ON THE SPECTRA OF SOME MATRICES DERIVED FROM TWO QUADRATIC MATRICES

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Communicated by Abbas Salemi Parizi

ABSTRACT. The relations between the spectrum of the matrix $Q + R$ and the spectra of matrices $(\gamma+\delta)Q+(\alpha+\beta)R-QR-RQ$, $QR-RQ$, $\alpha\beta R - QRQ$, $\alpha RQR - (QR)^2$, and $\beta R - QR$ have been given assuming that the matrix $Q + R$ is diagonalizable, where Q, R are $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, of order n.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For a positive integer n, let $\mathcal{M}_n$ be the set of all $n \times n$ matrices over $\mathbb{C}$. Moreover, let $\mathbf{0}$ and I_n denote the zero matrix (of any size) and the $n \times n$ identity matrix, respectively. First we recall some definitions, concepts, and properties from the linear algebra. For $A \in \mathcal{M}_n$, the trace, rank and spectrum of A will be denoted by $tr(A)$, $rk(A)$, and $\sigma(A)$, respectively. Let $p, q \in \mathbb{C}$. A matrix $A \in \mathcal{M}_n$ is said to be $\{p, q\}$-quadratic, if $(A - pI_n)(A - qI_n) = \mathbf{0}$ (see, e.g., [1]). If $p = q$, then we simply say that A is $\{p\}$-quadratic. The matrix A is called an idempotent matrix if $\{p, q\} = \{1, 0\}$, i.e., $A^2 = A$. The matrix A is called an involutive matrix if $\{p, q\} = \{1, -1\}$, i.e., $A^2 = I_n$. Hence, quadratic matrices are a wide class of matrices

containing idempotent, involutive, and several other types of matrices.

As we shall show (see Theorem 2.1), for any $\{p, q\}$-quadratic matrix A with $p \neq q$ there exist two idempotent matrices X and Y such that $A = pX + qY$, $I_n = X + Y$, and $XY = \mathbf{0}$. Therefore, to be quadratic is closely related to the degrees of freedom of quadratic forms in Statistics. For example, to be involutive gives the restriction that the sum of degrees of freedom of different independent quadratic forms must be equal to the dimension of the primary quadratic form matrix in the framework of statistical theory.

Recently, many studies concerning quadratic matrices have been done. For example, some properties of quadratic matrices have been given in [1]. In [6], generalized quadratic matrices have been introduced and some results about them have been presented. The spectral characterizations of generalized quadratic operators have been obtained in [5]. Idempotent and involutive matrices, which are special cases of quadratic matrices, have been extensively studied and there are many results about the spectra of some special types of matrices in the literature. For example, a linear combination of two involutive matrices has been studied in [10]. Some rank identities for involutive matrices have been obtained in [13]. The spectrum of a linear combination of two projections in C^*-algebras has been considered in [3]. In [2], the spectrum of a linear combination of two orthogonal projections has been studied by means of the CS decomposition, which is closely associated with the principal angles between two subspaces. It has been shown how the spectrum of the sum of orthogonal projectors determine the convergence of many parallel iterative algorithms in [4].

The concept of spectrum is interesting not only from the algebraic point of view but also from the role it plays in applied sciences. For example, eigenvalues, which form the spectrum, are used to study differential equations and continuous dynamical systems. They provide important information in engineering design, and they arise naturally in fields such as physics and chemistry. Moreover, they are practically the most important feature of any dynamical system and hold the key to the discrete evolution of a dynamical system (see, e.g., [8]). Eigenvalues are also used in the theory of diagonalization, difference equations, Fibonacci numbers, and Markov processes (see, e.g., [12]).

In [9], Liu and Benítez discussed the spectra of some matrices depending on two idempotents. We extend those results to a pair of $\{p, q\}$-quadratic matrices with $p \neq q$. Obviously, any idempotent matrix is a

$\{p, q\}$-quadratic matrices with $p \neq q$, therefore, the results given here are nontrivial extensions of the results given in [9].

2. **Preliminary Results**

In this section, first we establish a result stating some properties of quadratic matrices.

Theorem 2.1. *Let $A \in \mathcal{M}_n$. The following statements are equivalent.*

(1) *There exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq \beta$ and $(A - \alpha I_n)(A - \beta I_n) = \mathbf{0}$.*
(2) *A is diagonalizable and $\sigma(A) \subset \{\alpha, \beta\}$.*
(3) *There exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq \beta$ and two idempotents $X, Y \in \mathcal{M}_n$ such that $A = \alpha X + \beta Y$, $X + Y = I_n$, and $XY = YX = \mathbf{0}$.*
(4) *There exist $a, b \in \mathbb{C}$ and an idempotent matrix X such that $a \neq 0$ and $A = aX + bI_n$.*

Proof. **(1) implies (2).** It is seen from the fact that a matrix is diagonalizable if and only if its minimal polynomial has simple roots (Corollary 3.3.10, [7]).

(2) implies (3). By the hypotheses, there exists a nonsingular matrix S such that
$$A = S(\alpha I_p \oplus \beta I_q)S^{-1}$$
with $p, q \in \{0, 1, \ldots, n\}$ and $p + q = n$. Define
$$X = S(I_p \oplus \mathbf{0})S^{-1} \quad \text{and} \quad Y = S(\mathbf{0} \oplus I_q)S^{-1}.$$
Observe that $A = \alpha X + \beta Y$, $X + Y = I_n$, and $XY = YX = \mathbf{0}$ as desired.

(3) implies (4). Since $A = \alpha X + \beta Y$ and $Y = I_n - X$, we can write
$$A = (\alpha - \beta)X + \beta I_n,$$
and the desired result is obtained by taking $a = \alpha - \beta$ and $b = \beta$.

(4) implies (1). Since the matrix X is idempotent, there exists a nonsingular matrix S such that $X = S(I_p \oplus \mathbf{0})S^{-1}$ with $rk(X) = p$. From this, it can be written
$$A = aS(I_p \oplus \mathbf{0})S^{-1} + bS(I_p \oplus I_{n-p})S^{-1} = S((a + b)I_p \oplus bI_{n-p})S^{-1}$$
by the hypothesis. We define $\alpha = a + b$, $\beta = b$. Thus, we have
$$A - \alpha I_n = S(\mathbf{0} \oplus (\beta - \alpha)I_{n-p})S^{-1} \text{ and } A - \beta I_n = S((\alpha - \beta)I_p \oplus \mathbf{0})S^{-1}.$$
Hence, we arrive at the equality $(A - \alpha I_n)(A - \beta I_n) = \mathbf{0}$ as desired. $\square$

It is noteworthy the following result.

Let $A \in \mathcal{M}_n$ be a $\{q\}$-quadratic matrix. By the Jordan canonical form, there is a nonsingular matrix $S \in \mathcal{M}_n$ such that

$$A = S(J_1 \oplus \cdots \oplus J_k)S^{-1},$$

where

$$J_i = \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix} \qquad \text{or} \qquad J_i = \begin{pmatrix} q \end{pmatrix}$$

for $i = 1, \ldots, k$. Evidently, if in addition, A is diagonalizable, then $A = qI_n$.

Remark: If $X \in \mathcal{M}_n$ is a $\{p, q\}$-quadratic matrix and $c \in \mathbb{C}^*$, then cX is a $\{cp, cq\}$-quadratic matrix. This simple observation permits to study the spectrum of the sum of two $\{p, q\}$-quadratic matrices instead of arbitrary linear combinations of two $\{p, q\}$-quadratic matrices.

Now, we state the following theorem about the spectrum of a matrix $Q + R$ with Q and R commuting quadratic matrices.

Theorem 2.2. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, such that $QR = RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Then*

$$\sigma(Q + R) \subset \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}.$$

Proof. By Theorem 2.1(4), there exist idempotent matrices X and Y such that $Q = (\alpha - \beta)X + \beta I_n$ and $R = (\gamma - \delta)Y + \delta I_n$. Since $QR = RQ$, then $XY = YX$, and therefore, the matrix $(\alpha - \beta)X + (\gamma - \delta)Y$ is diagonalizable and $\sigma((\alpha - \beta)X + (\gamma - \delta)Y) \subset \{0, \gamma - \delta, \alpha - \beta, \gamma - \delta + \alpha - \beta\}$ by Theorem 1 of [9]. Combining the last inclusion with the equality $Q + R = (\alpha - \beta)X + (\gamma - \delta)Y + (\delta + \beta)I_n$ proves the theorem. $\qquad \square$

Theorem 2.3. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable. If $\lambda \in \sigma(Q + R) \backslash \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then there exists $\mu \in \sigma(Q + R)$ such that $\lambda + \mu = \alpha + \beta + \gamma + \delta$.*

Proof. By Theorem 2.1, we have the equalities $Q = (\alpha - \beta)X + \beta I_n$ and $R = (\gamma - \delta)Y + \delta I_n$ with $X^2 = X$, $Y^2 = Y$. Since $QR \neq RQ$, then $XY \neq YX$. There exist a nonsingular matrix T and a diagonal matrix Λ such that

$$(2.1) \qquad\qquad\qquad T\Lambda T^{-1} = Q + R$$

since the matrix $Q + R$ is diagonalizable. So, we get

$$(2.2) \qquad (\alpha - \beta)X + (\gamma - \delta)Y = T(\Lambda - (\delta + \beta)I_n)T^{-1}$$

considering the equalities $Q + R = (\alpha - \beta)X + (\gamma - \delta)Y + (\beta + \delta)I_n$ and (2.1). This states that the matrix $(\alpha - \beta)X + (\gamma - \delta)Y$ is diagonalizable.

Now take any $\lambda \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$. Thus $\lambda - (\delta + \beta) \notin \{0, \gamma - \delta, \alpha - \beta, \gamma - \delta + \alpha - \beta\}$. Moreover, $\lambda - (\delta + \beta) \in \sigma((\alpha - \beta)X + (\gamma - \delta)Y)$ by (2.2). Since the matrices X and Y are two noncommuting idempotent matrices, there exists $\mu^* \in \sigma((\alpha - \beta)X + (\gamma - \delta)Y)$ such that $\lambda - (\delta + \beta) + \mu^* = \alpha + \gamma - \beta - \delta$ by Corollary 1 of [9]. Hence, $\mu^* + \delta + \beta \in \sigma(Q + R)$ from the equality $Q + R = (\alpha - \beta)X + (\gamma - \delta)Y + (\delta + \beta)I_n$. Thus, we get $\lambda + \mu = \alpha + \beta + \gamma + \delta$ by taking $\mu = \mu^* + \delta + \beta$. $\qquad\square$

3. Main Results

In this section, it has been considered relationships between the spectrum of the matrix $Q + R$ and the spectra of the matrices $(\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ$, $QR - RQ$, $\alpha\beta R - QRQ$, $\alpha RQR - (QR)^2$, and $\beta R - QR$ respectively, where the matrices Q and R are noncommuting quadratic matrices.

Theorem 3.1. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable.*

  (i) *If $\mu \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then $\mu(\gamma + \delta + \alpha + \beta - \mu) - \gamma\delta - \alpha\beta \in \sigma((\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ)$.*
  (ii) *If $\lambda \in \sigma((\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ)\backslash\{\alpha\gamma + \beta\delta, \gamma\beta + \alpha\delta\}$, then the roots of the polynomial $x^2 - (\gamma + \delta + \alpha + \beta)x + \lambda + \gamma\delta + \alpha\beta$ are the eigenvalues of the matrix $Q + R$.*

Proof. By Theorem 2.1, there exist idempotent matrices X and Y such that $Q = (\alpha - \beta)X + \beta I_n$ and $R = (\gamma - \delta)Y + \delta I_n$. Since $QR \neq RQ$, then $XY \neq YX$. Also, the matrix $(\alpha - \beta)X + (\gamma - \delta)Y$ is diagonalizable because the matrix $Q + R$ is diagonalizable. Thus, by Lemma 1 (i) of [9], there exists a nonsingular matrix $S \in \mathcal{M}_n$ and idempotent matrices $X_0, \ldots, X_k, Y_0, \ldots, Y_k$ such that $X_i, Y_i \in \mathcal{M}_{m_i}$ for $i = 0, 1, \ldots, k$,

$$(3.1) \quad X = S((\oplus_{i=1}^{k} X_i) \oplus X_0)S^{-1} \quad and \quad Y = S((\oplus_{i=1}^{k} Y_i) \oplus Y_0)S^{-1}$$

with $X_0 Y_0 = Y_0 X_0$ and $X_i Y_i \neq Y_i X_i$ for $i = 1, \ldots, k$. Considering the equalities $Q = (\alpha - \beta)X + \beta I_n$ and $R = (\gamma - \delta)Y + \delta I_n$ together with

the expressions in (3.1) leads to

$$Q = S((\oplus_{i=1}^{k}((\alpha - \beta)X_i + \beta I_{m_i})) \oplus ((\alpha - \beta)X_0 + \beta I_{m_0}))S^{-1}$$

and

$$R = S((\oplus_{i=1}^{k}((\gamma - \delta)Y_i + \delta I_{m_i})) \oplus ((\gamma - \delta)Y_0 + \delta I_{m_0}))S^{-1}.$$

Now, for $i = 0, 1, \ldots, k$, we define the matrices Q_i and R_i as

$$(3.2) \qquad Q_i = (\alpha - \beta)X_i + \beta I_{m_i} \quad and \quad R_i = (\gamma - \delta)Y_i + \delta I_{m_i}.$$

It is clear that all the matrices Q_i are $\{\alpha, \beta\}$-quadratic matrices and all the matrices R_i are $\{\gamma, \delta\}$-quadratic matrices. Also, we get $Q_0 R_0 = R_0 Q_0$ and $Q_i R_i \neq R_i Q_i$ for $i = 1, \ldots, k$ since $X_0 Y_0 = Y_0 X_0$ and $X_i Y_i \neq Y_i X_i$ for $i = 1, \ldots, k$. Thus, under the hypotheses of the theorem, there exists a nonsingular matrix $S \in \mathcal{M}_n$, $\{\alpha, \beta\}$-quadratic matrices $Q_0, \ldots, Q_k$, and $\{\gamma, \delta\}$-quadratic matrices $R_0, \ldots, R_k$ such that $Q_i, R_i \in \mathcal{M}_{m_i}$ for $i = 0, 1, \ldots, k$,

$$(3.3) \quad Q = S((\oplus_{i=1}^{k} Q_i) \oplus Q_0)S^{-1} \quad and \quad R = S((\oplus_{i=1}^{k} R_i) \oplus R_0)S^{-1},$$

with $Q_0 R_0 = R_0 Q_0$ and $Q_i R_i \neq R_i Q_i$ for $i = 1, \ldots, k$. There exist distinct complex numbers $\xi_1, \eta_1; \ldots; \xi_k, \eta_k$ such that

$$(3.4) \quad \xi_i + \eta_i = \alpha + \gamma - \beta - \delta, \quad \sigma((\alpha - \beta)X_i + (\gamma - \delta)Y_i) = \{\xi_i, \eta_i\},$$

$$(3.5) \qquad\qquad (\alpha - \beta)(\gamma - \delta)(X_i - Y_i)^2 = \xi_i \eta_i I_{m_i}$$

for $i = 1, \ldots, k$ by Lemma 1 (ii) of [9]. From the equalities (3.2) and the second equality of (3.4), we obtain $\sigma(Q_i + R_i) = \{\xi_i + \delta + \beta, \eta_i + \delta + \beta\}$. For $i = 1, \ldots, k$, we define μ_i and ν_i as

$$(3.6) \qquad\qquad \mu_i = \xi_i + \delta + \beta \quad and \quad \nu_i = \eta_i + \delta + \beta.$$

Thus, we have $\sigma(Q_i + R_i) = \{\mu_i, \nu_i\}$ for $i = 1, \ldots, k$. Considering the first equality of (3.4) and the equalities (3.6), we arrive at

$$(3.7) \qquad\qquad \mu_i + \nu_i = \alpha + \beta + \gamma + \delta.$$

Using the expressions (3.2), (3.6), and (3.7) in the equality (3.5) we have

$$(3.8) \quad (\gamma + \delta)Q_i + (\alpha + \beta)R_i - Q_i R_i - R_i Q_i = (\mu_i \nu_i - \gamma\delta - \alpha\beta)I_{m_i}$$

for $i = 1, \ldots, k$. From the equalities (3.3), it can be shown that $(\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ = S((\oplus_{i=1}^{k}(\gamma + \delta)Q_i + (\alpha + \beta)R_i - Q_i R_i - R_i Q_i) \oplus ((\gamma + \delta)Q_0 + (\alpha + \beta)R_0 - 2Q_0 R_0))S^{-1}$.
Observe that $(\gamma + \delta)Q_0 + (\alpha + \beta)R_0 - 2Q_0 R_0 = (\gamma - \delta)(\alpha - \beta)(X_0 + Y_0 - 2X_0 Y_0) + (\gamma\beta + \alpha\delta)I_{m_0}$ and the matrix $X_0 + Y_0 - 2X_0 Y_0$ is idempotent. Moreover, $(\gamma - \delta)(\alpha - \beta) \neq 0$. Thus, the matrix $(\gamma + \delta)Q_0 + (\alpha + \beta)R_0 -$

$2Q_0R_0$ is a $\{\alpha\gamma + \delta\beta, \gamma\beta + \alpha\delta\}$-quadratic matrix.

Now, take any $\mu \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$. Then $\mu \notin \sigma(Q_0 + R_0)$ by Theorem 2.2. Hence, there exists $i \in \{1, \ldots, k\}$ such that $\mu \in \sigma(Q_i + R_i)$, and therefore $\sigma(Q_i + R_i) = \{\mu, \alpha + \beta + \gamma + \delta - \mu\}$ on account of Theorem 2.3. Thus, it can be written $\mu(\alpha + \beta + \gamma + \delta - \mu) - \gamma\delta - \alpha\beta \in \sigma((\gamma + \delta)Q_i + (\alpha + \beta)R_i - Q_iR_i - R_iQ_i)$ from (3.8). It is obtained the desired result in (i) by considering $\sigma((\gamma + \delta)Q_i + (\alpha + \beta)R_i - Q_iR_i - R_iQ_i) \subset \sigma((\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ)$.

Next, take any $\lambda \in \sigma((\gamma + \delta)Q + (\alpha + \beta)R - QR - RQ)\backslash\{\alpha\gamma + \beta\delta, \gamma\beta + \alpha\delta\}$. Since $\lambda \notin \{\alpha\gamma + \beta\delta, \gamma\beta + \alpha\delta\}$ and the matrix $(\gamma + \delta)Q_0 + (\alpha + \beta)R_0 - 2Q_0R_0$ is $\{\alpha\gamma + \beta\delta, \gamma\beta + \alpha\delta\}$-quadratic, then $\lambda \notin \sigma((\gamma + \delta)Q_0 + (\alpha + \beta)R_0 - 2Q_0R_0)$. Thus, there exists $i \in \{1, \ldots, k\}$ such that $\lambda = \mu_i\nu_i - \gamma\delta - \alpha\beta$ from the equality (3.8). By the equality (3.7), we have $\mu + \nu = \alpha + \beta + \gamma + \delta$ with $\mu = \mu_i$ and $\nu = \nu_i$. Considering the equality $\mu\nu = \lambda + \gamma\delta + \alpha\beta$ together with the last expression completes the proof of (ii). $\square$

If we consider Theorem 3.1 together with the Remark preceding Theorem 2.2, we get the following corollary.

Corollary 3.2. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let $c, d, c', d' \in \mathbb{C}^*$ such that the matrices $cR + dQ$ and $c'R + d'Q$ are diagonalizable. If $\mu \in \sigma(cR + dQ)\backslash\{c\delta + d\beta, c\gamma + d\beta, c\delta + d\alpha, c\gamma + d\alpha\}$, then the roots of the polynomial $x^2 - (c'(\gamma + \delta) + d'(\alpha + \beta))x + \frac{1}{cd}(\mu(c(\gamma + \delta) + d(\alpha + \beta) - \mu) - c^2\gamma\delta - d^2\alpha\beta)c'd' + c'^2\gamma\delta + d'^2\alpha\beta$ are the eigenvalues of the matrix $c'R + d'Q$.*

Theorem 3.3. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable.*

   (i) *If $\lambda \in \sigma(QR - RQ)\backslash\{0\}$, then there exist $\mu, \nu \in \sigma(R + Q)$ such that $\mu + \nu = \gamma + \delta + \alpha + \beta$ and $\lambda^2 = (\mu\nu - (\gamma + \alpha)(\beta + \delta))(\mu\nu - (\beta + \gamma)(\delta + \alpha))$.*

   (ii) *If $\mu \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then there exists $\lambda \in \sigma(QR - RQ)$ such that $\lambda^2 = (\mu(\gamma + \delta + \alpha + \beta - \mu) - (\gamma + \alpha)(\beta + \delta))(\mu(\gamma + \delta + \alpha + \beta - \mu) - (\beta + \gamma)(\delta + \alpha))$.*

Proof. Let Q and R be as in (3.3). So, we have the equality $QR - RQ = S((\oplus_{i=1}^k (Q_iR_i - R_iQ_i)) \oplus \mathbf{0})S^{-1}$. By Theorem 2.1, there exist

two idempotent matrices X and Y such that $Q = (\alpha - \beta)X + \beta I_n$ and $R = (\gamma - \delta)Y + \delta I_n$. Since $QR \neq RQ$, then $XY \neq YX$. Also, the matrix $(\alpha - \beta)X + (\gamma - \delta)Y$ is diagonalizable because $Q + R$ is diagonalizable. Thus, by in [9, Lemma 1,(iii)], for $i = 1, \ldots, k$, there exist nonsingular matrices S_i such that $X_i = S_i \begin{pmatrix} I_{x_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S_i^{-1}$, $Y_i = S_i \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} S_i^{-1}$ where $x_i = rk(X_i)$, $A_i \in \mathcal{M}_{m_i}$, $D_i \in \mathcal{M}_{m_i - x_i}$, and $A_i = (1 - \rho_i^*)I_{x_i}$, $B_i C_i = \rho_i^*(1 - \rho_i^*)I_{x_i}$, $C_i B_i = \rho_i^*(1 - \rho_i^*)I_{m_i - x_i}$, and $D_i = \rho_i^* I_{m_i - x_i}$ with $\rho_i^* = \frac{\xi_i \eta_i}{(\alpha - \beta)(\gamma - \delta)}$ and ξ_i, η_i are distinct complex numbers satisfying the equalities (3.4). Since $X_i = S_i \begin{pmatrix} I_{x_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S_i^{-1}$ and $Q_i = (\alpha - \beta)X_i + \beta I_n$ for $i = 1, \ldots, k$, we get

$$(3.9) \qquad Q_i = S_i \begin{pmatrix} \alpha I_{x_i} & \mathbf{0} \\ \mathbf{0} & \beta I_{m_i - x_i} \end{pmatrix} S_i^{-1}$$

Likewise, since $Y_i = S_i \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} S_i^{-1}$ and $R_i = (\gamma - \delta)Y_i + \delta I_n$ for $i = 1, \ldots, k$, we obtain

$$(3.10) \qquad R_i = S_i \begin{pmatrix} K_i & L_i \\ M_i & N_i \end{pmatrix} S_i^{-1}$$

with

$$(3.11) \qquad K_i = (\gamma - \delta)A_i + \beta I_{x_i} = ((1 - \rho_i^*)(\gamma - \delta) + \beta)I_{x_i},$$

$$(3.12) \qquad L_i M_i = (\gamma - \delta)^2 B_i C_i = (\gamma - \delta)^2 \rho_i^*(1 - \rho_i^*)I_{x_i},$$

$$(3.13) \qquad M_i L_i = (\gamma - \delta)^2 C_i B_i = (\gamma - \delta)^2 \rho_i^*(1 - \rho_i^*)I_{m_i - x_i},$$

and

$$(3.14) \qquad N_i = (\gamma - \delta)D_i + \delta I_{m_i - x_i} = ((\gamma - \delta)\rho_i^* + \delta)I_{m_i - x_i}.$$

By considering the equalities (3.6) and (3.7), it follows that

$$(3.15) \quad \rho_i^* = \frac{1}{(\alpha - \beta)(\gamma - \delta)}\xi_i \eta_i = \frac{1}{(\alpha - \beta)(\gamma - \delta)}(\mu_i \nu_i - (\gamma + \alpha)(\beta + \delta)).$$

Hence, we get

$$(3.16) \qquad (1 - \rho_i^*)(\gamma - \delta) + \delta = -\frac{\mu_i \nu_i - \alpha(\gamma + \beta) - \delta(\alpha + \gamma)}{\alpha - \beta}$$

Now, for $i = 1, \ldots, k$ we define ρ_i as

$$(3.17) \qquad \rho_i = \frac{\mu_i \nu_i - \alpha(\gamma + \beta) - \delta(\alpha + \gamma)}{\alpha - \beta}.$$

Thus, we obtain $(1 - \rho_i^*)(\gamma - \delta) + \delta = -\rho_i$, that is, $\gamma + \rho_i = \frac{\xi_i \eta_i}{\alpha - \beta}$, and therefore $\xi_i \eta_i = (\alpha - \beta)(\gamma + \rho_i)$. It is clear that $\rho_i^*(1 - \rho_i^*) = -\frac{(\gamma + \rho_i)(\delta + \rho_i)}{(\gamma - \delta)^2}$ in view of $\rho_i^* = \frac{\xi_i \eta_i}{(\alpha - \beta)(\gamma - \delta)}$ and $\xi_i \eta_i = (\alpha - \beta)(\gamma + \rho_i)$, for $i = 1, \ldots, k$. Hence, it follows that

$$(3.18)$$
$$L_i M_i = -(\gamma + \rho_i)(\delta + \rho_i) I_{x_i} \quad and \quad M_i L_i = -(\gamma + \rho_i)(\delta + \rho_i) I_{m_i - x_i}$$

by (3.12) and (3.13). From (3.9), (3.10), and (3.18), we get

$$(3.19) \qquad (Q_i R_i - R_i Q_i)^2 = (\alpha - \beta)^2 (\gamma + \rho_i)(\delta + \rho_i) I_{m_i}$$

for $i = 1, \ldots, k$.

Now, take any $\lambda \in \sigma(QR - RQ) \backslash \{0\}$. From this, we have $\lambda^2 \in \sigma[(QR - RQ)^2] \backslash \{0\}$. So, there exists $i \in \{1, \ldots, k\}$ such that $\lambda^2 \in \sigma[(Q_i R_i - R_i Q_i)^2]$. Considering the equalities (3.7), (3.19) and taking $\mu_i = \mu$ and $\nu_i = \nu$, we obtain the desired result in (i).

Next, take any $\mu \in \sigma(Q + R) \backslash \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$. Then there exists $i \in \{1, \ldots, k\}$ such that $\mu \in \sigma(R_i + Q_i)$. Thus it follows that $(\alpha - \beta)^2 (\gamma + \rho)(\delta + \rho) \in \sigma[(Q_i R_i - R_i Q_i)^2] \subset \sigma[(QR - RQ)^2]$ from (3.7) and (3.19) with $\rho = \frac{\mu(\alpha + \beta + \gamma + \delta - \mu) - \alpha(\gamma + \beta) - \delta(\alpha + \gamma)}{\alpha - \beta}$. Hence by the spectral mapping theorem (see, e.g., [11], Theorem 9.33), there exists $\lambda \in \sigma(QR - RQ)$ such that $\lambda^2 = (\alpha - \beta)^2 (\gamma + \rho)(\delta + \rho)$. So, the proof of (ii) is complete. $\qquad \square$

Theorem 3.4. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable.*

(i) *If $\lambda \in \sigma(\alpha\beta R - QRQ) \backslash \{\alpha\gamma(\beta - \alpha), \alpha\delta(\beta - \alpha), \delta\beta(\alpha - \beta), \gamma\beta(\alpha - \beta)\}$, then the roots of the polynomial $x^2 - (\alpha + \beta + \gamma + \delta)x + \alpha(\beta + \delta) + \gamma(\alpha + \delta) + \frac{1}{\alpha}\lambda$ or $x^2 - (\alpha + \beta + \gamma + \delta)x + \beta(\alpha + \delta) + \gamma(\beta + \delta) + \frac{1}{\beta}\lambda$ are the eigenvalues of the matrix $Q + R$.*

(ii) *If $\mu \in \sigma(Q + R) \backslash \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then there exist $\lambda_1, \lambda_2 \in \sigma(\alpha\beta R - QRQ)$ such that $\lambda_1 = \alpha(\mu(\gamma + \delta + \alpha + \beta - \mu) - \alpha(\delta + \beta) - \gamma(\alpha + \delta))$ and $\lambda_2 = \beta(\mu(\gamma + \delta + \alpha + \beta - \mu) - \beta(\alpha + \delta) - \gamma(\beta + \delta))$.*

Proof. Let the matrices Q and R be as in (3.3). Then we can write

$$\alpha\beta R - QRQ = S((\oplus_{i=1}^{k}(\alpha\beta R_i - Q_i R_i Q_i)) \oplus (\alpha\beta R_0 - Q_0 R_0 Q_0))S^{-1}.$$

Since $Q_0 = (\alpha-\beta)X_0 + \beta I_{m_0}$ and $R_0 = (\gamma-\delta)Y_0 + \delta I_{m_0}$ (with $X_0^2 = X_0$, $Y_0^2 = Y_0$, and $X_0 Y_0 = Y_0 X_0$), we get

$$\alpha\beta R_0 - Q_0 R_0 Q_0 \quad =$$
$$(\beta^2 - \alpha^2)((\gamma - \delta)X_0 Y_0 + \delta X_0)$$
$$+(\alpha\beta - \beta^2)((\gamma - \delta)Y_0 + \delta I_{m_0}).$$

Also, the matrices X_0 and Y_0 are simultaneously diagonalizable because $X_0 Y_0 = Y_0 X_0$. Thus, we obtain $\sigma(\alpha\beta R_0 - Q_0 R_0 Q_0) \subset \{\alpha\gamma(\beta-\alpha), \alpha\delta(\beta-\alpha), \delta\beta(\alpha-\beta), \gamma\beta(\alpha-\beta)\}$ from (3.20).

Now, take any $\lambda \in \sigma(\alpha\beta R - QRQ)\backslash\{\alpha\gamma(\beta - \alpha), \alpha\delta(\beta - \alpha), \delta\beta(\alpha - \beta), \gamma\beta(\alpha-\beta)\}$. So, we have $\lambda \notin \sigma(\alpha\beta R_0 - Q_0 R_0 Q_0)$. Thus, there exists $i \in \{1, \ldots, k\}$ such that $\lambda \in \sigma(\alpha\beta R_i - Q_i R_i Q_i)$. On the other hand, the equalities (3.9) and (3.10) lead to the equality

$$(3.20) \qquad \alpha\beta R_i - Q_i R_i Q_i = S_i \begin{pmatrix} (\alpha\beta - \alpha^2)K_i & \mathbf{0} \\ \mathbf{0} & (\alpha\beta - \beta^2)N_i \end{pmatrix} S_i^{-1}$$

for $i = 1, \ldots, k$. From (3.11) and (3.16) we have the relation

$$(3.21) \qquad\qquad\qquad\qquad K_i = -\rho_i I_{x_i}$$

with $\rho_i = \frac{\mu_i\nu_i - \alpha(\delta+\beta) - \gamma(\alpha+\delta)}{\alpha-\beta}$ for $i = 1, \ldots, k$. Similarly, from (3.14) and (3.15), we get

$$(3.22) \qquad\qquad N_i = \frac{\mu_i\nu_i - \gamma(\beta + \delta) - \beta(\alpha + \delta)}{\alpha - \beta}I_{m_i-x_i}.$$

Hence by combining (3.20) with (3.21) and (3.22) it is seen that $\lambda = \alpha(\mu_i\nu_i - \gamma(\alpha+\delta) - \alpha(\delta+\beta))$ or $\lambda = \beta(\mu_i\nu_i - \beta(\delta+\alpha) - \gamma(\beta+\delta))$. From this, we have $\mu_i\nu_i = \frac{1}{\alpha}\lambda + \alpha(\delta+\beta) + \gamma(\alpha+\delta)$ or $\mu_i\nu_i = \frac{1}{\beta}\lambda + \beta(\alpha+\delta) + \gamma(\beta+\delta)$. The equality (3.7) assures that the roots of the polynomial $x^2 - (\alpha + \beta + \gamma + \delta)x + \alpha(\delta + \beta) + \gamma(\alpha + \delta) + \frac{1}{\alpha}\lambda$ or $x^2 - (\alpha + \beta + \gamma + \delta)x + \beta(\delta + \alpha) + \gamma(\beta + \delta) + \frac{1}{\beta}\lambda$ are the eigenvalues of the matrix $Q + R$. So the proof of (i) is complete.

Next, take any $\mu \in \sigma(Q+R)\backslash\{\delta+\beta, \gamma+\beta, \delta+\alpha, \gamma+\alpha\}$. So there exists $i \in \{1, \ldots, k\}$ such that $\mu \in \sigma(Q_i + R_i)$. Since $\sigma(Q_i + R_i) = \{\mu_i, \nu_i\}$ and $\mu_i + \nu_i = \alpha + \beta + \gamma + \delta$, we can write $\sigma(Q_i + R_i) = \{\mu, \alpha + \beta + \gamma + \delta - \mu\}$. Hence by the equalities (3.20)-(3.22),
$\alpha(\mu(\alpha+\beta+\gamma+\delta-\mu)-\alpha(\beta+\delta)-\gamma(\alpha+\delta)), \quad \beta(\mu(\alpha+\beta+\gamma+\delta-\mu)-\beta(\alpha+$

$\delta) - \gamma(\beta + \delta)) \in \sigma(\alpha\beta R_i - Q_i R_i Q_i)$. The inclusion $\sigma(\alpha\beta R_i - Q_i R_i Q_i) \subset \sigma(\alpha\beta R - QRQ)$ gives (ii). $\square$

Theorem 3.5. *Let $Q \in \mathcal{M}_n$ and $R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic matrix and $\{\gamma, \delta\}$-quadratic matrix, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable.*

(i) *If $\lambda \in \sigma(\alpha RQR - (QR)^2) \backslash \{0, \beta\gamma^2(\alpha - \beta), \beta\delta^2(\alpha - \beta)\}$, then the roots of the polynomial*
$x^2 - (\alpha + \beta + \gamma + \delta)x + \frac{1}{2}((\alpha + 2\beta)(\gamma + \delta) \pm$
$\sqrt{\alpha^2(\gamma - \delta)^2 + 4(\alpha\beta\gamma\delta - \lambda)}) + \gamma\delta + \alpha\beta$
are eigenvalues of the matrix $Q + R$.

(ii) *If $\mu \in \sigma(Q + R) \backslash \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then*
$-(\mu(\gamma + \delta + \alpha + \beta - \mu) - \gamma\delta - \alpha\beta)^2 - \frac{1}{\alpha - \beta}((\gamma + \delta)(2\beta^2 - \alpha^2 - \alpha\beta)(\mu(\gamma + \delta + \alpha + \beta - \mu) - \gamma\delta - \alpha\beta) + \alpha(\gamma\beta + \alpha\delta)(\delta\beta + \alpha\gamma) - \beta^3(\gamma + \delta)^2) \in \sigma(\alpha RQR - (QR)^2)$.

Proof. By the expressions (3.3), we get

$\alpha RQR - (QR)^2 =$
$$S((\oplus_{i=1}^k (\alpha R_i Q_i R_i - (Q_i R_i)^2)) \oplus (\alpha R_0 Q_0 R_0 - (Q_0 R_0)^2)) S^{-1}.$$

By the equalities $Q_0 = (\alpha - \beta) X_0 + \beta I_{m_0}$, $R_0 = (\gamma - \delta) Y_0 + \delta I_{m_0}$, and $X_0 Y_0 = Y_0 X_0$, we obtain
(3.23)
$\alpha R_0 Q_0 R_0 - (Q_0 R_0)^2 = \beta(\beta - \alpha)(\gamma^2 - \delta^2)(X_0 Y_0 - Y_0) + \beta\delta^2(\beta - \alpha)(X_0 - I_{m_0})$.

Since the matrices X_0 and Y_0 are simultaneously diagonalizable, it follows from (3.23) that $\sigma(\alpha R_0 Q_0 R_0 - (Q_0 R_0)^2) \subset \{0, \beta\gamma^2(\alpha - \beta), \beta\delta^2(\alpha - \beta)\}$.

Now, take any $\lambda \in \sigma(\alpha RQR - (QR)^2) \backslash \{0, \beta\gamma^2(\alpha - \beta), \beta\delta^2(\alpha - \beta)\}$. So, $\lambda \notin \sigma(\alpha R_0 Q_0 R_0 - (Q_0 R_0)^2)$. Thus there exists $i \in \{1, \ldots, k\}$ such that $\lambda \in \sigma(\alpha R_i Q_i R_i - (Q_i R_i)^2)$. Since, for $i = 1, \ldots, k$,

$\alpha R_i Q_i R_i - (Q_i R_i)^2 =$
$S_i \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ (\alpha^2 - \alpha\beta)M_i K_i + (\alpha\beta - \beta^2)N_i M_i & (\alpha^2 - \alpha\beta)M_i L_i + (\alpha\beta - \beta^2)N_i^2 \end{pmatrix} S_i^{-1}$

by (3.9) and (3.10) and

$$(3.24) \qquad (\alpha^2 - \alpha\beta)M_i L_i + (\alpha\beta - \beta^2)N_i^2 = \{-(\mu_i\nu_i - \gamma\delta - \alpha\beta)^2$$

$$- \frac{1}{\alpha - \beta}((2\beta^2 - \alpha^2 - \alpha\beta)(\gamma + \delta)(\mu_i\nu_i - \gamma\delta - \alpha\beta)$$

$$+ \alpha(\gamma\beta + \alpha\delta)(\delta\beta + \alpha\gamma) - \beta^3(\gamma + \delta)^2)\}I_{m_i - x_i},$$

by the equalities (3.14), (3.15), (3.17), and (3.18), then it can be written
$\lambda = -(\mu_i\nu_i - \gamma\delta - \alpha\beta)^2 - \frac{1}{\alpha-\beta}((\gamma+\delta)(2\beta^2 - \alpha^2 - \alpha\beta)(\mu_i\nu_i - \gamma\delta - \alpha\beta) + \alpha(\gamma\beta + \alpha\delta)(\delta\beta + \alpha\gamma) - \beta^3(\gamma + \delta)^2)$. From this,
$\mu_i\nu_i = \frac{1}{2}((\alpha+2\beta)(\gamma+\delta)\pm\sqrt{\alpha^2(\gamma^2 + \delta^2 - 2\gamma\delta) + 4(\alpha\beta\gamma\delta - \lambda)})+\gamma\delta+\alpha\beta$.
Taking into account the equality (3.7) it is seen that assertion in (i) is
true.

Next, take any $\mu \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$. So, there
exists $i \in \{1, \ldots, k\}$ such that $\mu \in \sigma(Q_i + R_i)$. By Theorem 2.3, there
exists $\lambda \in \sigma(Q_i + R_i)$ such that $\lambda + \mu = \alpha + \beta + \gamma + \delta$. Hence, we have
$\sigma(Q_i + R_i) = \{\mu, \alpha + \beta + \gamma + \delta - \mu\}$.
Thus, from (3.24), the desired result in (ii) is obtained.   □

Theorem 3.6. *Let $Q, R \in \mathcal{M}_n$ be an $\{\alpha, \beta\}$-quadratic and $\{\gamma, \delta\}$-quadratic matrices, respectively, with $QR \neq RQ$, $\alpha \neq \beta$, and $\gamma \neq \delta$. Let the matrix $Q + R$ be diagonalizable.*

   (i) *If $\lambda \in \sigma(\beta R - QR)\backslash\{0, (\beta - \alpha)\gamma, (\beta - \alpha)\delta\}$,*
       then the roots of the polynomial
       $x^2 - (\alpha + \beta + \gamma + \delta)x + \lambda + \alpha(\beta + \delta) + \gamma(\alpha + \delta)$ are eigenvalues
       of the matrix $R + Q$.
   (ii) *If $\mu \in \sigma(Q + R)\backslash\{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$, then*
       $(\mu(\alpha + \beta + \gamma + \delta - \mu) - \alpha(\delta + \beta) - \gamma(\alpha + \delta) \in \sigma(\beta R - QR)$.

Proof. Let Q and R be as in (3.3). Then we have

$$\beta R - QR = S((\oplus_{i=1}^k (\beta R_i - Q_i R_i)) \oplus (\beta R_0 - Q_0 R_0))S^{-1}.$$

By the representations (3.9), (3.10), and the relation (3.21) we get

$$(3.25) \qquad \beta R_i - Q_i R_i = S_i \begin{pmatrix} (\alpha - \beta)\rho_i I_{x_i} & (\beta - \alpha)L_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S_i^{-1}$$

for all $i = 1, \ldots, k$ with $\rho_i = \frac{\mu_i\nu_i - \alpha(\delta+\beta) - \gamma(\alpha+\delta)}{\alpha - \beta}$. On the other hand,
$\sigma(\beta R_0 - Q_0 R_0) \subset \{0, (\beta - \alpha)\gamma, (\beta - \alpha)\delta\}$ since $\beta R_0 - Q_0 R_0 = (\beta - \alpha)(\gamma - \delta)X_0 Y_0 + (\beta - \alpha)\delta X_0$ and the matrices X_0 and Y_0 are commuting
idempotents.

Now, take any $\lambda \in \sigma(\beta R - QR) \setminus \{0, (\beta - \alpha)\gamma, (\beta - \alpha)\delta\}$. There exists $i \in \{1, \ldots, k\}$ such that $\lambda = (\alpha - \beta)\rho_i$ by the equality (3.25). Also, by (3.7) and (3.17) there exist $\mu, \nu \in \sigma(R + Q)$ such that $\rho_i = \frac{\mu\nu - \alpha(\beta+\delta) - \gamma(\alpha+\delta)}{\alpha - \beta}$ and $\mu + \nu = \alpha + \beta + \gamma + \delta$. So we have the equalities $\mu + \nu = \alpha + \beta + \gamma + \delta$ and $\mu\nu = \lambda + \alpha(\beta + \delta) + \gamma(\alpha + \delta)$. Thus, μ and ν are the roots of the polynomial mentioned in (i).

Next, take any $\mu \in \sigma(Q + R) \setminus \{\delta + \beta, \gamma + \beta, \delta + \alpha, \gamma + \alpha\}$. So, there exists $i \in \{1, \ldots, k\}$ such that $\mu, \alpha + \beta + \gamma + \delta - \mu \in \sigma(Q_i + R_i)$ from Theorem 2.3. Hence, in view of (3.25), it follows that $\mu(\alpha + \beta + \gamma + \delta - \mu) - \alpha(\delta + \beta) - \gamma(\alpha + \delta) \in \sigma(\beta R_i - Q_i R_i) \subset \sigma(\beta R - QR)$. $\qquad\square$

Observe that we obtain the same relations in Theorems 3, 4, 6, 5, respectively, of [9] when in Theorems 3.1, 3.3, 3.4, 3.6, respectively, we substitute $aP = R'$, $bQ = Q'$ and apply our results for $R' + Q'$.

Necessary conditions has been given in order that the matrix $c_1 P_1 + c_2 P_2$ be involutive in Theorem 2.5 (b) of [10] with $c_1, c_2 \in \mathbb{C}^*$ and P_1, P_2 two noncommuting idempotent matrices, in Theorem 2.4 (b) of [10] with $c_1, c_2 \in \mathbb{C}^*$ and P_1, P_2 two noncommuting involutive matrices. Take $c_1 P_1 = R$ and $c_2 P_2 = Q$ in Theorem 2.4 (b) of [10]. Then R is $\{c_1, -c_1\}$-quadratic and Q is $\{c_2, -c_2\}$-quadratic since P_1, P_2 are involutive matrices. Moreover, $c_1 \neq -c_1$ and $c_2 \neq -c_2$ since $c_1, c_2 \in \mathbb{C}^*$. Thus, following the proof of Theorem 3.1, it is seen that the converse of the result in Theorem 2.4 (b) of [10] is also true. Similarly, take again $c_1 P_1 = R$ and $c_2 P_2 = Q$ in Theorem 2.5 of [10]. Then R is $\{c_1, 0\}$-quadratic and Q is $\{c_2, 0\}$-quadratic since P_1, P_2 are idempotents. Furthermore, $c_1 \neq 0$ and $c_2 \neq 0$ because $c_1, c_2 \in \mathbb{C}^*$. Thus, following the proof of Theorem 3.1, it is seen that the converse of Theorem 2.5 (b) of [10] is also true.

Acknowledgments

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