

## ON SEMI-ARTINIAN WEAKLY CO-SEMISIMPLE MODULES

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**ABSTRACT.** We show that every semi-artinian module which is contained in a direct sum of finitely presented modules in  $\sigma[M]$ , is weakly co-semisimple if and only if it is regular in  $\sigma[M]$ . As a consequence, we observe that every semi-artinian ring is regular in the sense of von Neumann if and only if its simple modules are *FP*-injective.

### 1. Introduction

Throughout,  $R$  is always an associative ring with identity,  $M$  is a unitary  $R$ -module, and by  $\sigma[M]$ , we mean the category of  $M$ -subgenerated modules or *the Wisbauer category*. Here, we follow a recent suggestion made by Patrik. F. Smith. The construction of  $\sigma[M]$  is quite simple: for any module  $M$ , take direct sums  $M^{(\Lambda)}$ , for any index set  $\Lambda$ , factor modules of these ( $M$ -generated modules), and then submodules ( $M$ -subgenerated modules). Hence, the Wisbauer category is the smallest Grothendieck category subgenerated by  $M$ . The reader is referred to [24], for a systematic study of module theory in term of  $\sigma[M]$ . The category of all right  $R$ -modules will be denoted by  $R\text{-Mod}$ . A module

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$U$  in  $\sigma[M]$  is called *weakly  $M$ -injective*, if every diagram (in  $\sigma[M]$ ),

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M^{(\mathbb{N})} \\ & & \downarrow & & \\ & & U & & \end{array}$$

with exact row and  $K$  finitely generated, can be extended commutatively by a morphism  $M^{(\mathbb{N})} \rightarrow U$ , i.e.,  $\text{Hom}(-, U)$  is exact with respect to the given row. If  $M = R$ , then weakly  $R$ -injective modules are also called *FP-injective*, where, “FP” abbreviates “finitely presented”. A module  $P$  in  $\sigma[M]$  is called *finitely presented* in  $\sigma[M]$  if (i)  $P$  is finitely generated and (ii) in every short exact sequence,  $0 \rightarrow K \rightarrow L \rightarrow P \rightarrow 0$ , in  $\sigma[M]$ , with  $L$  finitely generated,  $K$  is also finitely generated. A finitely generated module which is projective in  $\sigma[M]$  is finitely presented in  $\sigma[M]$ . A short exact sequence  $(*) 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $\sigma[M]$  is called *pure* in  $\sigma[M]$ , if every finitely presented module  $P$  in  $\sigma[M]$  is projective with respect to this sequence, i.e., if every diagram

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \end{array}$$

can be extended commutatively by a morphism  $P \rightarrow B$ . Equivalently, we may demand the sequence

$$0 \rightarrow \text{Hom}(P, A) \rightarrow \text{Hom}(P, B) \rightarrow \text{Hom}(P, C) \rightarrow 0$$

to be exact. In this case,  $\text{Im} f = f(A)$  is called a *pure submodule* of  $B$ . A module  $A$  in  $\sigma[M]$  is called *absolutely pure*, if every exact sequence of the type  $(*)$  is pure in  $\sigma[M]$ . A module  $B$  in  $\sigma[M]$  is called *regular* in  $\sigma[M]$ , if every short exact sequence of the type  $(*)$  is pure in  $\sigma[M]$ . A module is called co-semisimple (or  $V$ -module), if every simple module is  $M$ -injective. A ring is called a right (left)  $V$ -ring, if it is co-semisimple as a right (left)  $R$ -module.

The *socle series* of a module is defined inductively. The second socle is the submodule  $\text{Soc}_2(M) \supseteq \text{Soc}(M)$  such that  $\text{Soc}(M/\text{Soc}(M)) = \text{Soc}_2(M)/\text{Soc}(M)$ . By transfinite induction, one may define  $\text{Soc}_\beta(M)$  for a limit ordinal as the union  $\bigcup_{\alpha < \beta} \text{Soc}_\alpha(M)$ , and  $\text{Soc}_{\alpha+1}(M)/\text{Soc}_\alpha(M) = \text{Soc}(M/\text{Soc}_\alpha(M))$ , for every non-limit ordinal  $\alpha$ . The least ordinal  $\alpha$  such that  $\text{Soc}_{\alpha+1}(M) = \text{Soc}_\alpha(M)$  is called the *socle length* of  $M$ . If  $M$  has a socle length  $\alpha$ , and if  $M = \text{Soc}_\alpha(M)$ , then  $M$  is said to be a *Loewy module of Loewy length  $\alpha$* . A module  $M$  is *semi-artinian* if for every submodule  $N \neq M$ , we have  $\text{Soc}(M/N) \neq 0$ . A ring  $R$  is right

semi-artinian, if  $R$  is a semi-artinian right  $R$ -module. A module  $M$  is semi-artinian if and only if  $M$  is a Loewy module.

A well-known theorem of Kaplansky states that a commutative ring is regular in the sense of von Neumann if and only if it is a  $V$ -ring. However, in a non-commutative setting,  $V$ -rings are very far from being regular. Cozzen's examples [10] provide simple noetherian  $V$ -rings that are not artinian. Consequently, a natural question arises: when does the class of  $V$ -rings and the class of regular rings coincide? Hence, several authors have worked on the question at different times. In [21], Sarath and Varadarajan have shown that if maximal right ideals of the ring  $R$  are two sided, then  $R$  is regular if and only if the  $R$  is a right  $V$ -ring. In [2], Armendariz and Fisher have shown that over a  $PI$ -ring (a ring with polynomial identity), these two classes of rings are the same. In [5], Baccella has proved that if  $R$  is a ring, whose right primitive factor rings are artinian, then  $R$  is a right  $V$ -ring if and only if  $R$  is regular (the if part was well-known). To see when a regular right self-injective ring is a  $V$ -ring, deep considerations have been made by Tyukavkin [23] and Herbera [14]. Semi-artinian rings, semi-artinian  $V$ -rings, and semi-artinian regular rings have also been extensively studied by algebraists (see for example [1, 3, 4, 6, 7, 8, 11, 17, 18, 19, 20]). For example, Trlifaj [22] has shown that for a regular ring  $R$ , the Cantor-Bendixon dimension of  $Zg_R$  (*the Ziegler spectrum*) is defined if and only if  $R$  is semi-artinian. Moreover, he proved that for a semi-artinian regular ring  $R$ , the Cantor-Bendixon dimension of  $Zg_R$  is equal to the Loewy length of  $R$ . Năstăsescu [17] and Baccella [3] have observed that semi-artinian  $V$ -rings are regular. In [24, 37.10 and 37.11], it has been observed that the celebrated theorem of Kaplansky is true for modules over commutative rings, that is, *if  $R$  is a commutative ring and  $M$  is an  $R$ -module which is a direct sum of finitely presented modules in  $\sigma[M]$ , then  $M$  is regular in  $\sigma[M]$  if and only if it is co-semisimple*. Dung and Smith [12] considered semi-artinian  $V$ -modules (in our terminology co-semisimple modules) and proved the non-trivial equivalence. Here, we prove that every semi-artinian weakly co-semisimple module  $M$  is regular in  $\sigma[M]$ . This generalizes the results of Alin and Armendariz [1], Năstăsescu [17] and Baccella [3], and shows that some of the modules considered by Dung and Smith [12] are indeed regular modules.

## 2. Semi-Aartinian Weakly Co-Semisimple Modules

We begin with a definition that generalizes the concept of co-semisimple module.

**Definition 2.1.** A module  $M$  is called weakly co-semisimple in  $\sigma[M]$ , if every simple module (in  $\sigma[M]$  or  $R\text{-Mod}$ ) is weakly  $M$ -injective. A ring is called a right (left)  $FP$ - $V$ -ring, if its simple right (left)  $R$ -modules are  $FP$ -injective.

In the sequel, we use the following three results from [24]. The reader is reminded that the second part of the following lemma is true for every (not necessarily countable) chain.

**Lemma 2.2.** ([24], 16.10) *For every  $R$ -module  $M$ , we have*

- (a) *the direct sum of any family of weakly  $M$ -injective  $R$ -modules  $\{U_\lambda\}_\Lambda$  is weakly  $M$ -injective.*
- (b) *If  $U_1 \subseteq U_2 \subseteq \cdots$  is an ascending chain of weakly  $M$ -injective submodules of a module  $N$ , then  $\bigcup_{\mathbb{N}} U_i$  is also weakly  $M$ -injective.*

**Lemma 2.3.** ([24], 35.2) *If  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  is an exact sequence in  $\sigma[M]$ , with  $N'$  and  $N''$  absolutely pure, then  $N$  is also absolutely pure.*

**Lemma 2.4.** ([24], 35.4) *If the  $R$ -module  $M$  is a submodule of a direct sum of finitely presented modules in  $\sigma[M]$ , then for  $K \in \sigma[M]$ , the followings are equivalent:*

- (a)  *$K$  is weakly  $M$ -injective.*
- (b)  *$K$  is absolutely pure in  $\sigma[M]$ .*

Based on these three results, we prove our first lemma.

**Lemma 2.5.** *Let  $M$  be a semiartinian weakly co-semisimple module and a submodule of a direct sum of finitely presented modules in  $\sigma[M]$ . Furthermore, suppose that  $N$  is a submodule of  $M$ , with a non-limit Loewy length. If*

$$(*) \quad 0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{\lambda+1} = L_\xi = N$$

*is the Loewy chain of  $N$ , then for every ordinal  $1 \leq \alpha \leq \xi$ ,  $L_\alpha$  is absolutely pure.*

*Proof.* We proceed by transfinite induction on  $\alpha$ . Suppose that  $\alpha = 1$ . Then,  $L_1 = \text{Soc}(N)$ , since each simple module (in  $\sigma[M]$ ) is weakly  $M$ -injective and direct sums of weakly  $M$ -injective modules are weakly

$M$ -injective (Lemma 2.2 (a)),  $L_1$  is weakly  $M$ -injective, and hence by Lemma 2.4 is absolutely pure. Now, suppose that our claim holds for every  $\beta < \alpha$ . We know that  $\alpha$  is either a non-limit ordinal or a limit ordinal. Suppose that  $\alpha = \beta + 1$ . In this case,  $\text{Soc}(N/L_\beta) = L_\alpha/L_\beta$ . Consider the following short exact sequence:

$$0 \longrightarrow L_\beta \longrightarrow L_\alpha \longrightarrow L_\alpha/L_\beta \longrightarrow 0.$$

Since  $L_\alpha/L_\beta$  is semisimple, it is weakly  $M$ -injective (by Lemma 2.2 (a) and  $M$  being weakly co-semisimple) and  $L_\beta$  is weakly  $M$ -injective by the induction hypothesis. By Lemma 2.4, we conclude that  $L_\beta$  and  $L_\alpha/L_\beta$  are absolutely pure in  $\sigma[M]$ . By Lemma 2.3,  $L_\alpha$  is absolutely pure. Now, suppose that  $\alpha$  is a non-limit ordinal. Then, by definition,  $L_\alpha = \bigcup_{\gamma < \alpha} L_\gamma$ , and since by induction, each  $L_\gamma$  is weakly  $M$ -injective, by Lemma 2.2 (b),  $L_\alpha$  is weakly  $M$ -injective, and hence, by Lemma 2.4, is absolutely pure.  $\square$

**Lemma 2.6.** *Suppose that  $M$ ,  $N$  and  $(*)$  are as in Lemma 2.5. Then, every short exact sequence of the form*

$$(**) \quad 0 \longrightarrow L_\alpha \longrightarrow L_{\alpha+1} \longrightarrow L_{\alpha+1}/L_\alpha \longrightarrow 0$$

*is pure in  $\sigma[M]$ .*

*Proof.* By Lemma 2.5, each  $L_\alpha$  is absolutely pure. Hence, by the definition of absolute purity, the short exact sequence  $(**)$  is pure in  $\sigma[M]$ .  $\square$

We need the next result.

**Lemma 2.7.** ([24], 37.1)

- (a) *Let  $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$  be an exact sequence in  $\sigma[M]$ . If the sequence is pure and  $L'$  and  $L''$  are regular, then  $L$  is regular.*
- (b) *Direct sums and direct limits of regular modules are again regular.*

**Lemma 2.8.** *Let  $M$ ,  $N$ ,  $(*)$  and  $(**)$  be as in Lemma 2.5 and Lemma 2.6. If  $L_\alpha$  and  $L_{\alpha+1}/L_\alpha$  are regular in  $\sigma[M]$ , then  $L_{\alpha+1}$  is regular.*

*Proof.* By Lemma 2.6,  $(**)$  is pure. Now, by Lemma 2.7 (a), the proof is complete.  $\square$

The following result will be used in our main theorem. It also reveals the connection between the regularity in  $\sigma[M]$  based on the concept of purity and the usual definition of regularity in the literature.

**Lemma 2.9.** ([24], 37.4) *Assume that the  $R$ -module  $M$  to be a submodule of a direct sum of finitely presented modules in  $\sigma[M]$ . Then, the following assertions are equivalent:*

- (a)  $M$  is regular in  $\sigma[M]$ .
- (b) Every finitely generated submodule of  $M$  (or  $M^{(\mathbb{N})}$ ) is a direct summand.
- (c) Every finitely generated submodule of a finitely presented module in  $\sigma[M]$  is a direct summand.
- (d) Every  $R$ -module (in  $\sigma[M]$ ) is weakly  $M$ -injective.

Based on these series of lemmas and results, we are ready to prove our main theorem.

**Theorem 2.10.** *Let  $M$  be a semi-artinian module which is contained in a direct sum of finitely presented modules in  $\sigma[M]$ . Then, the followings are equivalent:*

- (a)  $M$  is weakly co-semisimple.
- (b)  $M$  is regular in  $\sigma[M]$ .

*Proof.* (a) $\Rightarrow$ (b). The proof will be divided in two steps.

Step 1. Suppose that  $N \subseteq M$ , with a non-limit Loewy length. We show that  $N$  is regular in  $\sigma[M]$ . As in the proof of Lemma 2.5, we proceed by transfinite induction on  $\alpha$ . Let

$$0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{\lambda+1} = L_\xi = N$$

be the Loewy chain of  $N$ , where  $\lambda$  is an ordinal and  $N/L_\lambda$  is semisimple, by definition. We claim that, for every  $1 \leq \alpha \leq \xi$ ,  $L_\alpha$  is a regular module. Since  $L_1$  is semisimple, it is a regular module. Now, let  $\alpha \leq \xi$  and  $\beta < \alpha$ . If  $\alpha = \beta + 1$ , for some  $\beta$ , then we observe that  $L_\alpha/L_\beta = \text{Soc}(N/L_\beta)$  is semisimple, and hence regular and  $L_\beta$  is regular by the induction hypothesis. Now, by Lemma 2.8,  $L_\alpha$  is regular too. If  $\alpha$  is a limit ordinal, then  $L_\alpha$  is the union of a chain of regular modules in  $\sigma[M]$ , and hence, by Lemma 2.9 (b), or by Lemma 2.7 (b), it is regular in  $\sigma[M]$ . Since  $N = L_\xi$ , we conclude that  $N$  is regular.

Step 2. It is well-known that every module is a direct limit of its finitely generated submodules and any finitely generated module has a non-limit

Loewy length. Therefore, by the Step 1, each finitely generated submodule of  $M$  is regular in  $\sigma[M]$ . On the other hand, by Lemma 2.7 (b), the direct limit of regular modules is regular. Hence,  $M$  is regular in  $\sigma[M]$ .

(b) $\Rightarrow$ (a). Since  $M$  is regular in  $\sigma[M]$ , by Lemma 2.9(d), every module in  $\sigma[M]$  is weakly  $M$ -injective, and hence simple modules are also weakly  $M$ -injective, i.e.,  $M$  is weakly co-semisimple.  $\square$

In [1], [16, Corollary 4.3], and [3, Proposition 2.3], it has been observed that every right semi-artinian right (or left)  $V$ -ring is a regular ring. Furthermore, Baccella [3] has observed that, for a ring  $R$  such that every right primitive factor ring of  $R$  is artinian, the following assertions are equivalent: (i)  $R$  is a right semi-artinian right  $V$ -ring; (ii)  $R$  is a left semi-artinian left  $V$ -ring; (iii)  $R$  is a regular right and left semi-artinian ring. The following corollary is a generalization of these results.

**Corollary 2.11.** *Let  $R$  be a right and left semi-artinian ring. Then, the followings are equivalent:*

- (a)  $R$  is a right FP- $V$ -ring.
- (b)  $R$  is a left FP- $V$ -ring.
- (c)  $R$  is a von Neumann regular ring.

Clark [9] has given an example of a right hereditary ring for which the left socle is not projective. Since every semi-artinian module has a nontrivial socle, the following result may be of some value and shows that sometimes the maximality of socle implies being hereditary. This has been first proved for Boolean rings by Kutami and Oshiro (see [15, Lemma 5]).

**Proposition 2.12.** *Let  $M$  be a regular projective module in  $\sigma[M]$  which is contained in a direct sum of finitely presented modules in  $\sigma[M]$ . If  $\text{Soc}(M)$  is a maximal submodule, then  $M$  is hereditary in  $\sigma[M]$ .*

*Proof.* Let  $N \subseteq M$  If  $N \subseteq \text{Soc}(M)$ , then  $N$  is projective. If  $N \not\subseteq \text{Soc}(M)$ , then  $N + \text{Soc}(M) = M$ . Hence, we may write  $N \oplus K = M$ , where  $K$  is a submodule of  $\text{Soc}(M)$  (for  $N \cap \text{Soc}(M)$  is a summand of  $\text{Soc}(M)$ ). But,  $M$  is projective, and hence  $N$  is projective in  $\sigma[M]$ . This implies that  $M$  is hereditary in  $\sigma[M]$ .  $\square$

**Corollary 2.13.** *Let  $M$  be semi-artinian weakly co-semisimple projective (in  $\sigma[M]$ ) which is contained in a direct sum of finitely presented*

modules in  $\sigma[M]$ . If  $\text{Soc}(M)$  is a maximal submodule, then  $M$  is hereditary in  $\sigma[M]$ .

*Proof.* By Theorem 2.10,  $M$  is regular in  $\sigma[M]$ , and by the above proposition,  $M$  is hereditary in  $\sigma[M]$ .  $\square$

**Example 2.14.** Let  $\xi$  be an ordinal. Baccella [3] has constructed semi-artinian right  $V$ -rings (and hence regular), with the Loewy length  $\xi + 1$ , which are not left  $V$ -rings. By our main result, all of these examples are also examples of semi-artinian weakly co-semisimple modules (semi-artinian  $FP$ - $V$ -rings) which are not co-semisimple ( $V$ -rings).

**Example 2.15.** In [13], an  $R$ -module  $M$  is called *generalized co-semisimple* or *GCO-module*, if every singular simple  $R$ -module is  $M$ -injective or  $M$ -projective. The reader is reminded that semi-artinian weakly co-semisimple modules, defined in this article, are different from the GCO-modules, for every GCO-module is a Max module (see [13, Proposition 16.3]), while there exists a right and left semi-artinian regular ring (and hence an  $FP$ - $V$ -ring), which is not a right Max ring (see [7]).

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### REFERENCES

- [1] J. S. Alin and E. P. Armendariz, A class of rings having all singular simple modules injective, *Math. Scand.* **23** (1968) 233-240.
- [2] E. P. Armendariz and J. W. Fisher, Regular PI-rings, *Proc. Amer. Math. Soc.* **39** (1973) 247-251.
- [3] G. Baccella, Semi-Artinian  $V$ -rings and semi-Artinian Von Neumann regular rings, *J. Algebra* **173** (1995) 587-612.
- [4] G. Baccella, Exchange property and the natural preorder between simple modules over semi-Artinian rings, *J. Algebra* **253**(1) (2002) 133-166.
- [5] G. Baccella, von Neumann regularity of  $V$ -rings with Artinian primitive factor rings, *Proc. Amer. Math. Soc.* **103**(3) (1988) 747-749.
- [6] V. Camillo and K. Fuller, On Loewy length of rings, *Pacific J. Math.* **53** (1974) 347-354.
- [7] V. Camillo and K. Fuller, *A Note on Loewy Rings and Chain Conditions on Primitive Ideals*, Module theory (Proc. Special Session, Amer. Math. Soc., Univ.

- Washington, Seattle, Wash., 1977), pp. 75-86, Lecture Notes in Math., 700, Springer, Berlin, 1979.
- [8] J. Clark and P.F. Smith, On semi-Artinian modules and injectivity conditions, *Proc. Edinburgh Math. Soc. (2)* **39**(2) (1996) 263-270.
  - [9] J. Clark, A note on rings with projective socle, *C. R. Math. Rep. Acad. Sci. Canada* **18**(2-3) (1996) 85-87.
  - [10] J. Cozzens, Homological properties of the ring of differential polynomials, *Bull. Amer. Math. Soc.* **76** (1970) 75-79.
  - [11] H. Q. Dinh and P. F. Smith, A result on semi-Artinian rings, *Proc. Edinb. Math. Soc. (2)* **46**(1) (2003) 63-66.
  - [12] N. V. Dung and P.F. Smith, On semi-Artinian  $V$ -modules, *J. Pure Appl. Algebra* **82**(1) (1992) 27-37.
  - [13] N V. Dung, D. Van Huynh, P. F. Smith and R. Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics, Longman Scientific & Technical, Harlow, 1994.
  - [14] D. Herbera, *Simple Modules Over Small Rings*, Rings, modules and representations, 189-205, Contemp. Math., 480, Amer. Math. Soc., Providence, RI, 2009.
  - [15] M. Kutami and K. Oshiro, An example of a ring whose projective modules have the exchange property, *Osaka J. Math.* **17**(2) (1980) 415-420 .
  - [16] C. Năstăsescu, Quelques remarques sur la dimension homologique des anneaux, *Éléments réguliers*, (French), *J. Algebra* **19** (1971) 470-485.
  - [17] C. Năstăsescu and N. Popescu, Anneaux semi-artinien, (French) *Bull. Soc. Math. France* **96** (1968) 357-368.
  - [18] C. Năstăsescu, Semi-Artinian rings, (Romanian) *Stud. Cerc. Mat.* **22** (1970) 1435-1507.
  - [19] N. Popescu and T. Spiricu, Examples of semi-artinian rings, (Romanian) *Stud. Cerc. Mat.* **26** (1974) 1153-1157.
  - [20] N. Popescu and C. Vraciu, Some remarks about semi-Artinian rings, *Rev. Roumaine Math. Pures Appl.* **18** (1973) 1413-1422.
  - [21] B. Sarath and K. Varadarajan, Injectivity of certain classes of modules, *J. Pure. App. Algebra* **5** (1974) 293-305.
  - [22] J. Trlifaj, Uniform Modules,  $\Gamma$ -Invariants, and Ziegler Spectra of Regular rings, Abelian Groups and Modules (Dublin, 1998), 327-340, Trends Math., Birkhäuser, Basel, 1999.
  - [23] D. V. Tyukavkin, Regular self-injective rings and  $V$ -Rings, *Algebra and Logic* **33**(5) (1994) 315-321.
  - [24] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

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