TOPOLOGICALLY LEFT INVARIANT MEAN ON DUAL SEMIGROUP ALGEBRAS

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ABSTRACT. Let $S$ be a locally compact Hausdorff semitopological semigroup, and $M(S)$ be the Banach algebra of all bounded regular Borel measures on $S$. In this paper, we obtain a necessary and sufficient condition for $M(S)^*$ to have a topologically left invariant mean.

1. Introduction

Let $S$ be a locally compact Hausdorff semitopological semigroup with convolution measure algebra $M(S)$ and probability measures $M_p(S)$. We know that $M(S)$ is a Banach algebra with total variation norm and convolution, so we can define the first Arens product on $M(S)^{**}$, i.e. for $F,G \in M(S)^{**}$ and $f \in M(S)^*$

$$\langle FG, f \rangle = \langle F, Gf \rangle, \quad \langle Gf, \mu \rangle = \langle G, f\mu \rangle, \quad \langle f\mu, \nu \rangle = \langle f, \mu^* \nu \rangle$$

where $\mu, \nu \in M(S)$. On a Banach algebra $A$ a functional $f \in A^*$ is called weakly almost periodic if $W(f) = \{fa; a \in A, ||a|| \leq 1\}$ is relatively weakly compact in $A^*$ where $\langle fa, b \rangle = \langle f, ab \rangle$ for all $a, b \in A$ [8]. We denote by $wap(M(S))$ the set of all weakly almost periodic functionals on $M(S)$. Clearly $1 : M(S) \to \mathbb{C}$ given by $\langle 1, \mu \rangle = \mu(S)$ is weakly almost periodic. A functional $M \in M(S)^{**}$ (respectively $M \in wap(M(S))$) is called a mean on $M(S)^*$ (respectively on $wap(M(S))$) if $||M|| = \langle M, 1 \rangle = 1$, and $\langle M, f \rangle \geq 0$ where $f \in M(S)^*$ (respectively $f \in wap(M(S))$) and $f \geq 0$ ([10],

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A mean \( M \) is said to be a topologically left invariant if
\[
\langle M, f\mu \rangle = \langle M, f \rangle \quad \text{where} \quad f \in M(S)^* \quad \text{(respectively} \quad f \in \text{wap}(M(S)))
\]
and \( \mu \in M_o(S) \).

Wong studied topologically left invariant mean on \( M(S)^* \) and proved that \( M(S)^* \) has a topologically left invariant mean if and only if there is a net \( (\mu_\alpha) \) in \( M_o(S) \) such that \( \|\mu * \mu_\alpha - \mu_\alpha\| \to 0 \) \( (\mu \in M_o(S)) \)[12].

Also, Day \([2], [3]\) and Junghenn \([5]\) have studied topologically left invariant mean on \( M(S)^* \).

For a locally compact group \( G \), Wong \([13]\) has shown, there is a net \( (\mu_\alpha) \) in \( M_o(G) \) such that \( \|\mu * \mu_\alpha - \mu_\alpha\| \to 0 \) for all \( \mu \in M_o(G) \) if and only if there is a net \( (\mu_\alpha) \) in \( M_o(G) \) such that for all compact subset \( K \) of \( G \), \( \|\mu * \mu_\alpha - \mu_\alpha\| \to 0 \) uniformly over all \( \mu \) in \( M_o(G) \) which are supported in \( K \). But for a semigroup \( S \) this matter is not known.

In this paper, among other things, we will show that if \( M_o(S) \) has a measure \( \nu \) such that the map \( s \mapsto \delta_s * \nu \) from \( S \) into \( M(S) \) is continuous, then the last statement is valid. In fact with this condition we provide an answer to a problem raised by Lau \([7], \text{p. 322}\) and Day \([3]\).

### 2. Topologically left invariant mean

Suppose \( S \) is a locally compact Hausdorff semitopological semigroup. By the Eberlein-Smulian theorem \( \text{wap}(M(S)) \) is a Banach subspace of \( M(S)^* \). It is easy to see that for every \( f \in M(S)^* \), \( \{f\mu; \mu \in M(S), ||\mu|| \leq 1\} \) is relatively weakly compact if and only if \( \{\mu f; \mu \in M(S), ||\mu|| \leq 1\} \) is relatively weakly compact. So, if \( f \in \text{wap}(M(S)) \) then \( \{\mu f; \mu \in M(S), ||\mu|| \leq 1\} \) is relatively weakly compact. Lashkarizadeh in \([6]\) has proved \( \text{wap}(S) \subseteq \text{wap}(M(S)) \), where \( \text{wap}(S) = \{f \in C(S); \{L_s f, s \in S\} \text{ is relatively weakly compact in } C(S)\} \). Also he has shown that, if \( S \) is a foundation topological semigroup with identity, then \( \text{wap}(L(S)) = \text{wap}(S) \).

We recall that a semigroup \( S \) is said to be left amenable if there exists \( m \in B(S)^* \) such that \( m \geq 0, ||m|| = 1 \) and \( \langle m, L_s f \rangle = \langle m, f \rangle \) for all \( s \in S \) and all \( f \in B(S) \), where \( B(S) \) is the set of all bounded complex valued functions on \( S \) \([1]\). In the following Theorem, we give conditions on \( S \) and \( M_o(S) \) that are sufficient to guarantee topologically left amenability of \( \text{wap}(M(S)) \).
Theorem 2.1. Let $S$ be a locally compact Hausdorff semitopological semigroup. $\text{wap}(M(S))$ has a topologically left invariant mean if any one of the following conditions holds:

1. $S$ is left amenable and there exists $\nu \in M_1(S)$ such that the map $s \mapsto \delta_s * \nu$ from $S$ into $M(S)$ is weakly continuous and $\delta_s * \nu = \nu * \delta_s$ ($s \in S$).

2. $S$ has an identity $e$ and $X$ ($X$ is the set of all means on $\text{wap}(M(S))$) with $\sigma(X, \text{wap}(M(S)))$ topology) has a dense subset $Y$ such that $\delta_e \in Xy$ for all $y \in Y$.

Proof. We start by showing that for $f \in \text{wap}(M(S))$ the function $s \mapsto \langle f \nu, \delta_s \rangle$ is a continuous function on $S$. Indeed this is easy, because $\langle f \nu, \delta_s \rangle = \langle f, \nu * \delta_s \rangle = \langle f, \delta_s * \nu \rangle$. Notice too that if also $t \in S$, then $\langle f(\nu * \delta_s), \delta_t \rangle = \langle f \nu, \delta_s * \delta_t \rangle$. The iterated limit condition (or lots of other methods) now shows that $s \mapsto \langle f \nu, s \rangle$ is in $\text{wap}(S)$. This means in particular that if $M$ is a left invariant mean on $B(S)$ (and in fact we only need on $\text{wap}(S)$) then $\langle M, f \nu \rangle$ is well-defined. Let $M_1$ be any continuous linear extension of $M$ from $\text{wap}(S)$ to $\text{wap}(M(S))$. We claim that $\nu M_1$ is a left invariant mean on $\text{wap}(M(S))$. Indeed for $f \in \text{wap}(M(S))$,

$$\langle \delta_s(\nu M_1), f \rangle = \langle M_1, (f \delta_s) \nu \rangle = \langle M, f(\delta_s * \nu) \rangle = \langle M, f(\nu * \delta_s) \rangle$$

$$= \langle \delta_s M, f \nu \rangle = \langle M, f \nu \rangle = \langle M_1, f \nu \rangle = \langle \nu M_1, f \rangle.$$

To see that this is topologically left invariant, we simply integrate, using the fact the function $s \mapsto \langle \nu M_1, f \delta_s \rangle = \langle M_1 f, \delta_s * \nu \rangle$ is continuous:

$$\langle \mu(\nu M_1), f \rangle = \langle \nu M_1, f \mu \rangle = \int \langle \nu M_1, f \delta_s \rangle d\mu = \int \langle \nu M_1, f \rangle d\mu = \langle \nu M_1, f \rangle.$$

2) Let $(M_\alpha)$ be a net in $X$, $M \in X$ and $M_\alpha \to M$ in the $\sigma(X, \text{wap}(M(S)))$ topology. If $M_1 \in X$, $f \in \text{wap}(M(S))$, since $\{Mf; M \in X\}$ is relatively compact (of course we can define the first Arens product on $\text{wap}(M(S))$) and so $Mf$ is well defined), for every subnet $(M_\beta)$ of $(M_\alpha)$ there is a subnet $(M_\gamma)$ of $(M_\beta)$ such that $M_\gamma f \to Mf$ in the weak topology. Hence $\langle M_1 M_\gamma, f \rangle \to \langle M_1 M, f \rangle$. Consequently $\langle M_1 M_\alpha, f \rangle \to \langle M_1 M, f \rangle$, i.e. $X$ is a semitopological semigroup (in the $\sigma(X, \text{wap}(M(S)))$ topology). Now, let $E$, $E_1$ be two idempotents lying in the same minimal left ideal. Since $EE_1 = E$, by assumption and an argument similar to the proof in ([1], Theorem...
1.5.9), we have $E = E_1$. So the minimal left ideals of $X$ are groups. On the other hand the minimal left ideals of $X$ are affine. By Corollary 1.3.23 in [1] every minimal idempotent is right zero. Therefore if $E$ is a minimal idempotent, it is clear that $E$ is a topologically left invariant mean on $wap(M(S))$.

**Theorem 2.2.** Let $M_o(S)$ contains a measure $\nu$ such that the map $s \rightarrow \delta_s * \nu$ from $S$ into $M(S)$ is continuous when $M(S)$ has the norm topology. Then the following statements are equivalent:

1. $M(S)^*$ has a topologically left invariant mean.
2. There is a net $(\mu_\alpha)$ in $M_o(S)$ such that for every compact subset $K$ of $S$, $\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$ uniformly over all $\mu$ in $M_o(S)$ which are supported in $K$.
3. For every finite subset $\Omega$ of $M_o(S)$, $\epsilon > 0$, there exists a measure $\mu \in M_o(S)$ such that $\|\nu * \mu - \mu\| < \epsilon/2$ for all $\nu \in \Omega$.

**Proof.** Reader could point out that the hypothesis about the continuity of $s \rightarrow \delta_s * \nu$ is needed only for (1) $\rightarrow$ (2). Let $M(S)^*$ has a topologically left invariant mean. Then by ([12], Theorem 3.1), there is a net $(\nu_\alpha)$ in $M_o(S)$ such that $\|\mu * \nu_\alpha - \mu_\alpha\| \rightarrow 0$ for all $\mu \in M_o(S)$. Now, let $\nu \in M_o(S)$ and the map $s \rightarrow \delta_s * \nu$ be continuous. For all $\alpha$, we take $\mu_\alpha = \nu * \nu_\alpha$. It is easy to see that $\|\mu * \mu_\alpha - \mu_\alpha\| \rightarrow 0$ ($\mu \in M_o(S)$) and $s \rightarrow \delta_s * \mu_\alpha$ is continuous. So, if $K$ is a compact subset of $S$ and $\epsilon > 0$, then for $s \in K$ there is a neighbourhood $U_s$ of $s$ such that for every $\alpha$ and $t \in U_s$

$$\|\delta_t * \mu_\alpha - \delta_s * \mu_\alpha\| < \epsilon/2.$$  

But $K$ is a compact subset of $S$, hence we can choose a finite subset $\{s_1, s_2, ..., s_n\}$ of $K$ which $K \subseteq \bigcup_{j=1}^n U_{s_j}$. Also, we can find an $\alpha_0$ such that for every $\alpha \geq \alpha_0$ and $1 \leq i \leq n$,

$$\|\delta_{s_i} * \mu_\alpha - \mu_\alpha\| < \epsilon/2.$$  

Let $A_1 = U_{s_1}$, $A_i = U_{s_i} \setminus \bigcup_{j=1}^{i-1} U_{s_j}$, $2 \leq i \leq n$, and $\mu \in M_o(S)$. Since for every $\alpha$ the map $s \rightarrow \delta_s * \mu_\alpha$ is continuous, therefore by ([11], Chapter 3), $f_K \delta_s * \mu_\alpha d\mu(s) \in M(S)$ and $f_K \delta_s * \mu_\alpha d\mu(s) = \mu \chi_K * \mu_\alpha$. Consequently for $\mu \in M_o(S)$ with $\text{supp} \mu \subseteq K$ and $f \in M(S)^*$, we
can write

\[ |\langle f, \mu \ast \mu_\alpha \rangle - \langle f, \mu_\alpha \rangle| = \left| \sum_{i=1}^{n} \int_{A_i} \langle f, \delta_s \ast \mu_\alpha \rangle - \langle f, \mu_\alpha \rangle d\mu(s) \right| \]

\[ \leq \sum_{i=1}^{n} \int_{A_i} |\langle f, \delta_s \ast \mu_\alpha \rangle - \langle f, \delta_{s_i} \ast \mu_\alpha \rangle| d\mu(s) \]

\[ + \sum_{i=1}^{n} |\langle f, \delta_{s_i} \ast \mu_\alpha \rangle - \langle f, \mu_\alpha \rangle| \mu(A_i) \leq \]

\[ \sum_{i=1}^{n} \int_{A_i} ||f|| ||\delta_s \ast \mu_\alpha - \delta_{s_i} \ast \mu_\alpha|| d\mu(s) \]

\[ + \sum_{i=1}^{n} ||f|| ||\delta_{s_i} \ast \mu_\alpha - \mu_\alpha|| \mu(A_i) \leq ||f|| \epsilon \]

where \( \alpha \geq \alpha_0 \). So (1) implies (2).

(2) implies (3) is easy. Now, assume that (3) holds. For every finite subset \( \Omega \) of \( M_\circ(S) \) and \( \epsilon > 0 \), we associate the nonvoid subset \( A_{\Omega,\epsilon} = \{ \eta \in M_\circ(S); ||\mu \ast \eta - \eta|| < \epsilon \text{ for all } \mu \in \Omega \} \). Since the family \( \{A_{\Omega,\epsilon}; \Omega \text{ is a finite subset of } M_\circ(S) \text{ and } \epsilon > 0\} \) has the finite intersection property, so there exists \( M \in M(S)^{**} \) such that \( M \in \bigcap_{\Omega,\epsilon} \text{weak}^* \)-closure \( A_{\Omega,\epsilon} \). Now let \( f \in M(S)^* \) with \( ||f|| = 1 \) and \( \mu \in M_\circ(S) \). For \( \epsilon > 0 \), there exists \( \eta \in A_{\{\mu\},\epsilon/3} \) such that \( |\langle \eta, f \rangle - \langle M, f \rangle| < \epsilon/3 \) and \( |\langle \eta, f \mu \rangle - \langle M, f \mu \rangle| < \epsilon/3 \). So

\[ |\langle M, f \rangle - \langle M, f \mu \rangle| \leq |\langle M, f \rangle - \langle \eta, f \rangle| + |\langle \eta, f \rangle - \langle \mu \ast \eta, f \rangle| \]

\[ + |\langle \mu \ast \eta, f \rangle - \langle M, f \mu \rangle| < \epsilon. \]

Therefore \( \langle M, f \rangle = \langle M, f \mu \rangle \). It is trivial that \( M \) is a mean, and so (1) holds. This completes our proof.

Let \( S \) be a topological semigroup with identity. We define \( L(S) = \{ \mu \in M(S); s \rightarrow |\mu| \ast \delta_s \text{ and } s \rightarrow |\mu| \ast \delta_s \text{ are weakly continuous} \} \). In the following theorem, we may assume that \( S \) is a locally compact Hausdorff foundation topological semigroup, i.e. \( \bigcup \{ \text{supp } \mu; \mu \in L(S) \} \) is dense in \( S \). It is well known that \( L(S) \) is an ideal in \( M(S) \) and has an approximate identity [6]. We also note that for \( \mu \in L(S) \) both mapping \( x \rightarrow |\mu| \ast \delta_x \) and \( x \rightarrow \delta_x \ast |\mu| \) from \( S \) into \( M(S) \) are norm continuous [4].
Theorem 2.3. Let $S$ be a foundation topological semigroup with identity. Then the following are equivalent:

1. $M(S)^*$ has a topologically left invariant mean.
2. There is a net $(\nu_\beta)$ in $M_0(S)$ with finite support such that for all $\mu \in M_0(S)$ and $\nu \in L(S)$, $||\mu * \nu_\beta * \nu - \nu_\beta * \nu|| \to 0$.
3. There is a net $(\nu_\beta)$ in $M_0(S)$ with finite support such that for all compact subset $K$ of $S$ and $\nu \in L(S)$, $||\mu * \nu_\beta * \nu - \nu_\beta * \nu|| \to 0$ uniformly over all $\mu$ in $M_0(S)$ which are supported in $K$.

Proof. Let $M(S)^*$ has a topologically left invariant mean. Then there is a net $(\gamma_\alpha)$ in $M_0(S)$ such that $||\mu * \gamma_\alpha - \gamma_\alpha|| \to 0$ ([12], Theorem 3.1). Now, if $\eta \in M_0(S) \cap L(S)$, we take $\mu_\alpha = \gamma_\alpha * \eta$ (for all $\alpha$). Since $L(S)$ is an ideal in $M(S)$, so $\mu_\alpha \in L(S)$. Let $\epsilon > 0$ be given. For all $\alpha$, we choose $\eta_\alpha \in M_0(S) \cap L(S)$ with compact support and $||\eta_\alpha - \mu_\alpha|| < \epsilon/4$. On the other hand, $L(S)$ has a positive approximate identity of norm one ([6], Lemma 3.4). So there is a $\xi_\alpha \in M_0(S) \cap L(S)$ such that $||\eta_\alpha - \mu_\alpha * \xi_\alpha|| < \epsilon/2$.

For $s \in S$, there exists a neighbourhood $U_s$ of $s$ such that $||\delta_s * \xi_\alpha - \delta_t * \xi_\alpha|| < \epsilon/2$ ($t \in U_s$). But $supp \eta_\alpha$ is compact, hence we can find a finite subset $\{s_1, s_2, ..., s_n\}$ of $S$ with $supp \eta_\alpha \subseteq \bigcup_{i=1}^{n} U_{s_i}$. If $A_1 = U_{s_1}$ and $A_i = U_{s_i} \setminus \bigcup_{j=1}^{i-1} U_{s_j}$, $2 \leq i \leq n$, we define $\nu_\alpha = \sum_{i=1}^{n} \eta_\alpha(A_i)d_{s_i}$. So for $f \in L(S)^*$, $||f|| \leq 1$, we have

$$|\langle f, \eta_\alpha * \xi_\alpha \rangle - \langle f, \nu_\alpha * \xi_\alpha \rangle| = \left| \int (f, \delta_s * \xi_\alpha) - \langle f, \nu_\alpha * \xi_\alpha \rangle d\eta_\alpha(s) \right|$$

$$\leq \sum_{i=1}^{n} \int_{A_i} \left| \langle f, \delta_s * \xi_\alpha \rangle - \langle f, \delta_{s_i} * \xi_\alpha \rangle \right| d\eta_\alpha(s) < \epsilon/2.$$ 

Therefore $||\eta_\alpha * \xi_\alpha - \nu_\alpha * \xi_\alpha|| < \epsilon/2$ and $||\mu_\alpha * \xi_\alpha - \nu_\alpha * \xi_\alpha|| < \epsilon$. Consequently, since for all $\mu \in M_0(S)$, $||\mu * \mu_\alpha - \mu_\alpha|| \to 0$, we may therefore determine a net $(\nu_\beta)$ in $M_0(S)$ with finite support and a net $(\xi_\beta)$ as an approximate identity in $L(S)$ such that $||\mu * \nu_\beta * \xi_\beta - \nu_\beta * \xi_\beta|| \to 0$ for all $\mu \in M_0(S)$. Hence it is easy to see that $||\mu * \nu_\beta * \nu - \nu_\beta * \nu|| \to 0$ for all $\nu \in L(S)$ and $\mu \in M_0(S)$. So (1) implies (2).

If (2) holds, an argument similar to the proof of Theorem 2.2 implies (3).

By ([12], Theorem 3.1), (3) implies (1).
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References