EXISTENCE AND MEASURABILITY OF THE SOLUTION OF THE STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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ABSTRACT. Here, the existence and measurability of solutions for stochastic differential equations driven by fractional Brownian noise with Hurst parameter greater than $\frac{1}{2}$ is proved. Our method is based on approximating the main equation by delayed equations as in Peano's method in ODEs. This method makes the proofs easier and needs weaker assumptions for the existence part, compared with the previous works as in [25]. In addition the constructive nature of the proofs helps to develop some numerical methods for solving such SDEs.

1. Introduction

Fractional Brownian motion: Since the pioneering work of Hurst [14], [15] and Mandelbrot and Van Ness [24], the fractional Brownian motions have played an increasingly important role in many fields of application such as hydrology, economics, queuing theory, telecommunications and mathematical finance. Let $B = \{B_t, t \geq 0\}$ be a fractional

Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$. That is, B is a centered Gaussian process with the covariance function

$$(1.1) \qquad R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Notice that if $H = \frac{1}{2}$, the process B is a standard Brownian motion, but if $H \neq \frac{1}{2}$, then it does not have independent increments. If $H > \frac{1}{2}$, then the process $B = \{B_t, t \geq 0\}$ exhibits a long-range dependence, that is, $\Sigma(E[B_1(B_{n+1} - B_n)]) = \infty$.

A fractional Brownian motion is also self-similar with the Hurst parameter H; that is, $\{B_{at}^H, t \geq 0\}$ has the same probability law as $\{a^H B_t^H, t \geq 0\}$. This property is an immediate consequence of the fact that the covariance function (1.1) is homogeneous of order $2H$. Hurst parameter was first introduced by the British hydrologist Harold Edwin Hurst (1880-1978) in the statistical analysis of the yearly water run-offs of Nile river [14]. This process was introduced by Kolmogorov [19] and studied by Mandelbrot and Van Ness [24], where a stochastic integral representation in terms of a standard Brownian motion was established. From (1.1) it follows that the increment of the process in an interval $[s, t]$ has a normal distribution with zero mean and variance $E|B_t - B_s|^2 = |t - s|^{2H}$; i.e., the process has stationary increment property. As a consequence, by the Kolmogorov's continuity criterion, the process B has α-Hölder continuous paths for all $\alpha \in (0, H)$.

Stochastic Integral: If $H \neq \frac{1}{2}$, then fractional Brownian motion (fBm) is not a semimartingale and hence the Itô approach to construct a stochastic integral with respect to fBm is not valid. Two main approaches have been suggested in order to define stochastic integrals with respect to fBm:

(i) One can construct stochastic integrals with respect to the fractional Brownian motion based on the Malliavin calculus. In fact, as in the case of the Brownian motion, the divergence operator with respect to B can be interpreted as a stochastic integral called the Skorohod integral [29]. This idea has been developed by Decreusefond and Üstünel [9], Carmona and Coutin [3], Alós et al. [1], [2], Duncan et al. [10] and Hu and Øksendal [13].

(ii) A pathwise method using Young's integral [30] can be used in the case $H > \frac{1}{2}$. The theory of rough path analysis introduced by Lyons [23] provides a pathwise approach to stochastic integration and stochastic differential equations with respect to the fBm in the case $H > \frac{1}{4}$ [6]. An alternative pathwise method based on fractional calculus has been

introduced by Zähle [31]. Nualart and Răşcanu follow the approach of Zähle, and prove a general result on the existence and uniqueness of the solution for multidimensional, time dependent, stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ [25]. Here, we follow their approach.

In the case $H > \frac{1}{2}$, one can use a pathwise approach to define integrals with respect to the fractional Brownian motion. In fact, if u_t is λ-Hölder continuous with $\lambda > 1 - H$, then the Riemann-Stieltjes integral $\int_0^T u_s dB_s$ exists, due to the results by Young [30].

More refined results have been obtained in [4] by Ciesielski et al. [4] (see also [26]) for processes u_t with trajectories in the Besov space $\mathcal{B}_{p,1}^{1-H}$, where $\frac{1}{p} < H < 1 - \frac{1}{p}$.

Here, we are interested in multidimensional stochastic differential equations of the form

$$(1.2) \qquad X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

where B is an fBm with Hurst parameter $H > \frac{1}{2}$, and the integral with respect to B is a pathwise Riemann-Stieltjes integral. This kind of equation has been studied by several authors ([18], [20], [21], [22], [27]).

In [22], Lyons considered integral equations

$$(1.3) \qquad x_t = x_0 + \sum_{j=1}^{m} \int_0^t \sigma^j(x_s)dg_s^j,$$

$0 \leq t \leq T$, where each g^j is a continuous function with bounded p-variation on $[0, T]$ for some $p \in [1, 2)$. He proved that if each σ^j is differentiable and its derivative is α-Hölder for some $\alpha > p - 1$, then a unique solution exists. Also, he proved if each σ^j is α-Hölder, then a solution exists, but it is not unique in general. Taking into account that the fBm of Hurst parameter H has locally bounded p-variation for $p > 1/H$, the result proved in [22] can be applied to equations driven by an fBm provided the nonlinear coefficient has a Hölder continuous derivative of order $\alpha > 1/H - 1$. Using this approach, based on the notion of p-variation and the general limit theorem proved by Lyons in [23] for differential equations driven by geometric rough paths, Coutin and Qian [5], [6] have established the existence of strong solutions and a Wong-Zakai type approximation limit for stochastic differential equations driven by a fractional Brownian motion with parameter $H > \frac{1}{4}$.

A. M. Davie in [8] develops an alternative approach to study equation (1.3), using (modified) Euler approximations, and investigates its applicability to stochastic differential equations driven by fractional Brownian motion with parameter $H > \frac{1}{3}$. He defines $x(t)$ to be a solution of (1.3) if $x^i(0) = x_0^i$ and there exists a continuous increasing function θ on $[0, T]$ and a non-negative function w on $[0, \infty)$ such that $\theta(\delta) = o(\delta)$ as $\delta \to 0$ and such that

$$(1.4) \qquad |x^i(t) - x^i(s) - f_j^i(x(s))(\sigma^j(t) - \sigma^j(s))| \le \theta(w(t) - w(s)),$$

for all s and t with $0 \le s < t \le T$. Being a solution in the sense of Davie is weaker than the classical one.

In [27] Ruzmaikina establishes an existence-uniqueness theorem for ordinary differential equations with Hölder continuous forcing. The global solution is constructed, first in small time intervals, where the contraction principle can be applied, provided the Hölder constant is small enough. The integral $\int_0^T f dg$ is defined in the sense of Young [30], assuming that the functions f and g are Hölder continuous of orders β and α, respectively, with $\beta + \alpha > 1$. This result is applied to stochastic differential equations driven by a fractional Brownian motion with parameter $H > \frac{1}{2}$.

Motivations: Here, we present a general result on the existence, measurability and adaptivity of solutions for the multidimensional time-dependent stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, following the approach of Zähle [30] and Nualart and Răşcanu [24]. They suppose b and σ are Lipchitz and σ is differentiable and its partial derivatives are Hölder. In the beginning, we were trying to prove an existence-uniqueness theorem, assuming b is monotone nonlinearities rather than Lipchitz. Similar results are proved for ordinary stochastic differential equations as in Zamani and Zangeneh [32], Zangeneh ([33], [34], [35], [36], [37], [38]), Jahanipur and Zangeneh [16], Jahanipur [17], Hamedani and Zangeneh [12], Dadashi and Zangeneh [7]. The main idea, that we hope to be applied in the case of fractional noise as well, is to use energy-type inequalities like the ones used in Zangeneh ([35], [36]) Jahanipur and Zangeneh [16] and Hamedani and Zangeneh [11]. Towards this goal, we found out that the conditions for the existence part of Nualarts theorem can be weakened. We proved that for the existence of a solution, b need not be Lipchitz and σ is not necessarily needed to be neither differentiable nor Lipchitz, but being Hölder of an order less than one is sufficient.

Our method based on delay equations and Peano's method (in ODE), is simpler than Nualart and Răşcanus approach and is more capable to be used as a numerical method to find a solution. Here, we focus on the existence part and then prove that by assuming "any condition" that assures the uniqueness, the solution is measurable and adaptive. For example, as mentioned before, this condition might be monotonicity and not Lipschitz for b.

2. Preliminaries

Let $B_t(\omega) = (B_t^j(\omega))_{j=1}^m$ and $B^j = \{B_t^j, t \geq 0\}$, for $j = 1, 2, ..., m$, be independent fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$. Consider the following stochastic differential equation in $\mathbb{R}^d$,

$$(2.1) \qquad X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

We are going to formulate and prove an existence theorem about this equation with some assumptions on b and σ. Our method is to transform the equation into a deterministic one. We know that if $\alpha < H$, then with probability one, B is α times differentiable and α-Hölder.

For $f : [a, b] \to \mathbb{R}$, the right-sided and left-sided fractional derivatives of order $0 < \rho < 1$ are defined by

$$(2.2) \qquad D_{a+}^\rho f(t) = \frac{1}{\Gamma(1-\rho)}\left(\frac{f(t)}{(t-a)^\rho} + \rho\int_a^t \frac{f(t)-f(s)}{(t-s)^{\rho+1}}ds\right)$$

and

$$(2.3) \qquad D_{b-}^\rho f(t) = \frac{(-1)^\rho}{\Gamma(1-\rho)}\left(\frac{f(t)}{(b-t)^\rho} + \rho\int_t^b \frac{f(t)-f(s)}{(s-t)^{\rho+1}}ds\right).$$

Using fractional derivatives one can generalize the Riemann-Stieltjes integral [28]. But we only use formulation of this generalization when the integral exists in the sense of classical Riemann-Stieltjes integral.

Lemma 2.1. *(Integration by fractional derivative) If $u, g : [a, b] \to \mathbb{R}$ are continuous and respectively ρ and $1 - \rho$ times differentiable, then u is integrable with respect to g in the sense of Riemann-Stieltjes and*

$$(2.4) \quad \int_a^b udg = (-1)^\rho \int_a^b D_{a+}^\rho u_{a+}(t)D_{b-}^{1-\rho}g_{b-}(t)dt + u(a)(g(b) - g(a)).$$

A proof for this lemma can be found in [28].

Let $u : [0, T] \to \mathbb{R}^d$ be an α-Hölder function. For $t \in [0, T]$, define:

$$(2.5) \qquad M_\alpha(u)(t) = \sup_{0 \le r < s \le t} \frac{|u(s) - u(r)|}{|s - r|^\alpha}$$

and

$$(2.6) \qquad M_\alpha(u) = M_\alpha(u)(T).$$

Lemma 2.2. *Let* $u, g : [a, b] \to \mathbb{R}^d$ *be, respectively,* α'-*Hölder and* α-*Hölder. If* $\alpha + \alpha' > 1$, *then* u *is integrable with respect to* g *and there exists* $C = C(\alpha, \alpha')$ *such that*

$$(2.7) \qquad |\int_a^b u \, dg - u(a)(g(b) - g(a))| \le C M_{\alpha'}(u) M_\alpha(g)(b - a)^{\alpha + \alpha'}.$$

Proof. For any $\rho \in (1 - \alpha, \alpha')$, u and g are, respectively, ρ and $1 - \rho$ times differentiable and

$$|D_{a+}^\rho u_{a+}(t)| \le \frac{1}{\Gamma(1 - \rho)} \left(\frac{|u(t) - u(a)|}{(t - a)^\rho} + \rho \int_a^t \frac{|u(t) - u(s)|}{(t - s)^{\rho+1}} dy \right)$$

$$(2.8) \qquad\qquad \le \frac{\alpha' M_{\alpha'}(u)}{(\alpha' - \rho)\Gamma(1 - \rho)} (t - a)^{\alpha' - \rho}.$$

Similarly

$$(2.9) \qquad |D_{b-}^{1-\rho} g_{b-}(t)| \le \frac{\alpha M_\alpha(g)}{(\alpha - 1 + \rho)\Gamma(\rho)} (b - t)^{\alpha - 1 + \rho}.$$

So, by Lemma 2.1,

$$|\int_a^b u \, dg - u(a)(g(b) - g(a))| \le \int_a^b |D_{a+}^\rho u_{a+}(t) D_{b-}^{1-\rho} g_{b-}(t)| dt$$

$$\le \frac{\alpha' \alpha M_{\alpha'}(u) M_\alpha(g)}{(\alpha' - \rho)(\alpha - 1 + \rho)\Gamma(1 - \rho)\Gamma(\rho)} \int_a^b (t - a)^{\alpha' - \rho}(b - t)^{\alpha - 1 + \rho} dt$$

$$(2.10)$$

$$= \frac{\alpha' \alpha Sin(\pi\rho)\Gamma(\alpha' - \rho)\Gamma(\alpha - 1 + \rho)}{\pi \Gamma(\alpha' + \alpha + 1)} M_{\alpha'}(u) M_\alpha(g)(b - a)^{\alpha' + \alpha}.$$

If we choose $\rho = \frac{\alpha' - \alpha + 1}{2}$, then

$$(2.11)$$

$$|\int_a^b u \, dg - u(a)(g(b) - g(a))| \le \frac{\alpha \alpha' \Gamma(\frac{\alpha + \alpha' - 1}{2})^2}{\pi \Gamma(\alpha + \alpha' + 1)} M_{\alpha'}(u) M_\alpha(g)(b - a)^{\alpha + \alpha'}.$$

So, it suffices to put

(2.12) $$C = C(\alpha, \alpha') = \frac{\alpha\alpha'\Gamma(\frac{\alpha+\alpha'-1}{2})^2}{\pi\Gamma(\alpha+\alpha'+1)}.$$

$\square$

3. Deterministic differential equation

For $g_t = B(t, \omega)$, a realization of B, we have the deterministic differential equation,

(3.1) $$x_t = x_0 + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dg_s \quad t \in [0, T].$$

For $u : [0, T] \to \mathbb{R}^d$, define

(3.2) $$F(u)(t) = \int_0^t b(s, u_s)ds$$

and

(3.3) $$G(u)(t) = \int_0^t \sigma(s, u_s)dg_s,$$

if these integrals exist. Now, the deterministic equation is:

(3.4) $$x = x_0 + F(x) + G(x).$$

4. Existence theorem

What do we need to prove the existence theorem? Naturally, we assume that b and σ are measurable functions which are continuous in space and their growth in time is at most linear. In the special case of $b = 0$ and $\sigma = 1$, the above equation becomes $dx = dg$. So, in general, x may not be smoother than g. If σ is β-Hölder in time and γ-Hölder in space, then we can expect $\sigma(t, x_t)$, as a function of t, to be at most $\min\{\beta, \gamma\alpha\}$-Hölder. To guarantee $\int_0^t \sigma(s, x_s)dg_s$ to exist, $\alpha + \min\{\beta, \gamma\alpha\}$ should be greater than one. In other words, $\beta > 1 - \alpha$ and $\gamma > \frac{1-\alpha}{\alpha}$.

In addition, we need some assumptions to guarantee $\int_0^t b(s, x_s)ds$ to be α-Hölder. Using Hölder's inequality, one can show that the integral of a function in $L^\rho([0, T])$ is $(1 - \rho^{-1})$-Hölder. So, if $b(t, x_t) \in L^{\frac{1}{1-\alpha}}([0, T])$, then we are done.

Theorem 4.1. *Suppose that $g : [0, T] \to \mathbb{R}$ is α-Hölder. If measurable functions b and σ are continuous in space and satisfy the assumptions (H_b) and (H_σ), below then equation (3.4) has at least one α-Hölder solution:*

(H_b) *There is a function $b_0 \in L^{\frac{1}{1-\alpha}}([0, T])$ and a constant $L \geq 0$ such that for every $x \in \mathbb{R}^d$ and $t \in [0, T]$,*

$$(4.1) \qquad\qquad |b(t, x)| \leq L|x| + b_0(t).$$

(H_σ) *For some $\beta > 1 - \alpha$ and $\gamma > \frac{1-\alpha}{\alpha}$, there is a constant $M > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t, s \in [0, T]$,*

$$(4.2) \qquad\qquad |\sigma(t, x) - \sigma(s, y)| \leq M(|t - s|^\beta + |x - y|^\gamma).$$

Our approach to prove this theorem to be given later at the end of this section, is to use delay equations as in Peano's method. For $0 \leq \delta \leq T$ and a function $u : [0, T] \to \mathbb{R}^d$, define the δ-version of operator F and G, say F_δ and G_δ, as follows:

$$(4.3) \qquad F_\delta(u)(t) = \begin{cases} 0 & t \in [0, \delta] \\ \\ F(u)(t - \delta) & t \in [\delta, T] \end{cases}$$

and

$$(4.4) \qquad G_\delta(u)(t) = \begin{cases} 0 & t \in [0, \delta] \\ \\ G(u)(t - \delta) & t \in [\delta, T]. \end{cases}$$

Now, the delay form of equation (3.4) becomes:

$$(4.5) \qquad\qquad y = x_0 + F_\delta(y) + G_\delta(y).$$

Obviously, $F_0 = F$ and $G_0 = G$. In what follows, we will prove that for any $\delta > 0$, equation (4.5) has a unique solution $y^{(\delta)}$ and that the $y^{(\delta)}$ are uniformly bounded and uniformly α-Hölder (i.e., simultaneously α-Hölder with the same coefficient), and consequently equi-continuous; So, there exists a convergent subsequence of the $y^{(\delta)}$ as $\delta \to 0$. We prove that the limit of this subsequence solves equation (3.4).

To start, let us take a closer look at the operators F_δ and G_δ.

Lemma 4.2. *Assume $0 \leq \delta \leq T$. If $u : [0, T] \to \mathbb{R}^d$ is continuous, then $F_\delta(u)$ exists and is an α-Hölder function. In addition, for any*

$t \in [0, T - \delta]$,

$$(4.6) \qquad \big\| F_\delta(u)|_{[0,t+\delta]} \big\|_\infty \leq L \int_0^t |u(s)| ds + \|b_0\|_{\frac{1}{1-\alpha}} t^\alpha$$

and

$$(4.7) \qquad M_\alpha(F_\delta(u))(t + \delta) \leq L t^{1-\alpha} \big\| u|_{[0,t]} \big\|_\infty + \|b_0\|_{\frac{1}{1-\alpha}} .$$

Proof. For $t \in [0, T - \delta]$, we have,

$$(4.8) \ |F_\delta(u)(t + \delta)| \ \leq \ \int_0^t |b(s, u(s))| ds \leq \int_0^t (L|u(s)| + b_0(s)) ds$$

$$\leq \ L \int_0^t |u(s)| ds + \|b_0\|_{\frac{1}{1-\alpha}} t^\alpha.$$

The right hand side of this inequality is increasing in t, and so we can replace $|F_\delta(u(t + \delta))|$ by $|F_\delta(u(t' + \delta))|$, for any $0 \leq t' \leq t$, and thus,

$$(4.9) \qquad \big\| F_\delta(u)|_{[0,t+\delta]} \big\|_\infty \leq L \int_0^t |u(s)| ds + \|b_0\|_{\frac{1}{1-\alpha}} t^\alpha.$$

To prove the second inequality, suppose $0 \leq t_1 < t_2 \leq t$. Then,

$$|F_\delta(u)(t_2 + \delta) - F_\delta(u)(t_1 + \delta)| = |\int_{t_1}^{t_2} b(s, u(s)) ds|$$

$$\leq \int_{t_1}^{t_2} (L|u(s)| + b_0(s)) ds$$

$$(4.10) \qquad \qquad \leq L(t_2 - t_1) \big\| u|_{[0,t]} \big\|_\infty + \|b_0\|_{\frac{1}{1-\alpha}} (t_2 - t_1)^\alpha.$$

For any $s \leq \delta$, we have $F_\delta(u)(s) = 0$. So,

$$(4.11) \qquad M_\alpha(F_\delta(u))(t + \delta) \leq L t^{1-\alpha} \big\| u|_{[0,t]} \big\|_\infty + \|b_0\|_{\frac{1}{1-\alpha}} .$$

$\square$

Lemma 4.3. *Assume that $0 \leq \delta \leq T$. If $u : [0, T] \to \mathbb{R}^d$ is α-Hölder, then $G_\delta(u)$ exists and is an α-Hölder function. In addition, for any $t \in [0, T - \delta]$,*

$$\big\| G_\delta(u)|_{[0,t+\delta]} \big\|_\infty \leq (|\sigma(0, u(0))| + M(1 + C)$$

$$(4.12) \qquad \qquad \times (t^\beta + M_\alpha(u)(t)^\gamma t^{\alpha\gamma})) M_\alpha(g) t^\alpha$$

and

$$M_\alpha(G_\delta(u))(t+\delta) \leq (|\sigma(0,u(0))| + M(1+C)$$
$$(4.13) \qquad \times (t^\beta + M_\alpha(u)(t)^\gamma t^{\alpha\gamma}))M_\alpha(g).$$

Proof. First, we prove the second inequality. Suppose that $0 \leq t_1 < t_2 \leq t$. Using Lemma 2.2,

$$|G_\delta(u)(t_2+\delta) - G_\delta(u)(t_1+\delta)| = |\int_{t_1}^{t_2} \sigma(s,u(s))dg_s|$$
$$\leq |\sigma(t_1,u(t_1))||g_{t_2} - g_{t_1}|$$
$$+ CM_{\alpha'}(\sigma(.,u.))(t)M_\alpha(g)(t_2-t_1)^{\alpha+\alpha'}$$
$$\leq (|\sigma(0,u(0))| + |\sigma(t_1,u(t_1)) - \sigma(0,u(0))|)M_\alpha(g)(t_2-t_1)^\alpha$$
$$(4.14) \qquad + CM_{\alpha'}(\sigma(.,u.))(t)M_\alpha(g)(t_2-t_1)^{\alpha+\alpha'}.$$

By (H_σ) and some calculations,

$$(4.15) \qquad M_{\alpha'}(\sigma(.,u.))(t) \leq M(t^{\beta-\alpha'} + (M_\alpha(u)(t))^\gamma t^{\alpha\gamma-\alpha'}).$$

$G_\delta(u)(s) = 0$, for any $s \leq \delta$. So,

$$M_\alpha(G_\delta(u))(t+\delta) \leq (|\sigma(0,u(0))| + M(t^\beta + (M_\alpha(u)(t))^\gamma t^{\alpha\gamma}))M_\alpha(g)$$
$$+ CM(t^\beta + (M_\alpha(u)(t))^\gamma t^{\alpha\gamma})M_\alpha(g)$$
$$= (|\sigma(0,u(0))| + M(1+C)$$
$$(4.16) \qquad \times (t^\beta + (M_\alpha(u)(t))^\gamma t^{\alpha\gamma}))M_\alpha(g).$$

The first inequality is an easy consequence of the second one. We have $G_\delta(u)(\delta) = 0$, and so for any $t \in [0, T-\delta]$,

$$|G_\delta(u)(t+\delta)| \leq M_\alpha(G_\delta)(u)(t)t^\alpha \leq (|\sigma(0,u(0))|$$
$$(4.17) \qquad + M(1+C)(t^\beta + (M_\alpha(u)(t))^\gamma t^{\alpha\gamma})M_\alpha(g)t^\alpha.$$

$\square$

Now, using previous lemmas, the existence of a solution to (4.5) can be easily proved.

Lemma 4.4. *For any $\delta > 0$, equation (4.5) has a unique solution and this solution is α-Hölder.*

Proof. By induction, we show that for $n = 1, 2, ..., \lceil \frac{T}{\delta} \rceil$, equation (4.5) has a unique solution on $[0, n\delta] \cap [0, T]$ which is α-Hölder.

The case $n = 1$, is trivial, because the constant function $y_t \equiv x_0$ on $[0, \delta]$ is the solution.

Suppose that $n = k < \frac{T}{\delta}$ and $y : [0, k\delta] \to \mathbb{R}$ is an α-Hölder solution for (4.5) on interval $[0, k\delta]$. For $t \in [k\delta, (k + 1)\delta] \cap [0, T]$, y_t should be equal to

$$x_0 + F_\delta(y)(t) + G_\delta(y)(t).$$

So, it is sufficient to show that $F_\delta(y)$ and $G_\delta(y)$ exist and are α-Hölder. Notice that the values of $F_\delta(y)$ and $G_\delta(y)$ in the interval $[k\delta, (k + 1)\delta]$ depend on y only in the interval $[0, k\delta]$. So, by lemmas 4.2 and 4.3, y is α-Hölder on $[0, (k + 1)\delta] \cap [0, T]$. $\square$

Remark 4.5. *For any $\delta > 0$, the unique solution to (4.5), as a stochastic process, is measurable and if b and σ are adaptive, then the solution is adaptive as well.*

According to what we just proved, one can speak of $y^{(\delta)}$ as the unique solution to (4.5). Our next step is to prove that the $y^{(\delta)}$ are uniformly bounded and that they are all α-Hölder with the same coefficient, independent of δ. We will do that using some Gronwall inequalities simultaneously for the bound of the $y^{(\delta)}$ and their α-Hölder coefficients which are related to each other. In the following lemmas, δ is fixed and hence we drop the superscript δ and write y instead of $y^{(\delta)}$.

Lemma 4.6. *Suppose that the sequence $\{u_m : [0, T] \to \mathbb{R}^d\}$ of continuous functions converges uniformly to a function u. Then, for any $\delta \geq 0$, $\{F_\delta(u_m)\}$ converges pointwise to $F_\delta(u)$.*

Proof. For any $t \in [0, T - \delta]$,

$$(4.19) \qquad F_\delta(u_m)(t + \delta) = \int_0^t b(s, u_m(s)) ds$$

and

$$(4.20) \qquad |b(s, u_m(s))| \leq L |u_m(s)| + b_0(s).$$

Note that $\{u_m\}$ is a bounded sequence and b_0 is integrable, and thus the statement is proved according to Lebesgue's dominated convergence theorem.

Lemma 4.7. *Suppose that $u, v : [0, T] \to \mathbb{R}^d$ are α-Hölder, and $u(0) = v(0)$. For any $\rho \in (1 - \alpha, \min(\beta, \alpha\gamma))$, there is a constant A_ρ such that*

for any $\theta \in (0, 1]$,

(4.21) $\|G_\delta(u) - G_\delta(v)\|_\infty \le A_\rho(\|u - v\|_\infty^\gamma + \theta^\beta + M_0\theta^{\alpha\gamma})\theta^{-\rho},$

where $M_0 = \max(M_\alpha(u), M_\alpha(v))$.

Proof. For any $t \in [0, T - \delta]$,

$$|G_\delta(u)(t + \delta) - G_\delta(v)(t + \delta)| = |\int_0^t (\sigma(s, u(s)) - \sigma(s, v(s)))dg_s|$$

$$\le \int_0^t |D_{0+}^\rho(\sigma(., u(.)) - \sigma(., v(.)))_{0+}(s)D_{t-}^{1-\rho}g_{t-}(s)|ds$$

(4.22) $$\le \frac{\alpha M_\alpha(g)t^{\alpha-1+\rho}}{(\alpha - 1 + \rho)\Gamma(\rho)}\int_0^t |D_{0+}^\rho(\sigma(., u(.)) - \sigma(., v(.)))_{0+}(s)|ds.$$

For any $s \le t$,

(4.23) $$|D_{t-}^{1-\rho}g_{t-}(s)| \le \frac{1-\rho}{\Gamma(\rho)}\int_s^t \frac{|g(s) - g(r)|}{(r-s)^{2-\rho}}dr \le \frac{-M_\alpha(g)t^{\alpha+\rho-1}}{(\alpha+\rho-1)\Gamma(\rho-1)}$$

and

$$|D_{0+}^\rho(\sigma(., u(.)) - \sigma(., v(.)))_{0+}(s)| \le \frac{\rho}{\Gamma(1-\rho)}$$

$$\times \int_0^s \frac{|\sigma(s, u(s)) - \sigma(s, v(s)) - \sigma(r, u(r)) + \sigma(r, v(r))|}{(s-r)^{1+\rho}}dr$$

$$\le \frac{-M}{\Gamma(-\rho)}(\int_0^{s-\theta s} \frac{(|u(s) - v(s)|^\gamma + |u(r) - v(r)|^\gamma)}{(s-r)^{1+\rho}}dr$$

$$+ \int_{s-\theta s}^s \frac{2|s-r|^\beta + |u(s) - u(r)|^\gamma + |v(s) - v(r)|^\gamma}{(s-r)^{1+\rho}}dr)$$

(4.24) $$\le \frac{-2M}{\Gamma(-\rho)}(\frac{\|u-v\|_\infty^\gamma}{\rho(\theta s)^\rho} + \frac{(\theta t)^{\beta-\rho}}{\beta-\rho} + \frac{M_0(\theta t)^{\alpha\gamma-\rho}}{\alpha\gamma-\rho}).$$

So,

$$
\begin{aligned}
|G_\delta(u)(t+\delta) - G_\delta(v)(t+\delta)| &\leq \frac{2MM_\alpha(g)t^{\alpha+\rho-1}}{(\alpha+\rho-1)\Gamma(-\rho)\Gamma(\rho-1)} \\
&\times \int_0^t \Big(\frac{\|u-v\|_\infty^\gamma}{\rho(\theta s)^\rho} + \frac{(\theta t)^{\beta-\rho}}{\beta-\rho} + \frac{M_0(\theta t)^{\alpha\gamma-\rho}}{\alpha\gamma-\rho}\Big)ds \\
&= \frac{2MM_\alpha(g)t^\alpha}{(\alpha+\rho-1)\Gamma(-\rho)\Gamma(\rho-1)}\Big(\frac{\|u-v\|_\infty^\gamma}{\rho(1-\rho)} \\
&+ \frac{(\theta t)^\beta}{\beta-\rho} + \frac{M_0(\theta t)^{\alpha\gamma}}{\alpha\gamma-\rho}\Big)\theta^{-\rho}.
\end{aligned}
$$

(4.25)

Let
(4.26)

$$
A_\rho = \frac{2MM_\alpha(g)T^\alpha}{(\alpha+\rho-1)\Gamma(-\rho)\Gamma(\rho-1)} \max\Big\{\frac{1}{\rho(1-\rho)}, \frac{T^\beta}{\beta-\rho}, \frac{T^{\alpha\gamma}}{\alpha\gamma-\rho}\Big\}.
$$

Then, we have,

$$
(4.27) \qquad \|G_\delta(u) - G_\delta(v)\|_\infty \leq A_\rho(\|u-v\|_\infty^\gamma + \theta^\beta + M_0\theta^{\alpha\gamma})\theta^{-\rho}.
$$

$\square$

Let $\bar{y}, z : [0, T] \to \mathbb{R}$ be:

$$
(4.28) \qquad \bar{y}(t) = \big\||y|_{[0,t]}\big\|_\infty = \sup_{0\leq s\leq t} |y_s|
$$

and

$$
(4.29) \qquad z(t) = M_\alpha(y)(t) = \sup_{0\leq r<s\leq t} \frac{|y_s - y_r|}{(s-r)^\alpha}.
$$

Lemma 4.8. *There are $A : \mathbb{R}^d \to \mathbb{R}$ and a constant B such that for any $\delta > 0$ and $t \in [0, T-\delta]$,*

$$
(4.30) \quad \bar{y}(t+\delta) \leq |x_0| + A(x_0)t^\alpha + L\int_0^t \bar{y}(s)ds + M'(1+C)t^{\alpha(\gamma+1)}z(t)^\gamma
$$

and

$$
(4.31) \qquad z(t+\delta) \leq B + M'\bar{y}(t)^\gamma + Lt^{1-\alpha}\bar{y}(t) + M'Ct^{\alpha\gamma}z(t)^\gamma,
$$

where $M' = MM_\alpha(g)$.

Proof. For any $t \in [0, T - \delta]$,

$$|y_{t+\delta}| \leq |x_0| + |F_\delta(y)(t)| + |G_\delta(y)(t)|$$

$$\leq |x_0| + L \int_0^t |y_s| ds + \|b_0\|_{\frac{1}{1-\alpha}} t^\alpha + (|\sigma(0, x_0)|$$

$$+ (1 + C) M(t^\beta + z(t)^\gamma t^{\alpha\gamma})) M_\alpha(g) t^\alpha$$

$$(4.32) \qquad \leq |x_0| + A(x_0) t^\alpha + L \int_0^t \bar{y}(s) ds + M' C t^{\alpha(\gamma+1)} z(t)^\gamma,$$

where,

$$(4.33) \qquad A(x_0) = \|b_0\|_{\frac{1}{1-\alpha}} + |\sigma(0, x_0)| M_\alpha(g) + M' C T 0^\beta.$$

The right hand side of inequality (4.32) is increasing in t. So we can replace $|y_{t+\delta}|$ by $|y_{t'+\delta}|$ for any $0 \leq t' \leq t$, and by the definition of $\bar{y}$ one has,

$$(4.34) \quad \bar{y}(t + \delta) \leq |x_0| + A(x_0) t^\alpha + L \int_0^t \bar{y}(s) ds + M' C t^{\alpha(\gamma+1)} z(t)^\gamma.$$

To prove the second inequality, one has,

$$M_\alpha(y)(t + \delta) \leq M_\alpha(F(y))(t + \delta) + M_\alpha(G(y))(t + \delta)$$

$$\leq L t^{1-\alpha} \left\| y|_{[0,t]} \right\|_\infty + \|b_0\|_{\frac{1}{1-\alpha}} + (|\sigma(0, y_0)|$$

$$+ (1 + C) M(t^\beta + (M_\alpha(y)(t))^\gamma t^{\alpha\gamma})) M_\alpha(g)$$

$$\leq L t^{1-\alpha} \bar{y}(t) + \|b_0\|_{\frac{1}{1-\alpha}} + (|\sigma(0, x_0)|$$

$$(4.35) \qquad + (1 + C) M(t^\beta + z(t)^\gamma t^{\alpha\gamma})) M_\alpha(g).$$

So,

$$z(t + \delta) \leq L t^{1-\alpha} \bar{y}(t) + \|b_0\|_{\frac{1-\alpha}{\alpha}} + (|\sigma(0, x_0)|$$

$$+ (1 + C) M(t^\beta + z(t)^\gamma t^{\alpha\gamma})) M_\alpha(g)$$

$$\leq B + L t^{1-\alpha} \bar{y}(t) + M'(1 + C) t^{\alpha\gamma} z(t)^\gamma,$$

$$(4.36)$$

where,

$$(4.37) \qquad B = \|b_0\|_{\frac{1-\alpha}{\alpha}} + |\sigma(0, x_0)| M_\alpha(g) + (1 + C) M' T^\beta,$$

and as above, $M' = M M_\alpha(g)$. $\square$

Lemma 4.9. *There are $L' > 0$, $T_1 \leq T$, and a function $A' : \mathbb{R}^d \to \mathbb{R}$ such that for any $t \in [0, T_1]$,*

$$(4.38) \qquad \bar{y}(t) \leq A'(x_0) + 2L \int_0^t \bar{y}(s)ds$$

and

$$(4.39) \qquad z(t) \leq C' + 2Lt^{1-\alpha}\bar{y}(t).$$

Proof. Since $\gamma \leq 1$, then $a^\gamma \leq 1 + a$, for any nonnegative a. Since z is an increasing function, then one has,

$$(4.40) \qquad z(t) \leq B + Lt^{1-\alpha}\bar{y}(t) + M'(1 + C)t^{\alpha\gamma}(1 + z(t)).$$

If we choose $T_1 \leq T$ such that $M'(1 + C)T_1^{\alpha\gamma} \leq \frac{1}{2}$, then we have,

$$(4.41) \qquad z(t) \leq C' + 2Lt^{1-\alpha}\bar{y}(t),$$

where $C' = 2(B + M'(1 + CT^{\alpha\gamma}))$. This proves (4.39). To prove (4.38) for any $t \in [0, T]$, notice that

$$(4.42) \quad \bar{y}(t) \leq |x_0| + A(x_0)t^\alpha + L \int_0^t \bar{y}(s)ds + M'Ct^{\alpha(\gamma+1)}(1 + z(t)).$$

If $T_1 \leq T$ is small enough so that $LM'CT_1^{\alpha\gamma+1} \leq \frac{1}{4}$, then

$$(4.43) \qquad \bar{y}(t) \leq A'(x_0) + 2L \int_0^t \bar{y}(s)ds,$$

where,

$$(4.44) \qquad A'(x_0) = 2(|x_0| + A(x_0)T^\alpha + (1 + C')M'CT^{\alpha(\gamma+1)}).$$

$\square$

Remark 4.10. *In Lemma 4.9 one can choose $T_1 \leq (4M'(1 + C + 2LC))^{\frac{-1}{\alpha\gamma}}$.*

Lemma 4.11. *For any $t \in [0, T_1]$,*

$$(4.45) \qquad \bar{y}(t) \leq A'(x_0)e^{2Lt}$$

and

$$(4.46) \qquad z(t) \leq C' + 2Lt^{1-\alpha}A'(x_0)e^{2Lt}.$$

Proof. It is a consequence of Gronwall inequality and Lemma 4.9. $\square$

Lemma 4.12. *Uniformly on interval* $[0, T_1]$,

(4.47) $$\lim_{\delta \to 0} |F(y^{(\delta)}) - F_\delta(y^{(\delta)})| = 0$$

and

(4.48) $$\lim_{\delta \to 0} |G(y^{(\delta)}) - G_\delta(y^{(\delta)})| = 0.$$

Proof. If $\delta \leq t \leq T_1$, then

$$|F(y^{(\delta)})(t) - F_\delta(y^{(\delta)})(t)| = |F(y^{(\delta)})(t) - F(y^{(\delta)})(t - \delta)|$$
$$\leq M_\alpha(F(y^{(\delta)}))(t)\delta^\alpha \leq (Lt^{1-\alpha}\bar{y}^{(\delta)}(t) + \|b_0\|_{\frac{1}{1-\alpha}})\delta^\alpha$$

(4.49) $$\leq (LT^{1-\alpha}A'(x_0)e^{2LT} + \|b_0\|_{\frac{1}{1-\alpha}})\delta^\alpha.$$

If $0 \leq t \leq \delta$, then

$$|F(y^{(\delta)})(t) - F_\delta(y^{(\delta)})(t)| = |F(y^{(\delta)})(t) - F(y^{(\delta)})(0)|$$
$$\leq (Lt^{1-\alpha}\bar{y}^{(\delta)}(t) + \|b_0\|_{\frac{1}{1-\alpha}})t^\alpha$$

(4.50) $$\leq (LT^{1-\alpha}A'(x_0)e^{2LT} + \|b_0\|_{\frac{1}{1-\alpha}})\delta^\alpha.$$

So, uniformly on interval $[0, T_1]$,

(4.51) $$\lim_{\delta \to 0} |F(y^{(\delta)}) - F_\delta(y^{(\delta)})| = 0.$$

If $\delta \leq t \leq T_1$, then

$$|G(y^{(\delta)})(t) - G_\delta(y^{(\delta)})(t)| = |G(y^{(\delta)})(t) - G(y^{(\delta)})(t - \delta)|$$
$$\leq M_\alpha(G(y^{(\delta)}))(t)\delta^\alpha \leq ((|\sigma(0, y(0))| + (1 + C)$$
$$\times (t^\beta + (M_\alpha(y^{(\delta)})(t))^\gamma t^{\alpha\gamma})M)M_\alpha(g))\delta^\alpha$$
$$\leq ((|\sigma(0, x_0)| + (1 + C)$$

(4.52) $$\times (T^\beta + (C' + 2LT^{1-\alpha}A'(x_0)e^{2LT})^\gamma T^{\alpha\gamma})M)M_\alpha(g))\delta^\alpha.$$

If $0 \leq t \leq \delta$, then

$$|G(y^{(\delta)})(t) - G_\delta(y^{(\delta)})(t)| = |G(y^{(\delta)})(t) - 0| = |G(y^{(\delta)})(t) - G(y^{(\delta)})(0)|$$
$$\leq M_\alpha(G(y^{(\delta)}))(t)t^\alpha \leq ((|\sigma(0, y^{(\delta)}(0))| + (1 + C)$$
$$\times (t^\beta + (M_\alpha(y^{(\delta)})(t))^\gamma t^{\alpha\gamma})M)M_\alpha(g))t^\alpha$$
$$\leq ((|\sigma(0, x_0)| + (1 + C)$$

(4.53) $$\times (T^\beta + (C' + 2LT^{1-\alpha}A'(x_0)e^{2LT})^\gamma T^{\alpha\gamma})M)M_\alpha(g))\delta^\alpha.$$

So, uniformly on interval $[0, T_1]$,

$$(4.54) \qquad \lim_{\delta \to 0} |G(y^{(\delta)}) - G_\delta(y^{(\delta)})| = 0.$$

Lemma 4.13. *The family $\{y^{(\delta)}\}_{\delta > 0}$ is bounded and α-Hölder with the same coefficient on $[0, T_1]$. In addition, every sequence of positive numbers converging to zero has a subsequence like $\{\delta_m\}$ such that $\{y^{(\delta_m)}\}$ is uniformly convergent to an α-Hölder function on $[0, T_1]$ and any function which is the limit of such a sequence is a solution to (3.4) on $[0, T_1]$.*

Proof. Lemma 4.11 implies that the family $\{y^{(\delta)}\}_{\delta > 0}$ is bounded by $A'(x_0)e^{2LT}$ on the interval $[0, T_1]$ and

$$(4.55) \qquad M_\alpha(y^{(\delta)})(T_1) \leq C' + 2LT^{1-\alpha}A'(x_0)e^{2LT}.$$

So, this family is equi-continuous on the interval $[0, T_1]$. Arzela Ascoli Theorem implies that any sequence in this family has a uniform convergent subsequence. Therefore, the so-called sequence in this lemma has a subsequence $\{\delta_m\}$ such that $\{y_m = y^{(\delta_m)}\}$ uniformly converges on the interval $[0, T_1]$ to a function, say x. It is clear that the limit of an α-Hölder sequence with the same Hölder coefficients is α-Hölder. So, x is in the domain of F_δ and G_δ. For any $t \in [0, T_1]$, one has

$$\begin{aligned}
|x - x_0 - F(x) - G(x)|(t) &\leq |x - y_m|(t) \\
&\quad + |F_{\delta_m}(y_m) - F(y_m)|(t) + |G_{\delta_m}(y_m) - G(y_m)|(t) \\
&\quad + |F(y_m) - F(x)|(t) + |G(y_m) - G(x)|(t) \\
&\leq \|x - y_m\|_\infty + \|F_{\delta_m}(y_m) - F(y_m)\|_\infty \\
&\quad + \|G_{\delta_m}(y_m) - G(y_m)\|_\infty + \|F(y_m) - F(x)\|_\infty \\
&\quad + \|G(y_m) - G(x)\|_\infty.
\end{aligned}$$

$$(4.56)$$

So,

$$\begin{aligned}
|x - x_0 - F(x) - G(x)|(t) &\leq \liminf_{m \to \infty}(\|x - y_m\|_\infty \\
&\quad + \|F_{\delta_m}(y_m) - F(y_m)\|_\infty + \|G_{\delta_m}(y_m) - G(y_m)\|_\infty \\
&\quad + \|F(y_m) - F(x)\|_\infty + \|G(y_m) - G(x)\|_\infty) \\
&\leq A_\rho \liminf_{m \to \infty}(\|x - y_m\|_\infty^\gamma + \theta^\beta + M_0\theta^{\alpha\gamma})\theta^{-\rho} \\
&\leq A_\rho(\theta^{\beta-\rho} + M_0\theta^{\alpha\gamma-\rho}),
\end{aligned}$$

$$(4.57)$$

where $\rho \in (1 - \alpha, \min(\beta, \alpha\gamma))$, $\theta \in (0, 1]$, and $M_0 = C' + 2LT^{1-\alpha}A'(x_0)$ e^{2LT}. Hence, it is obvious that x is the solution to (3.4) as θ tends to zero. $\square$

Proof. (Of Theorem 4.1) According to Lemma 4.13, equation (3.4) has a solution on interval $[0, T_1]$. Note that T_1 is independent of x_0. Iterating this argument, the solution can be extended to $[0, 2T_1]$ and so on. This way, the solution on the whole interval $[0, T]$ can be constructed after some steps. $\square$

5. Measurability and adaptivity

The assumptions (H_b) and (H_σ) do not guarantee uniqueness of the solution for equation (3.4). In [25], Nualart and Răşcanu prove that if moreover, b is locally Lipschitz in x, σ is differentiable in x and $\frac{\partial \sigma}{\partial x}$ is locally γ-Hölder for some $\gamma > \frac{1}{H} - 1$, then equation (3.4) has a unique solution. It might be possible to strengthen their theorem or prove uniqueness with different assumptions, for example assuming that b is monotone. However, we are going to prove measurability and adaptivity assuming uniqueness.

In the previous section, we proved the existence of a solution by constructing a sequence and showing that there existed some $T_1 > 0$ such that the sequence had a convergent subsequence on $[0, T_1]$. The difficulty to prove measurability is that although this sequence is measurable, but both T_1 and the so-called subsequence depend on ω. Here, we assume some uniqueness assumptions and construct a measurable sequence that converges on the whole interval $[0, T]$.

Theorem 5.1. *If b and σ satisfy all the assumptions in Theorem 4.1 and one of the following conditions, then the stochastic equation (2.1) has a unique measurable solution and if b and σ are adaptive, then the solution is also adaptive:*

(U) For almost every $g_t = B(t, \omega)$, a realization of fractional Brownian motion, the equation (3.1) has at most one solution.

(U') For every α-Hölder g_t, the equation (3.1) has at most one solution.

The proof will be given at the end of this section after giving the following two lemmas.

Lemma 5.2. *Suppose that b and σ satisfy all assumptions in theorem 4.1 and for a realization of fractional Brownian motion like $g_t = B(t, \omega)$, the equation (3.1) has exactly one solution. If $\{\delta_m\}$ is a sequence of positive numbers converging to zero, then $\{y^{(\delta_m)}\}$ converges to the solution of (3.1) on $[0, T_1]$.*

Proof. Suppose that for some $t_0 \in [0, T_1]$, the sequence $\{y^{(\delta_m)}(t_0)\}$ does not converge to $x(t_0)$. Therefore, there exists an $\epsilon > 0$ and a subsequence of $\{\delta_m\}$ like $\{\delta'_m\}$ such that $|y^{(\delta'_m)}(t_0) - x(t_0)| \geq \epsilon$. But, by Lemma 4.13 we know that $\{y^{(\delta'_m)}\}$ has a subsequence converging to a solution of (3.1), which is a contradiction since we had assumed uniqueness.   □
Now, define,

$$(5.1) \qquad \tau(\omega) = \min(T, (4M'(1 + C + 2LC))^{\frac{-1}{\alpha\gamma}}).$$

By Lemma 5.2, Lemma 4.9 and Remark 4.10, we know that for any initial value $x(0) = x_0$, the sequence of solutions of delay equations (i.e., $\{y^{(\delta_m)}\}$) converges to the solution of (3.1) on the interval $[0, \tau(\omega)]$. Similarly, this sequence converges to the solution of (3.1) on the interval $[\tau(\omega), 2\tau(\omega)]$ with initial value $x(\tau(\omega))$ in $t = \tau(\omega)$. For $k = 0, 1, 2, ...$ and $\delta < \tau(\omega)$, consider the following equation,
(5.2)

$$\hat{y}^{(\delta)}(t) = \begin{cases} x(k\tau(\omega)) & t - k\tau(\omega) \in [0, \delta) \\ \\ \begin{aligned} x(k\tau(\omega)) \quad &+ \quad \int_{k\tau(\omega)}^{t-\delta} b(s, \hat{y}^{(\delta)}(s))ds \\ &+ \quad \int_{k\tau(\omega)}^{t-\delta} \sigma(s, \hat{y}^{(\delta)}(s))dg_s \end{aligned} & t - k\tau(\omega) \in [\delta, \tau(\omega)) \end{cases}$$

and for any $\delta \geq \tau(\omega)$, let

$$(5.3) \qquad\qquad \hat{y}^{(\delta)}(t) = x(k\tau(\omega)).$$

Lemma 5.3. *For a.s. ω and any $\delta < \tau(\omega)$, (respectively $\delta \geq \tau(\omega)$) equation (5.2) (respectively equation (5.3)) has a unique solution. In addition, $\hat{y}^{(\delta)}$, as a stochastic process, is measurable and if b and σ are adaptive, then it is adaptive too.*

Proof. For a.s. ω, $g_t = B(t, \omega)$ is α-Hölder. The proof for the case $\delta < \tau(\omega)$ is similar to Lemma 4.4. The case $\delta \geq \tau(\omega)$ is trivial.   □

Remark 5.4. $\hat{y}^{(\delta)}$ *may not be continuous at* $t = k\tau(\omega)$*, but it is* α*-Hölder on every subintervals* $[k\tau(\omega), (k+1)\tau(\omega))$*.*

Proof. (Of Theorem 5.1) It is clear that (U′) implies (U). Consider a positive sequence like $\{\delta_m\}$ which converges to zero. For a.s. ω, using Lemma 5.2 on any subinterval $[k\tau(\omega), (k+1)\tau(\omega))$ implies that $\hat{y}^{(\delta_m)}(t) \to x(t)$. Therefore, x is measurable and if b and σ are adaptive, then x will be adaptive too. □

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