptotically nonexpansive mappings, *Math. Comput. Modelling* **45** (2007) 1157–1169.

[25] R. E. Bruck, Nonexpansive retracts of Banach spaces, *Bull. Amer. Math. Soc.* **76** (1970) 384-386.

[26] T. Suzuki, Strong convergence of Krasnoselskii and Mann's sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005) 227–239 .

[27] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* **298** (2004) 279–291.

[28] L. C. Ceng and J. C. Yao, Hybrid viscosity approximation scheme for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, *Appl. Math. Comput.* **198** (2008) 729–741.

[29] H. Zhou, Convergence theorems for $\lambda-$strict pseudo-contraction in $2-$uniformly smooth Banach spaces, *Nonlinear Anal.* **69** (2008) 3160–3173.

Li-ping Yang
Faculty of Applied Mathematics, Guangdong University of Technology, Guangzhou 510090 China
Email: yanglping2003@126.com