RANKS OF THE COMMON SOLUTION TO SOME QUATERNION MATRIX EQUATIONS WITH APPLICATIONS

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ABSTRACT. We derive the formulas of the maximal and minimal ranks of four real matrices X_1, X_2, X_3 and X_4 in common solution $X = X_1 + X_2i + X_3j + X_4k$ to quaternion matrix equations $A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3$. As applications, we establish necessary and sufficient conditions for the existence of the common real and complex solutions to the matrix equations. We give the expressions of such solutions to this system when the solvability conditions are met. Moreover, we present necessary and sufficient conditions for the existence of real and complex solutions to the system of quaternion matrix equations $A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3, A_4XB_4 = C_4$. The findings of this paper extend some known results in the literature.

1. Introduction

Throughout this paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, \, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

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by $\mathbb{H}^{m \times n}$, the identity matrix with the appropriate size by I. For a matrix A, over $\mathbb{H}$, we denote the column right space, the row left space of A by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, the dimension of $\mathcal{R}(A)$ by $\dim \mathcal{R}(A)$, a generalized inverse of a matrix A by A^- which satisfies $AA^-A = A$, a reflexive inverse of matrix A over $\mathbb{H}$ by A^+ which satisfies simultaneously $AA^+A = A$ and $AA^+A = A$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^+A$ and $R_A = I - AA^+$ induced by A, where A^+ is any but fixed reflexive inverse of A. Clearly, R_A and L_A are idempotent and one of its reflexive inverses is itself. By [14], for a quaternion matrix A, $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$ is called the rank of A and is denoted by $r(A)$.

Introduced by W. R. Hamilton in 1843, quaternion has made further appearance ever since in associative algebra, analysis, topology, and physics. Nowadays quaternion matrices play an important role in computer science, quantum physics, signal and color image processing, and so on (see, e.g. [1–3, 13, 18, 22, 39, 40]).We know that matrix equation is one of the very active topics in the research of matrix theory and its applications, and a large number of papers have presented several methods for solving several matrix equations (e.g. [5]- [12], [17], [26]-[31], [33], [36]- [38]). Researches on extreme ranks of solutions to linear matrix equations have been actively ongoing for more than 30 years (e.g. [15, 16], [19]- [21], [23]- [25], [32, 34, 35]). It is worthy to say that minimal and maximal ranks of a general solution to a matrix equation are very useful in linear programming computations (e.g. [19]- [21]).

Recall that Mitra in [19] investigated the solutions of minimum possible rank to the system of matrix equations

$$(1.1) \qquad\qquad A_1X = C_1, XB_2 = C_2.$$

Tian [23] gave the maximal and minimal ranks of two real matrices X_0 and X_1 in solution $X = X_0 + iX_1$ to the classical matrix equation

$$(1.2) \qquad\qquad AXB = C$$

over the complex number field $\mathbb{C}$ and gave its applications. P. Bhimasankaram [4] presented a necessary and sufficient condition for the system of matrix equations

$$(1.3) \qquad\qquad A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3$$

to have a general solution, and gave the representation of the general solution over $\mathbb{C}$. Lin and Wang in [16] established a practical solvability

condition and a new expression of system (1.3), and investigated the extremal ranks of the general solution to (1.3) in [15]. To our knowledge, so far there has been little information on the necessary and sufficient conditions for (1.3) and (1.4) over $\mathbb{H}$ to have real and complex solutions. Motivated by the work mentioned above and keeping applications and interests of quaternion matrices in view, in this paper we aim to investigate the real and complex solutions to system (1.3) over $\mathbb{H}$, the extreme ranks of such solutions, and their applications.

The remainder of this paper is organized as follows. In Section 2, we first derive formulas of extremal ranks of four real matrices X_1, X_2, X_3 and X_4 in quaternion solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.3) over $\mathbb{H}$, then we give necessary and sufficient conditions for (1.3) over $\mathbb{H}$ to have real and complex solutions as well as the expressions of the real and complex solutions. As an application, in Section 3 we establish necessary and sufficient conditions for the existence of the real and complex solutions to the system

$$(1.4) \qquad A_1 X = C_1, X B_2 = C_2, A_3 X B_3 = C_3, A_4 X B_4 = C_4$$

over $\mathbb{H}$, which was once investigated in [26] and [35].

2. The real and complex solutions to system (1.3) over $\mathbb{H}$

In this section, we first give a solvability condition and an expression of the general solution to (1.3) over $\mathbb{H}$, then consider the maximal and minimal ranks of four real matrices X_1, X_2, X_3 and X_4 in solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.3) over $\mathbb{H}$, last, investigate the real and complex solutions to (1.3) over $\mathbb{H}$.

For an arbitrary matrix $M_t = M_{t1} + M_{t2} i + M_{t3} j + M_{t4} k \in \mathbb{H}^{m \times n}$ where $M_{t1}, M_{t2}, M_{t3}, M_{t4}$ are real matrices, we define a map $\phi(\cdot)$ from $\mathbb{H}^{m \times n}$ to $\mathbb{R}^{4m \times 4n}$ by

$$(2.1) \qquad \phi(M_t) = \begin{bmatrix} M_{t1} & M_{t2} & M_{t3} & M_{t4} \\ -M_{t2} & M_{t1} & M_{t4} & -M_{t3} \\ -M_{t3} & -M_{t4} & M_{t1} & M_{t2} \\ -M_{t4} & M_{t3} & -M_{t2} & M_{t1} \end{bmatrix}.$$

By (2.1), it is easy to verify that $\phi(\cdot)$ satisfies the following properties:

(a) $M = N \iff \phi(M) = \phi(N)$.
(b) $\phi(kM + lN) = k\phi(M) + l\phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $k, l \in \mathbb{R}$.

(c) $\phi(M) = T_m^{-1}\phi(M)T_n = R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where $t = m, n$,

$$T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix}, R_t = \begin{bmatrix} 0 & 0 & -I_t & 0 \\ 0 & 0 & 0 & -I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{bmatrix},$$

$$S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}.$$

(d) $r\left[\phi(M)\right] = 4r(M)$.

We will use the following lemma to give the basic theorem of this paper.

Lemma 2.1. *(See Theorem 2.4 in [16]) Suppose that $A_1 \in \mathbb{H}^{m \times n}$, $A_3 \in \mathbb{H}^{k \times n}$, $B_2 \in \mathbb{H}^{r \times s}$, $B_3 \in \mathbb{H}^{r \times p}$, $C_1 \in \mathbb{H}^{m \times r}$, $C_2 \in \mathbb{H}^{n \times s}$, $C_3 \in \mathbb{H}^{k \times p}$ are known, $X \in \mathbb{H}^{n \times r}$ is unknown, and $K = A_3 L_{A_1}$, $H = R_{B_2} B_3$, $Q_1 = C_3 - A_3 A_1^+ C_1 B_3 - K C_2 B_2^+ B_3$, $K^+ Q_1 H^+ = Q_2$, then the following statements are equivalent:*

(1) *The system* (1.3) *is consistent.*

(2)
$$Q_1 = K Q_2 H,$$
$$A_1 A_1^+ C_1 = C_1, \ C_2 B_2^+ B_2 = C_2, \ A_1 C_2 = C_1 B_2, \ A_3 A_3^+ C_3 B_3^+ B_3 = C_3.$$

(3)
$$A_1 C_2 = C_1 B_2, \ r\left[A_1, C_1\right] = r(A_1), \ r\begin{bmatrix} B_2 \\ C_2 \end{bmatrix} = r(B_2),$$

$$r\left[\begin{array}{cc} A_3 & C_3 \end{array}\right] = r(A_3), \ r\begin{bmatrix} B_3 \\ C_3 \end{bmatrix} = r(B_3),$$

$$r\begin{bmatrix} A_1 & C_1 B_3 \\ A_3 & C_3 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}, \ r\begin{bmatrix} B_2 & B_3 \\ A_3 C_2 & C_3 \end{bmatrix} = r\left[\begin{array}{cc} B_2 & B_3 \end{array}\right].$$

In that case, the general solution of (1.3) *can be expressed as*

$$X = A_1^+ C_1 + L_{A_1} C_2 B_2^+ + L_{A_1} Q_2 R_{B_2} + L_{A_1} L_K Z R_{B_2} + L_{A_1} W R_H R_{B_2}$$

where Z, W are arbitrary matrices over $\mathbb{H}$ with compatible sizes.

Equipping with the above preliminaries, we can now give the foundational theorem of this paper.

Theorem 2.2. *System* (1.3) *is consistent over* $\mathbb{H}$ *if and only if the system of matrix equations*

(2.2) $\phi(A_1)Y = \phi(C_1), Y\phi(B_2) = \phi(C_2), \phi(A_3)Y\phi(B_3) = \phi(C_3)$

is consistent over $\mathbb{R}$. *In that case, the general solution of* (1.3) *over* $\mathbb{H}$ *can be expressed as*

$$X = X_1 + X_2 i + X_3 j + X_4 k$$

where

$$X_1 = \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 + \frac{1}{4}P_3\phi(X_0)Q_3 + \frac{1}{4}P_4\phi(X_0)Q_4$$

(2.3) $+\, [P_1, P_2, P_3, P_4]\, L_{\phi(A_1)}(L_{\phi(K)}Z + WR_{\phi(H)})R_{\phi(B_2)} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$

$$X_2 = \frac{1}{4}P_1\phi(X_0)Q_2 - \frac{1}{4}P_2\phi(X_0)Q_1 + \frac{1}{4}P_3\phi(X_0)Q_4 - \frac{1}{4}P_4\phi(X_0)Q_3$$

(2.4) $+\, [P_1, -P_2, P_3, -P_4]\, L_{\phi(A_1)}(L_{\phi(K)}Z + WR_{\phi(H)})R_{\phi(B_2)} \begin{bmatrix} Q_2 \\ Q_1 \\ Q_4 \\ Q_3 \end{bmatrix},$

$$X_3 = \frac{1}{4}P_1\phi(X_0)Q_3 - \frac{1}{4}P_3\phi(X_0)Q_1 + \frac{1}{4}P_4\phi(X_0)Q_2 - \frac{1}{4}P_2\phi(X_0)Q_4$$

(2.5) $+\, [P_1, -P_3, P_4, -P_2]\, L_{\phi(A_1)}(L_{\phi(K)}Z + WR_{\phi(H)})R_{\phi(B_2)} \begin{bmatrix} Q_3 \\ Q_1 \\ Q_2 \\ Q_4 \end{bmatrix},$

$$X_4 = \frac{1}{4}P_1\phi(X_0)Q_4 - \frac{1}{4}P_4\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_3 - \frac{1}{4}P_3\phi(X_0)Q_2$$

(2.6) $+\, [P_1, -P_4, P_2, -P_3]\, L_{\phi(A_1)}(L_{\phi(K)}Z + WR_{\phi(H)})R_{\phi(B_2)} \begin{bmatrix} Q_4 \\ Q_1 \\ Q_3 \\ Q_2 \end{bmatrix}.$

where

$$P_1 = [I_p, 0, 0, 0],\;\; P_2 = [0, I_p, 0, 0],\;\; P_3 = [0, 0, I_p, 0],\;\; P_4 = [0, 0, 0, I_p],$$

$$Q_1 = \begin{bmatrix} I_q \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ Q_2 = \begin{bmatrix} 0 \\ I_q \\ 0 \\ 0 \end{bmatrix}, \ Q_3 = \begin{bmatrix} 0 \\ 0 \\ I_q \\ 0 \end{bmatrix}, \ Q_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_q \end{bmatrix},$$

$\phi(X_0)$ *is a particular solution to* (2.2), Z *and* W *are arbitrary real matrices with compatible sizes.*

Proof. Suppose that (1.3) has a solution X over $\mathbb{H}$. Applying properties (a) and (b) of $\phi(\cdot)$ to (1.3) yields

$$\phi(A_1)\phi(X) = \phi(C_1), \ \phi(X)\phi(B_2) = \phi(C_2), \ \phi(A_3)\phi(X)\phi(B_3) = \phi(C_3),$$

implying that $\phi(X)$ is a solution to (2.2).

Conversely, suppose that (2.2) has a solution

$$\hat{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{24} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix},$$

i.e.,

$$\phi(A_1)\hat{Y} = \phi(C_1), \ \hat{Y}\phi(B_2) = \phi(C_2), \ \phi(A_3)\hat{Y}\phi(B_3) = \phi(C_3),$$

then by property (c) of the map $\phi(\cdot)$,

$$\begin{aligned} T_m^{-1}\phi(A_1)T_p\hat{Y} &= T_m^{-1}\phi(C_1)T_n, \hat{Y}T_q^{-1}\phi(B_2)T_n \\ &= T_m^{-1}\phi(C_2)T_n, T_m^{-1}\phi(A_3)T_p\hat{Y}T_q^{-1}\phi(B_3)T_n \\ &= T_m^{-1}\phi(C_3)T_n, \end{aligned}$$

$$\begin{aligned} R_m^{-1}\phi(A_1)R_p\hat{Y} &= R_m^{-1}\phi(C_1)R_n, \hat{Y}R_q^{-1}\phi(B_2)R_n \\ &= R_m^{-1}\phi(C_2)R_n, R_m^{-1}\phi(A_3)R_p\hat{Y}R_q^{-1}\phi(B_3)R_n \\ &= R_m^{-1}\phi(C_3)R_n, \end{aligned}$$

$$\begin{aligned} S_m^{-1}\phi(A_1)S_p\hat{Y} &= S_m^{-1}\phi(C_1)S_n, \hat{Y}S_q^{-1}\phi(B_2)S_n \\ &= S_m^{-1}\phi(C_2)S_n, S_m^{-1}\phi(A_3)S_p\hat{Y}S_q^{-1}\phi(B_3)S_n \\ &= S_m^{-1}\phi(C_3)S_n. \end{aligned}$$

Hence,

$$\phi(A_1)T_p\hat{Y}T_q^{-1} = \phi(C_1), \ T_p\hat{Y}T_q^{-1}\phi(B_2) = \phi(C_2),$$
$$\phi(A_3)T_p\hat{Y}T_q^{-1}\phi(B_3) = \phi(C_3), \phi(A_1)R_p\hat{Y}R_q^{-1} = \phi(C_1),$$

$$R_p \hat{Y} R_q^{-1} \phi(B_2) = \phi(C_2), \quad \phi(A_3) R_p \hat{Y} R_q^{-1} \phi(B_3) = \phi(C_3),$$

$$\phi(A_1) S_p \hat{Y} S_q^{-1} = \phi(C_1), \quad S_p \hat{Y} S_q^{-1} \phi(B_2) = \phi(C_2),$$

$$\phi(A_3) S_p \hat{Y} S_q^{-1} \phi(B_3) = \phi(C_3),$$

implying that $T_p \hat{Y} T_q^{-1}$, $R_p \hat{Y} R_q^{-1}$ and $S_p \hat{Y} S_q^{-1}$ are also solutions of (2.2). Thus, $\frac{1}{4}(\hat{Y} + T_p \hat{Y} T_q^{-1} + R_p \hat{Y} R_q^{-1} + S_p \hat{Y} S_q^{-1})$ is a solution of (2.2), where

$$\hat{Y} + T_p \hat{Y} T_q^{-1} + R_p \hat{Y} R_q^{-1} + S_p \hat{Y} S_q^{-1}$$

$$= \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{24} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} + \begin{bmatrix} Y_{22} & -Y_{21} & -Y_{24} & Y_{23} \\ -Y_{12} & Y_{11} & Y_{14} & -Y_{13} \\ -Y_{42} & Y_{41} & Y_{44} & -Y_{43} \\ Y_{32} & -Y_{31} & -Y_{34} & Y_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} Y_{33} & Y_{34} & -Y_{31} & -Y_{32} \\ Y_{43} & Y_{44} & -Y_{41} & -Y_{42} \\ -Y_{13} & -Y_{14} & Y_{11} & Y_{12} \\ -Y_{23} & -Y_{24} & Y_{21} & Y_{22} \end{bmatrix} + \begin{bmatrix} Y_{44} & -Y_{43} & Y_{42} & -Y_{41} \\ -Y_{34} & Y_{33} & -Y_{32} & Y_{31} \\ Y_{24} & -Y_{23} & Y_{22} & -Y_{21} \\ -Y_{14} & Y_{13} & -Y_{12} & Y_{11} \end{bmatrix}$$

$$= [Y_1, Y_2, Y_3, Y_3],$$

with

$$Y_1 = \begin{bmatrix} Y_{11} + Y_{22} + Y_{33} + Y_{44} \\ Y_{21} - Y_{12} - Y_{34} + Y_{43} \\ Y_{31} - Y_{13} - Y_{42} + Y_{24} \\ Y_{41} - Y_{14} - Y_{23} + Y_{32} \end{bmatrix}, Y_2 = \begin{bmatrix} Y_{12} - Y_{21} + Y_{34} - Y_{43} \\ Y_{11} + Y_{22} + Y_{33} + Y_{44} \\ Y_{41} - Y_{14} - Y_{23} + Y_{32} \\ Y_{13} - Y_{31} + Y_{42} - Y_{24} \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} Y_{13} - Y_{31} + Y_{42} - Y_{24} \\ Y_{14} - Y_{41} + Y_{23} - Y_{32} \\ Y_{11} + Y_{22} + Y_{33} + Y_{44} \\ -Y_{12} + Y_{21} - Y_{34} + Y_{43} \end{bmatrix}, Y_4 = \begin{bmatrix} Y_{14} - Y_{41} + Y_{23} - Y_{32} \\ Y_{31} - Y_{13} - Y_{42} + Y_{24} \\ Y_{12} - Y_{21} + Y_{34} - Y_{43} \\ Y_{11} + Y_{22} + Y_{33} + Y_{44} \end{bmatrix}.$$

Let

$$\hat{X} = \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{34} - Y_{43})i$$

$$+ \frac{1}{4}(Y_{13} - Y_{31} + Y_{42} - Y_{24})j + \frac{1}{4}(Y_{14} - Y_{41} + Y_{23} - Y_{32})k.$$

Then by (2.1),

$$\phi(\hat{X}) = \frac{1}{4}(\hat{Y} + T_p \hat{Y} T_q^{-1} + R_p \hat{Y} R_q^{-1} + S_p \hat{Y} S_q^{-1}).$$

Hence, by the property (a), we know that $\hat{X}$ is a solution of (1.3). The discussion referred shows that the two matrix equations (1.3) and (2.2) have the same solvability condition and their solutions satisfy

$$X = X_1 + X_2 i + X_3 j + X_4 k$$
$$= \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{34} - Y_{43})i$$

(2.7) $\qquad + \frac{1}{4}(Y_{13} - Y_{31} + Y_{42} - Y_{24})j + \frac{1}{4}(Y_{14} - Y_{41} + Y_{23} - Y_{32})k.$

Observe that Y_{1t}, Y_{2t}, Y_{3t} and Y_{4t}, $t = 1, 2, 3, 4$ in (2.2) can be written as

$$Y_{11} = P_1 \hat{Y} Q_1, \; Y_{12} = P_1 \hat{Y} Q_2, \; Y_{13} = P_1 \hat{Y} Q_3, \; Y_{14} = P_1 \hat{Y} Q_4,$$

$$Y_{21} = P_2 \hat{Y} Q_1, \; Y_{22} = P_2 \hat{Y} Q_2, \; Y_{23} = P_2 \hat{Y} Q_3, \; Y_{24} = P_2 \hat{Y} Q_4,$$

$$Y_{31} = P_3 \hat{Y} Q_1, \; Y_{32} = P_3 \hat{Y} Q_2, \; Y_{33} = P_3 \hat{Y} Q_3, \; Y_{34} = P_3 \hat{Y} Q_4,$$

$$Y_{41} = P_4 \hat{Y} Q_1, \; Y_{42} = P_4 \hat{Y} Q_2, \; Y_{43} = P_4 \hat{Y} Q_3, \; Y_{44} = P_4 \hat{Y} Q_4.$$

From Lemma 2.1, the general solution to (2.2) can be written as

$$\hat{Y} = \phi(X_0) + 4L_{\phi(A_1)}L_{\phi(K)}ZR_{\phi(B_2)} + 4L_{\phi(A_1)}WR_{\phi(H)}R_{\phi(B_2)}$$

where $Z, W \in \mathbb{R}^{p \times q}$, are arbitrary. Hence,

$$Y_{1t} = P_1\phi(X_0)Q_t + 4P_1 L_{\phi(A_1)}L_{\phi(K)}ZR_{\phi(B_2)}Q_t$$
$$+ 4P_1 L_{\phi(A_1)}WR_{\phi(H)}R_{\phi(B_2)}Q_t,$$

$$Y_{2t} = P_2\phi(X_0)Q_t + 4P_2 L_{\phi(A_1)}L_{\phi(K)}ZR_{\phi(B_2)}Q_t$$
$$+ 4P_2 L_{\phi(A_1)}WR_{\phi(H)}R_{\phi(B_2)}Q_t,$$

$$Y_{3t} = P_3\phi(X_0)Q_t + 4P_3 L_{\phi(A_1)}L_{\phi(K)}ZR_{\phi(B_2)}Q_t$$
$$+ 4P_3 L_{\phi(A_1)}WR_{\phi(H)}R_{\phi(B_2)}Q_t,$$

$$Y_{4t} = P_4\phi(X_0)Q_t + 4P_4 L_{\phi(A_1)}L_{\phi(K)}ZR_{\phi(B_2)}Q_t$$
$$+ 4P_4 L_{\phi(A_1)}WR_{\phi(H)}R_{\phi(B_2)}Q_t$$

where $t = 1, 2, 3, 4$. Substituting them into (2.7) yields the four real matrices X_1, X_2, X_3 and X_4 in (2.3)-(2.6). $\qquad\square$

In order to investigate the real and complex solution of system (1.3), we need to consider first the maximal and minimal ranks of four real matrices X_1, X_2, X_3 and X_4 in solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.3) over $\mathbb{H}$. We have the following.

Lemma 2.3. (Lemma 2.4 in [32]) *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$. Then they satisfy the following rank equalities*

(a) $r(CL_A) = r \begin{bmatrix} A \\ C \end{bmatrix} - r(A)$.

(b) $r \begin{bmatrix} B & AL_C \end{bmatrix} = r \begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$.

(c) $r \begin{bmatrix} C \\ R_B A \end{bmatrix} = r \begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$.

(d) $r \begin{bmatrix} A & BL_D \\ R_E C & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E)$.

The following lemma is due to Tian [24], which can be generalized to $\mathbb{H}$.

Lemma 2.4. *Let*

$$f(X_1, X_2) = A - B_1 X_1 C_1 - B_2 X_2 C_2$$

be a matrix expression over $\mathbb{H}$. Then the extremal ranks of $f(X_1, X_2)$ are the following

$$\max_{X_1, X_2} r[f(X_1, X_2)]$$

$$= \min \left\{ r \begin{bmatrix} A & B_1 & B_2 \end{bmatrix}, \; r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix}, \right.$$

$$\left. r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}, \; r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\},$$

$$\min_{X_1,X_2} r[f(X_1, X_2)]$$

$$= r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r \begin{bmatrix} A & B_1 & B_2 \end{bmatrix}$$

$$+ \max \left\{ r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}, \right.$$

$$\left. r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} \right\}.$$

If $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{N}(C_2) \subseteq \mathcal{N}(C_1)$, then

(2.8) $$\max_{X_1,X_2} r[f(X_1, X_2)]$$

$$= \min \left\{ r[A, B_2], \ r \begin{bmatrix} A \\ C_1 \end{bmatrix}, \ r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} \right\},$$

(2.9)

$$\min_{X_1,X_2} r[f(X_1, X_2)]$$

$$= r[A, B_2] + r \begin{bmatrix} A \\ C_1 \end{bmatrix} + r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix}.$$

Now we consider the maximal and minimal ranks of four real matrices X_1, X_2, X_3 and X_4 in solution $X = X_1 + X_2i + X_3j + X_4k$ to (1.3) over $\mathbb{H}$.

Theorem 2.5. *Suppose that system (1.3) over $\mathbb{H}$ is consistent and $A_t = A_{t1} + A_{t2}i + A_{t3}j + A_{t4}k \in \mathbb{H}^{m \times p}, t = 1, 3, B_t = B_{t1} + B_{t2}i + B_{t3}j + B_{t4}k \in \mathbb{H}^{q \times n}, t = 1, 2, C_t = C_{t1} + C_{t2}i + C_{t3}j + C_{t4}k \in \mathbb{H}^{m \times n} \ t = 1, 2, 3. \ Put$*

$$S_1 = \left\{ X_1 \in \mathbb{R}^{p \times q} \ \middle| \ \begin{array}{c} A_1 X = C_1, X B_2 = C_2, A_3 X B_3 = C_3, \\ X = X_1 + X_2i + X_3j + X_4k \end{array} \right\},$$

$$S_2 = \left\{ X_2 \in \mathbb{R}^{p \times q} \ \middle| \ \begin{array}{c} A_1 X = C_1, X B_2 = C_2, A_3 X B_3 = C_3, \\ X = X_1 + X_2i + X_3j + X_4k \end{array} \right\},$$

$$S_3 = \left\{ X_3 \in \mathbb{R}^{p \times q} \ \middle| \ \begin{array}{c} A_1 X = C_1, X B_2 = C_2, A_3 X B_3 = C_3, \\ X = X_1 + X_2i + X_3j + X_4k \end{array} \right\},$$

$$S_4 = \left\{ X_4 \in \mathbb{R}^{p \times q} \;\middle|\; \begin{array}{c} A_1 X = C_1, \, X B_2 = C_2, \, A_3 X B_3 = C_3, \\ X = X_1 + X_2 i + X_3 j + X_4 k \end{array} \right\},$$

$$L_{21} = [C_{21}, C_{22}, C_{23}, C_{24}],$$

$$L_{11} = \begin{bmatrix} C_{11} \\ -C_{12} \\ -C_{13} \\ -C_{14} \end{bmatrix}, \; L_{12} = \begin{bmatrix} C_{12} \\ C_{11} \\ -C_{14} \\ C_{13} \end{bmatrix},$$

$$L_{13} = \begin{bmatrix} C_{13} \\ C_{14} \\ C_{11} \\ -C_{12} \end{bmatrix}, L_{14} = \begin{bmatrix} C_{14} \\ -C_{13} \\ C_{12} \\ C_{11} \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} A_{12} & A_{13} & A_{14} \\ A_{11} & A_{14} & -A_{13} \\ -A_{14} & A_{11} & A_{12} \\ A_{13} & -A_{12} & A_{11} \end{bmatrix}, \; M_{31} = \begin{bmatrix} A_{32} & A_{33} & A_{34} \\ A_{31} & A_{34} & -A_{33} \\ -A_{34} & A_{31} & A_{32} \\ A_{33} & -A_{32} & A_{31} \end{bmatrix},$$

$$N_{31} = \begin{bmatrix} -B_{32} & B_{31} & B_{34} & -B_{33} \\ -B_{33} & -B_{34} & B_{31} & B_{32} \\ -B_{34} & B_{33} & -B_{32} & B_{31} \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} -B_{22} & B_{21} & B_{24} & -B_{23} \\ -B_{23} & -B_{24} & B_{21} & B_{22} \\ -B_{24} & B_{23} & -B_{22} & B_{21} \end{bmatrix},$$

$$N_{32} = \begin{bmatrix} B_{31} & B_{32} & B_{33} & B_{34} \\ -B_{33} & -B_{34} & B_{31} & B_{32} \\ -B_{34} & B_{33} & -B_{32} & B_{31} \end{bmatrix},$$

$$N_{22} = \begin{bmatrix} B_{21} & B_{22} & B_{23} & B_{24} \\ -B_{23} & -B_{24} & B_{21} & B_{22} \\ -B_{24} & B_{23} & -B_{22} & B_{21} \end{bmatrix},$$

$$N_{33} = \begin{bmatrix} B_{31} & B_{32} & B_{33} & B_{34} \\ -B_{32} & B_{31} & B_{34} & -B_{33} \\ -B_{34} & B_{33} & -B_{32} & B_{31} \end{bmatrix},$$

$$N_{23} = \begin{bmatrix} B_{21} & B_{22} & B_{23} & B_{24} \\ -B_{22} & B_{21} & B_{24} & -B_{23} \\ -B_{24} & B_{23} & -B_{22} & B_{21} \end{bmatrix},$$

$$N_{34} = \begin{bmatrix} B_{31} & B_{32} & B_{33} & B_{34} \\ -B_{32} & B_{31} & B_{34} & -B_{33} \\ -B_{33} & -B_{34} & B_{31} & B_{32} \end{bmatrix},$$

$$N_{24} = \begin{bmatrix} B_{21} & B_{22} & B_{23} & B_{24} \\ -B_{22} & B_{21} & B_{24} & -B_{23} \\ -B_{23} & -B_{24} & B_{21} & B_{22} \end{bmatrix}.$$

Then the maximal and minimal ranks of X_i, $i = 1, 2, 3, 4$ in solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.3) are given by

$$(2.10) \qquad \max_{X_i \in S_i} r(X_i) = \min \{t_{1i}, t_{2i}, t_{3i}\},$$

$$(2.11) \qquad \min_{X_i \in S_i} r(X_i) = t_{1i} + t_{2i} + t_{3i} - t_{4i} - t_{5i}$$

where

$$t_{1i} = r \begin{bmatrix} -L_{21} \\ N_{2i} \end{bmatrix} - 4r(B_2) + q,$$

$$t_{2i} = r \begin{bmatrix} -L_{1i} & M_{11} \end{bmatrix} - 4r(A_1) + p,$$

$$t_{3i} = r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix} - 4r \begin{bmatrix} A_3 \\ A_1 \end{bmatrix}$$
$$- 4r[B_3, B_2] + p + q,$$

$$t_{4i} = r \begin{bmatrix} 0 & N_{2i} \\ M_{31} & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(A_1)\phi(C_2) \end{bmatrix} - 4r \begin{bmatrix} A_3 \\ A_1 \end{bmatrix} - 4r(B_2) + p + q,$$

$$t_{5i} = r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix} - 4r[B_3, B_2] - 4r(A_1) + p + q.$$

Proof. We only prove the case that $i = 1$. Similarly, we can get the results for $i = 2, 3, 4$. Let

$$\frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 + \frac{1}{4}P_3\phi(X_0)Q_3 + \frac{1}{4}P_4\phi(X_0)Q_4 = A,$$

$$\left[P_1 L_{\phi(A_1)}L_{\phi(K)}, P_2 L_{\phi(A_1)}L_{\phi(K)}, P_3 L_{\phi(A_1)}L_{\phi(K)}, P_4 L_{\phi(A_1)}L_{\phi(K)}\right] = B_1,$$

$$\left[P_1 L_{\phi(A_1)}, P_2 L_{\phi(A_1)}, P_3 L_{\phi(A_1)}, P_4 L_{\phi(A_1)}\right] = B_2,$$

$$\begin{bmatrix} R_{\phi(B_2)}Q_1 \\ R_{\phi(B_2)}Q_2 \\ R_{\phi(B_2)}Q_3 \\ R_{\phi(B_2)}Q_4 \end{bmatrix} = C_1, \qquad \begin{bmatrix} R_{\phi(H)}R_{\phi(B_2)}Q_1 \\ R_{\phi(H)}R_{\phi(B_2)}Q_2 \\ R_{\phi(H)}R_{\phi(B_2)}Q_3 \\ R_{\phi(H)}R_{\phi(B_2)}Q_4 \end{bmatrix} = C_2,$$

Then (2.3) can be written as

$$(2.12) \qquad X_1 = A + B_1 Z C_1 + B_2 W C_2.$$

Applying (2.8) and (2.9) in Lemma 2.4 to the two variant matrices Z and W in (2.12) yields

$$(2.13) \qquad \max_{X_1 \in S_1} r(X_1) = \min \left\{ r[A, B_2], \; r \begin{bmatrix} A \\ C_1 \end{bmatrix}, \; r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} \right\},$$

$$(2.14) \qquad \min_{X_1 \in S_1} r(X_1) = r[A, B_2] + r \begin{bmatrix} A \\ C_1 \end{bmatrix} + r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}$$
$$- r \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix}.$$

Note that $\phi(X_0)$ is a particular solution to (2.2). Let

$$[P_1, P_2, P_3, P_4] = P, \quad \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = Q,$$

$$a_i = \begin{bmatrix} \phi(A_i) & 0 & 0 & 0 \\ 0 & \phi(A_i) & 0 & 0 \\ 0 & 0 & \phi(A_i) & 0 \\ 0 & 0 & 0 & \phi(A_i) \end{bmatrix},$$

$$b_i = \begin{bmatrix} \phi(B_i) & 0 & 0 & 0 \\ 0 & \phi(B_i) & 0 & 0 \\ 0 & 0 & \phi(B_i) & 0 \\ 0 & 0 & 0 & \phi(B_i) \end{bmatrix}, \quad c_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi(C_i) \end{bmatrix},$$

where $i = 1, 2, 3$. It follows from Lemma 2.3, block Gaussian elimination, and property (d) of $\phi(\cdot)$ that

$$
\begin{aligned}
r[A, B_2] &= r \begin{bmatrix} A & P \\ 0 & a_1 \end{bmatrix} - 4r[\phi(A_1)] \\
&= r \begin{bmatrix} 0 & P \\ -\frac{1}{4}\phi(C_1)Q & a_1 \end{bmatrix} - 4r[\phi(A_1)] \\
&= r \begin{bmatrix} 0 & P_1 & 0 & 0 & 0 \\ -\phi(C_1)Q_1 & \phi(A_1) & 0 & 0 & 0 \\ 0 & 0 & \phi(A_1) & 0 & 0 \\ 0 & 0 & 0 & \phi(A_1) & 0 \\ 0 & 0 & 0 & 0 & \phi(A_1) \end{bmatrix} - 4r[\phi(A_1)] \\
&= r \begin{bmatrix} -L_{11} & M_{11} \end{bmatrix} - 4r[\phi(A_1)] + 3r[\phi(A_1)] + p \\
&= r \begin{bmatrix} -L_{11} & M_{11} \end{bmatrix} - 4r(A_1) + p;
\end{aligned}
$$

(2.15)

$$
\begin{aligned}
r \begin{bmatrix} A \\ C_1 \end{bmatrix} &= r \begin{bmatrix} A & 0 \\ Q & b_2 \end{bmatrix} - 4r[\phi(B_2)] \\
&= r \begin{bmatrix} 0 & -\frac{1}{4}P\phi(C_2) \\ Q & b_2 \end{bmatrix} - 4r[\phi(B_2)] \\
&= \begin{bmatrix} 0 & -P_1\phi(C_2) & 0 & 0 & 0 \\ Q_1 & \phi(B_2) & 0 & 0 & 0 \\ 0 & 0 & \phi(B_2) & 0 & 0 \\ 0 & 0 & 0 & \phi(B_2) & 0 \\ 0 & 0 & 0 & 0 & \phi(B_2) \end{bmatrix} - 4r[\phi(B_2)] \\
&= r \begin{bmatrix} -L_{21} \\ N_{21} \end{bmatrix} - 4r[\phi(B_2)] + 3r[\phi(B_2)] + q \\
&= r \begin{bmatrix} -L_{21} \\ N_{21} \end{bmatrix} - 4r(B_2) + q;
\end{aligned}
$$

$$r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} = r \begin{bmatrix} A & P & 0 & 0 \\ Q & 0 & b_3 & b_2 \\ 0 & a_3 & 0 & 0 \\ 0 & a_1 & 0 & 0 \end{bmatrix}$$

$$- 4r[\phi(A_1)] - 4r[\phi(K)] - 4r[\phi(B_2)] - 4r[\phi(H)]$$

$$= r \begin{bmatrix} 0 & P_1 & 0 & 0 \\ Q_1 & 0 & b_3 & b_2 \\ 0 & a_3 & \phi(C_3) & \phi(A_3)\phi(C_2) \\ 0 & a_1 & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$- 4r \begin{bmatrix} \phi(A_3) \\ \phi(A_1) \end{bmatrix} - 4r \begin{bmatrix} \phi(B_3) & \phi(B_2) \end{bmatrix}$$

$$= r \begin{bmatrix} 0 & N_{31} & N_{21} \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$- 4r \begin{bmatrix} \phi(A_3) \\ \phi(A_1) \end{bmatrix} - 4r \begin{bmatrix} \phi(B_3) & \phi(B_2) \end{bmatrix}$$

$$+ 3r \begin{bmatrix} \phi(A_3) \\ \phi(A_1) \end{bmatrix} + 3r \begin{bmatrix} \phi(B_3) & \phi(B_2) \end{bmatrix} + p + q$$

$$= r \begin{bmatrix} 0 & N_{31} & N_{21} \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$- 4r \begin{bmatrix} A_3 \\ A_1 \end{bmatrix} - 4r \begin{bmatrix} B_3 & B_2 \end{bmatrix} + p + q.$$

Similarly, we can obtain the following

$$r \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & N_{21} \\ M_{31} & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(A_1)\phi(C_2) \end{bmatrix}$$

$$- 4r \begin{bmatrix} A_3 \\ A_1 \end{bmatrix} - 4r(B_2) + p + q,$$

$$(2.16) \qquad r \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & N_{31} & N_{21} \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$- 4r[B_3, B_2] - 4r(A_1) + p + q.$$

Substituting (2.15)-(2.16) into (2.13) and (2.14) yields (2.10) and (2.11) for $i = 1$.

Now we consider the real and complex solutions to (1.3) over $\mathbb{H}$.

Theorem 2.6. *Let system* (1.3) *over* $\mathbb{H}$ *be consistent. Then we have the following:*
(a) System (1.3) *has a real solution if and only if*

$$
r \begin{bmatrix} -L_{1i} & M_{11} \end{bmatrix} + r \begin{bmatrix} -L_{21} \\ N_{2i} \end{bmatrix} + r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}
$$

(2.17)

$$
= r \begin{bmatrix} 0 & N_{2i} \\ M_{31} & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(A_1)\phi(C_2) \end{bmatrix} + r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix},
$$

$i = 2, 3, 4$. *In that case, the real solution of* (1.3) *can be expressed as* $X = X_1$ *in* (2.3).
(b) System (1.3) *has a complex solution if and only if*

$$
r \begin{bmatrix} -L_{1i} & M_{11} \end{bmatrix} + r \begin{bmatrix} -L_{21} \\ N_{2i} \end{bmatrix} + r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}
$$

(2.18)

$$
= r \begin{bmatrix} 0 & N_{2i} \\ M_{31} & \phi(A_3)\phi(C_2) \\ M_{11} & \phi(A_1)\phi(C_2) \end{bmatrix} + r \begin{bmatrix} 0 & N_{3i} & N_{2i} \\ M_{11} & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}
$$

hold when $i = 3, 4$ *or* $i = 2, 4$ *or* $i = 2, 3$. *In that case, the complex solutions of* (1.3) *can be expressed as* $X = X_1 + X_2 i$ *or* $X = X_1 + X_3 j$ *or* $X = X_1 + X_4 k$, *where* X_1, X_2, X_3 *and* X_4 *are expressed as* (2.3), (2.4), (2.5) *and* (2.6), *respectively.*

Proof. From (2.11) in Theorem 2.5, we can get the necessary and sufficient conditions and expressions for $X_i = 0, i = 1, 2, 3, 4$. $\qquad\square$

3. Solvability conditions for real and complex solutions to (1.4) over $\mathbb{H}$

In this section, using the results of Theorem 2.2, Theorem 2.5 and Theorem 2.6, we give necessary and sufficient conditions for (1.4) over $\mathbb{H}$ to have real and complex solutions.

The following lemma is due to Tian [25], which can be generalized to $\mathbb{H}$.

Lemma 3.1. *Let $A \in \mathbb{H}^{m \times n}$, $B_1 \in \mathbb{H}^{m \times p_1}$, $B_3 \in \mathbb{H}^{m \times p_3}$, $B_4 \in \mathbb{H}^{m \times p_4}$, $C_2 \in \mathbb{H}^{q_2 \times n}$, $C_3 \in \mathbb{H}^{q_3 \times n}$ and $C_4 \in \mathbb{H}^{q_4 \times n}$ be given. Then the matrix equation*

$$B_1 X_1 + X_2 C_2 + B_3 X_3 C_3 + B_4 X_4 C_4 = A$$

is consistent if and only if

$$r \begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix},$$

$$r \begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix},$$

$$r \begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

$$r \begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix}.$$

Theorem 3.2. *Let $A_1, A_2, A_3 \in \mathbb{H}^{m \times p}$, $B_1, B_2, B_3 \in \mathbb{H}^{q \times n}$ and $C_1, C_2, C_3 \in \mathbb{H}^{m \times n}$ be known, and suppose that the system (1.3) and the matrix equation $A_4 Y B_4 = C_4$ are solvable, where $X, Y \in \mathbb{H}^{p \times q}$ unknown. Then*
(a) The system (1.4) over $\mathbb{H}$ has a real solution if and only if (2.17) holds and

$$(3.1) \qquad r \begin{bmatrix} 0 & N_{4i} \\ M_{41} & \phi(C_4) \end{bmatrix} = r(M_{41}) + r(N_{4i}), \ i = 2, 3, 4,$$

$$(3.2) \quad r \begin{bmatrix} 0 & N_{41} & 0 \\ 0 & N_{411} & \phi(B_2) \\ M_{41} & \phi(C_4) & \phi(A_4)\phi(C_2) \end{bmatrix} = r(M_{41}) + r \begin{bmatrix} N_{41} & 0 \\ N_{411} & \phi(B_2) \end{bmatrix},$$

$$(3.3) \quad r \begin{bmatrix} 0 & 0 & N_{41} \\ M_{41} & M_{411} & \phi(C_4) \\ 0 & \phi(A_1) & \phi(C_1)\phi(B_4) \end{bmatrix} = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r(N_{41}),$$

$$r \begin{bmatrix} 0 & 0 & N_{41} & 0 & 0 \\ 0 & 0 & N_{411} & \phi(B_3) & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 \\ 0 & \phi(A_3) & 0 & \phi(C_3) & \phi(A_3)\phi(C_2) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$(3.4) \qquad = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_3) \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{41} & 0 & 0 \\ N_{411} & \phi(B_3) & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{41} & 0 \\ 0 & 0 & N_{411} & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 \\ 0 & \phi(A_1) & 0 & \phi(A_1)\phi(C_2) \end{bmatrix}$$

$$(3.5) \qquad = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{41} & 0 \\ N_{411} & \phi(B_2) \end{bmatrix}.$$

(b) The system (1.4) over $\mathbb{H}$ has a complex solution if and only if (2.18) hold when $i = 3, 4$, (3.2)-(3.5) holds and

$$(3.6) \qquad r \begin{bmatrix} 0 & N_{4i} \\ M_{41} & \phi(C_4) \end{bmatrix} = r(M_{41}) + r(N_{4i}), \ i = 3, 4,$$

$$r \begin{bmatrix} 0 & N_{42} & 0 \\ 0 & N_{422} & \phi(B_2) \\ M_{41} & \phi(C_4) & \phi(A_4)\phi(C_2) \end{bmatrix} = r(M_{41}) + r \begin{bmatrix} N_{42} & 0 \\ N_{422} & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{42} \\ M_{41} & M_{411} & \phi(C_4) \\ 0 & \phi(A_1) & \phi(C_1)\phi(B_4) \end{bmatrix} = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r(N_{42}),$$

$$r \begin{bmatrix} 0 & 0 & N_{42} & 0 & 0 \\ 0 & 0 & N_{422} & \phi(B_3) & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 \\ 0 & \phi(A_3) & 0 & \phi(C_3) & \phi(A_3)\phi(C_2) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_3) \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{42} & 0 & 0 \\ N_{422} & \phi(B_3) & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{42} & 0 \\ 0 & 0 & N_{422} & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 \\ 0 & \phi(A_1) & 0 & \phi(A_1)\phi(C_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{42} & 0 \\ N_{422} & \phi(B_2) \end{bmatrix},$$

or

(2.18) *holds when* $i = 2, 4$, (3.2)-(3.5) *hold and,*

(3.7) $\qquad r \begin{bmatrix} 0 & N_{4i} \\ M_{41} & \phi(C_4) \end{bmatrix} = r(M_{41}) + r(N_{4i}), \ i = 2, 4,$

$$r \begin{bmatrix} 0 & N_{43} & 0 \\ 0 & N_{433} & \phi(B_2) \\ M_{41} & \phi(C_4) & \phi(A_4)\phi(C_2) \end{bmatrix} = r(M_{41}) + r \begin{bmatrix} N_{43} & 0 \\ N_{433} & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{43} \\ M_{41} & M_{411} & \phi(C_4) \\ 0 & \phi(A_1) & \phi(C_1)\phi(B_4) \end{bmatrix} = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r(N_{43}),$$

$$r \begin{bmatrix} 0 & 0 & N_{43} & 0 & 0 \\ 0 & 0 & N_{433} & \phi(B_3) & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 \\ 0 & \phi(A_3) & 0 & \phi(C_3) & \phi(A_3)\phi(C_2) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_3) \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{43} & 0 & 0 \\ N_{433} & \phi(B_3) & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{43} & 0 \\ 0 & 0 & N_{433} & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 \\ 0 & \phi(A_1) & 0 & \phi(A_1)\phi(C_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{43} & 0 \\ N_{433} & \phi(B_2) \end{bmatrix},$$

or
(2.18) *holds when* $i = 2, 3$, *(3.2)-(3.5) hold and*

$$(3.8) \qquad r \begin{bmatrix} 0 & N_{4i} \\ M_{41} & \phi(C_4) \end{bmatrix} = r(M_{41}) + r(N_{4i}), \ \ i = 2, 3,$$

$$r \begin{bmatrix} 0 & N_{44} & 0 \\ 0 & N_{444} & \phi(B_2) \\ M_{41} & \phi(C_4) & \phi(A_4)\phi(C_2) \end{bmatrix} = r(M_{41}) + r \begin{bmatrix} N_{44} & 0 \\ N_{444} & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{44} \\ M_{41} & M_{411} & \phi(C_4) \\ 0 & \phi(A_1) & \phi(C_1)\phi(B_4) \end{bmatrix} = r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r(N_{44}),$$

$$r \begin{bmatrix} 0 & 0 & N_{44} & 0 & 0 \\ 0 & 0 & N_{444} & \phi(B_3) & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 \\ 0 & \phi(A_3) & 0 & \phi(C_3) & \phi(A_3)\phi(C_2) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi(B_3) & \phi(C_1)\phi(B_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_3) \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{44} & 0 & 0 \\ N_{444} & \phi(B_3) & \phi(B_2) \end{bmatrix},$$

$$r \begin{bmatrix} 0 & 0 & N_{444} & 0 \\ 0 & 0 & N_{444} & \phi(B_2) \\ M_{41} & M_{411} & \phi(C_4) & 0 \\ 0 & \phi(A_1) & 0 & \phi(A_1)\phi(C_2) \end{bmatrix}$$

$$= r \begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_1) \end{bmatrix} + r \begin{bmatrix} N_{44} & 0 \\ N_{444} & \phi(B_2) \end{bmatrix},$$

where

$$M_{41} = \begin{bmatrix} A_{42} & A_{43} & A_{44} \\ A_{41} & A_{44} & -A_{43} \\ -A_{44} & A_{41} & A_{42} \\ A_{43} & -A_{42} & A_{41} \end{bmatrix}, \quad M_{411} = \begin{bmatrix} A_{21} & 0 & 0 & 0 \\ -A_{22} & 0 & 0 & 0 \\ -A_{23} & 0 & 0 & 0 \\ -A_{24} & 0 & 0 & 0 \end{bmatrix},$$

$$N_{41} = \begin{bmatrix} -B_{42} & B_{41} & B_{44} & -B_{43} \\ -B_{43} & -B_{44} & B_{41} & B_{42} \\ -B_{44} & B_{43} & -B_{42} & B_{41} \end{bmatrix},$$

$$N_{411} = \begin{bmatrix} B_{41} & B_{42} & B_{43} & B_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_{42} = \begin{bmatrix} B_{41} & B_{42} & B_{43} & B_{44} \\ -B_{43} & -B_{44} & B_{41} & B_{42} \\ -B_{44} & B_{43} & -B_{42} & B_{41} \end{bmatrix},$$

$$N_{422} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -B_{42} & B_{41} & B_{44} & -B_{43} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_{43} = \begin{bmatrix} B_{41} & B_{42} & B_{43} & B_{44} \\ -B_{42} & B_{41} & B_{44} & -B_{43} \\ -B_{44} & B_{43} & -B_{42} & B_{41} \end{bmatrix},$$

$$N_{433} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -B_{43} & -B_{44} & B_{41} & B_{42} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_{44} = \begin{bmatrix} B_{41} & B_{42} & B_{43} & B_{44} \\ -B_{42} & B_{41} & B_{44} & -B_{43} \\ -B_{43} & -B_{44} & B_{41} & B_{42} \end{bmatrix},$$

$$N_{444} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -B_{44} & B_{43} & -B_{42} & B_{41} \end{bmatrix}.$$

Proof. From Theorem 2.6, the system (1.3) over $\mathbb{H}$ has a real solution if and only if (2.17) hold. By (2.3), the real solutions of (1.3) over $\mathbb{H}$ can be expressed as

$$
X_1 = \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 + \frac{1}{4}P_3\phi(X_0)Q_3 + \frac{1}{4}P_4\phi(X_0)Q_4
$$

$$
+ [P_1, P_2, P_3, P_4] L_{\phi(A_1)}(L_{\phi(K)}Z + WR_{\phi(H)})R_{\phi(B_2)}
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},
$$

where Z and W are arbitrary matrices with compatible sizes.

Let $A_1, C_1 = 0$, $B_2, C_2 = 0$, $A_3 = A_4, B_3 = B_4$ and $C_3 = C_4$ in Theorem 2.6 and (2.3). It is easy to verify that the quaternion matrix equation $A_4 Y B_4 = C_4$ has a real solution if and only if (3.1) holds and the real solutions can be expressed as

$$
Y_1 = \frac{1}{4}P_1\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_1 + \frac{1}{4}P_1\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_1
$$

$$
+ \frac{1}{4}P_3\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_3 + \frac{1}{4}P_4\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_4
$$

$$
+ [P_1, P_2, P_3, P_4] L_{\phi(A_4)}U + VR_{\phi(B_4)}
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},
$$

where U and V are arbitrary matrices with compatible sizes. Let

$$
[P_1, P_2, P_3, P_4] = P, \quad \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = Q,
$$

$$
G = \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 + \frac{1}{4}P_3\phi(X_0)Q_3 + \frac{1}{4}P_4\phi(X_0)Q_4
$$

$$
- \frac{1}{4}P_1\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_1 - \frac{1}{4}P_1\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_1
$$

$$
- \frac{1}{4}P_3\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_3 - \frac{1}{4}P_4\phi(A_4)^-\phi(C_4)\phi(B_4)^-Q_4.
$$

Equating X_1 and Y_1, we obtain the following equation

$$G = [P_1, P_2, P_3, P_4] L_{\phi(A_4)} U + V R_{\phi(B_4)} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

$$(3.9) \qquad - [P_1, P_2, P_3, P_4] L_{\phi(A_1)} (L_{\phi(K)} Z + W R_{\phi(H)}) R_{\phi(B_2)} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

We know by Lemma 3.1 that (3.9) is solvable if and only if the following four rank equalities hold

$$(3.10) \qquad r \begin{bmatrix} G & PL_{\phi(A_4)} \\ R_{\phi(B_4)}Q & 0 \\ R_{\phi(B_2)}Q & 0 \end{bmatrix} = r \begin{bmatrix} 0 & PL_{\phi(A_4)} \\ R_{\phi(B_4)}Q & 0 \\ R_{\phi(B_2)}Q & 0 \end{bmatrix},$$

$$r \begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)} \\ R_{\phi(B_4)}Q & 0 & 0 \end{bmatrix}$$

$$(3.11) \qquad = r \begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)} \\ R_{\phi(B_4)}Q & 0 & 0 \end{bmatrix},$$

$$r \begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(K)} \\ R_{\phi(B_4)}Q & 0 & 0 \\ R_{\phi(H)}R_{\phi(B_2)}Q & 0 & 0 \end{bmatrix}$$

$$(3.12) \qquad = r \begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(K)} \\ R_{\phi(B_4)}Q & 0 & 0 \\ R_{\phi(H)}R_{\phi(B_2)}Q & 0 & 0 \end{bmatrix},$$

$$r \begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)} \\ R_{\phi(B_4)}Q & 0 & 0 \\ R_{\phi(B_2)}Q & 0 & 0 \end{bmatrix}$$

$$(3.13) \qquad = r \begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)} \\ R_{\phi(B_4)}Q & 0 & 0 \\ R_{\phi(B_2)}Q & 0 & 0 \end{bmatrix}.$$

Under the conditions that the system (1.3) and the matrix equation $A_4 Y B_4 = C_4$ over $\mathbb{H}$ are solvable, it is not difficult to show by Lemma 2.3 and block Gaussian elimination that (3.10)-(3.13) are equivalent to the four rank equalities (3.2)-(3.5), respectively. Note that the processes

are too much tedious, and we omit them here. Obviously, system (1.3) and the matrix equation $A_4YB_4 = C_4$ over $\mathbb{H}$ have a common real solution if and only if (3.2))-(3.5) hold. Thus, system (1.4) over $\mathbb{H}$ has a real solution if and only if (2.17) and (3.1)-(3.5) hold.

Similarly, from Theorem 2.6, we know that the system (1.3) over $\mathbb{H}$ has a complex solution if and only if (2.18) holds when $i = 3, 4$ or $i = 2, 4$ or $i = 2, 3$, its complex solutions can be expressed as $X = X_1 + X_2i$ or $X = X_1 + X_3j$ or $X = X_1 + X_4k$. The quaternion matrix equation $A_4YB_4 = C_4$ has a complex solution if and only if (3.6) or (3.7) or (3.8) hold, its complex solution can be expressed as $Y = Y_1 + Y_2i$ or $Y = Y_1+Y_3j$ or $Y = Y_1+Y_4k$. By equating X_1 and Y_1, X_2 and Y_2, X_3 and Y_3, X_4 and Y_4, respectively, we can derive the necessary and sufficient conditions for the system (1.4) over $\mathbb{H}$ to have a complex solution. $\square$

Remark 3.3. *The results of [23] can be regarded as the special cases of this paper.*

Acknowledgments

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