MODIFIED NOOR ITERATIONS FOR INFINITE FAMILY OF STRICT PSEUDO-CONTRACTION MAPPINGS

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ABSTRACT. We introduce a modified Noor iteration scheme generated by an infinite family of strict pseudo-contractive mappings and prove the strong convergence theorems of the scheme in the framework of q-uniformly smooth and strictly convex Banach space. Results shown here are extensions and refinements of previously known results.

1. Introduction

Let $E$ be a real Banach space, and $K$ be a nonempty closed convex subset of $E$. Recall that a mapping $f : K \to K$ is said to be a contraction on $K$ if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, for all $x, y \in K$. We use $\Pi K$ to denote the collection of all contractions on $K$; that is, $\Pi K = \{f \mid f : K \to K$ is a contraction with constant $\alpha\}$. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. In the sequel, $F(T) = \{x \in K : Tx = x\}$ denotes the fixed point set of $T$. For fixed

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$x_1 \in K$, one classical way to study nonexpansive mappings is to use the following Mann iteration process [1]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad \forall n \geq 1,$$

where, $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$. For the approximation of a fixed point of a nonexpansive mapping, the Mann iteration process is always applicable and has been studied extensively by several authors.

Another iteration process found to be successful for the approximation of a fixed point of a nonexpansive map is the Halpern-type process. Let $K$ be a nonempty closed convex subset of a Hilbert space and $T : K \to K$ be a nonexpansive mapping. For an arbitrary $u \in K$ and any initial value $x_0 \in K$, define a sequence $\{x_n\} \subset K$ in an explicit iterative way by

$$(1.1) \quad x_{n+1} = \alpha_nu + (1 - \alpha_n)Tx_n, \quad n \geq 0.$$ 

In 1967, Halpern [2] proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of $T$ if $\{\alpha_n\}$ satisfies the following conditions: (C1) $\lim_{n \to \infty} \alpha_n = 0$; (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, or equivalently, $\prod_{n=0}^{\infty} (1 - \alpha_n) = 0$. However, it is unclear whether the conditions (C1) and (C2) are sufficient. Consequently, several authors have concentrated to study the convergence of Halpern iteration under a different restriction on the parameter $\{\alpha_n\}$. For example, Lions [3] proved strong convergence of Halpern iteration $\{x_n\}$ defined by (1.1) to a fixed point of $T$ in Hilbert space if $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions:

$$(C1) \lim_{n \to \infty} \alpha_n = 0; \quad (C2) \sum_{n=0}^{\infty} \alpha_n = \infty; \quad (C3) \lim_{n \to \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$ 

It was observed that both Halpern’s and Lions’ conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n + 1)$. This was rectified in 1992 by Wittmann [4], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of $T$ if $\{\alpha_n\}$ satisfies the conditions (C1), (C2) and the condition, (C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Xu [5] (see also [6]) improved Lions’ result in two respects. First, he suggested the following control condition (C5) instead of the conditions (C3) or (C4):

$$(C5) \lim_{n \to \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0, \text{or equivalently, } \lim_{n \to \infty} \frac{\alpha_{n-1}}{\alpha_n} = 1.$$
so that the canonical choice of \( \alpha_n = 1/(n+1) \) is made possible. Second, he proved the strong convergence of Halpern-type process in the framework of real uniformly smooth Banach space. And Xu [6] showed that condition (C3) and condition (C5) are not comparable.

In 2005, Kim and Xu [7] introduced the following iterative algorithm:

\[
\begin{align*}
  x_0 &= x \in K \text{ chosen arbitrarily}, \\
  y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\
  x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 0,
\end{align*}
\]

where \( T \) is a nonexpansive mapping of \( K \) into itself and \( u \in K \) is a given point. They proved that the sequence \( \{x_n\} \) defined by (1.2) converges strongly to a fixed point of \( T \) provided that the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy the conditions (C1), (C2), (C4) and

\[
\begin{align*}
  (B1) \lim_{n \to \infty} \beta_n &= 0; \\
  (B2) \sum_{n=0}^{\infty} \beta_n &= \infty; \\
  (B3) \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| &< \infty.
\end{align*}
\]

Recently, Yao et al. [8] modified the recursion formula (1.2) to have strong convergence by using the viscosity approximation method. They introduced the following iteration scheme:

\[
\begin{align*}
  x_0 &= x \in K \text{ chosen arbitrarily}, \\
  y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad f \in \Pi_K, \quad n \geq 0.
\end{align*}
\]

They proved that the sequence \( \{x_n\} \) converges strongly to a fixed point of \( T \), where the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy the conditions (C1), (C2) and (B4): \( 0 < \lim inf_{n \to \infty} \beta_n \leq \lim sup_{n \to \infty} \beta_n < 1 \).

Let \( E^* \) denote the dual space of a Banach space \( E \). The generalized duality mapping \( J_q : E \to 2^{E^*} \) is defined by

\[
J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,
\]

where \( q > 1 \) is a real number. In particular, \( J = J_2 \) is said to be normalized duality mapping and \( J_q(x) = \|x\|^{q-2}J_2(x) \) for \( x \neq 0 \). If \( E \) is a Hilbert space, then \( J = I \) (the identity mapping). It is well-known that if \( E \) is smooth, then \( J_q \) is single-valued, which is denoted by \( j_q \).

A mapping \( T \) is said to be a pseudo-contraction, if there exists some \( j_q(x-y) \in J_q(x-y) \) such that

\[
\langle Tx - Ty, j_q(x-y) \rangle \leq \|x-y\|^q, \quad \forall x, y \in K.
\]

A mapping \( T \) is said to be a \( \lambda \)-strict pseudo-contraction in the terminology of Browder and Petryshyn [9], if there exists a constant \( \lambda > 0 \)
such that
\[
(Tx - Ty, j_q(x - y)) \leq \|x - y\|^q - \lambda\|\left((I - T)x - (I - T)y\right)\|^q
\]
(1.4)
for every \(x, y \in K\) and for some \(j_q(x - y) \in J_q(x - y)\).

A mapping \(T\) is said to be a strong pseudo-contraction if there exists \(k \in (0, 1)\) such that \(\langle Tx - Ty, j_q(x - y)\rangle \leq k\|x - y\|^q, \quad \forall x, y \in K\).

It can be proved that if \(T\) is \(\lambda\)-strict pseudo-contraction in the terminology of Browder and Petryshyn, then \(T\) is Lipschitz continuous with the Lipschitz constant \(L = (1 + \lambda)/\lambda\). The class of strong pseudo-contractive mappings is independent of the class of \(\lambda\)-strict pseudo-contractive mappings.

Iterative methods for nonexpansive mapping have been extensively studied. Iterative methods for the \(\lambda\)-strict pseudo-contractive mapping, introduced by Browder and Petryshyn [9] in 1967, are far less developed than those for nonexpansive mapping; the reason is probably that the second term appearing in the right-hand side of (1.4) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping. On the other hand, \(\lambda\)-strict pseudo-contractive mapping has more powerful applications than nonexpansive mapping do in solving inverse problems (see, for example, Scherzer [10]). Therefore, it is more interesting to study the theory of iterative methods for \(\lambda\)-strict pseudo-contractive mappings.

Zhou and Su [11] proved the relation between the \(\lambda\)-strict pseudo-contraction and the nonexpansive mappings using the following lemma.

**Lemma 1.1.** (See [11, Lemma 2.2]) Let \(K\) be a nonempty convex subset of a real \(q\)-uniformly smooth Banach space \(E\) and \(T : K \to K\) be a \(\lambda\)-strict pseudo-contraction. For \(\alpha \in (0, 1)\), define \(T_\alpha x = (1 - \alpha)x + \alpha Tx\). Then, for \(\alpha \in (0, \mu]\), \(\mu = \min\{1, (\frac{\lambda^2}{\epsilon_q})^{\frac{1}{q - 1}}\}\), \(T_\alpha : K \to K\) is nonexpansive such that \(F(T_\alpha) = F(T)\).

Here, for any \(n \in \mathbb{N}\) (the set of positive integers), we consider the mapping \(W_n\) to be defined by
where $I$ is the identity operator on $E$, and $\gamma_1$, $\gamma_2$, ... are real numbers such that $0 \leq \gamma_n \leq 1$, for every $i \in \mathbb{N}$, $S_i = t_i T_i + (1 - t_i) I$, where $T_i$ is $\lambda_i$-strict pseudo-contractive mapping of $K$ into itself and $t_i \in (0, \mu]$, $\mu \in \min\{1, (\frac{\lambda}{\alpha})^{q-1}\}$. Such a mapping $W_n$ is called the $W$-mapping, generated by $T_n$, $T_{n-1}$, ..., $T_1$ and $\gamma_n$, $\gamma_{n-1}$, ..., $\gamma_1$, $t_n$, $t_{n-1}$, ..., $t_1$. It follows from Lemma 1.1 that non-expansivity of each $S_i$ ensures the non-expansivity of $W_n$.

Cho et al. [12] proposed the following iterative scheme:

\[
\begin{align*}
  x_0 &= x \in K \text{ chosen arbitrarily}, \\
  y_n &= \beta_n x_n + (1 - \beta_n) W_n x_n, \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad f \in \Pi_K, \quad n \geq 0.
\end{align*}
\]  

(1.6)

Under the conditions (C1), (C2) and (B4), they also proved the strong convergence of the sequence $\{x_n\}$, defined by (1.6), and extended the results of [8].

Recently, Yao et al. [13] considered the following iterative algorithm:

\[
\begin{align*}
  x_0 &= x \in K \text{ chosen arbitrarily}, \\
  x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n,
\end{align*}
\]  

(1.7)

where $f \in \Pi_K$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\beta_n\}$ is a sequence in $[0,1)$ and $\alpha_n + \beta_n < 1$, for all $n \in \mathbb{N}$. They proved that the iterative algorithm (1.7) converges strongly to a common fixed points of an infinite countable family of nonexpansive mappings $\{T_i\}_{i=1}^\infty$.

Shimoji and Takahashi [14] first introduced an iterative algorithm given by an infinite family of nonexpansive mappings. Furthermore, they considered the feasibility problem of finding a solution of infinite convex inequalities and the problem of finding a common fixed point of infinite nonexpansive mappings. Bauschke and Borwein [15] pointed out that
the well-known convex feasibility problem reduced to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest having practical importance (see [16]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [16,17]).

Noor [18] first introduced a three-step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [19] applied a three-step iterative sequence for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory. In [19], they showed that the three-step iterative schemes performed better than the Ishikawa type and Mann type iterative methods. Haubruge et al. [20] studied the convergence analysis of the three-step iterations to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus, three-step iteration process and multi-step iteration process have been investigated extensively by some authors (see [21–24] and the references therein).

Motivated and inspired by these facts, as the viscosity approximation method, we consider a new modified Noor iteration scheme for an infinite family of \( \lambda_i \)–strict pseudo-contractive mappings \( \{T_i\}_{i=1}^{\infty} \):

\[
\begin{align*}
  x_0 &= x \in K, \\
  z_n &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n, \\
  y_n &= (1 - b_n) z_n + b_n W_n z_n, \\
  x_{n+1} &= (1 - c_n) y_n + c_n W_n y_n, \quad \forall n \geq 0,
\end{align*}
\]

(1.8)

where \( \{\alpha_n\}, \{b_n\}, \{\beta_n\}, \{c_n\}, \{\alpha_n + \beta_n\} \subset (0, 1), f \in \Pi_K, \) and \( W_n \) is a mapping defined by (1.5). If \( b_n = c_n = 0 \), for all \( n \geq 0 \), then (1.8) reduces to (1.7). By using viscosity approximation methods, our purpose here is to study some sufficient and necessary conditions of the three-step iterative algorithm (1.8) for finding approximate common fixed points of an infinite countable family of \( \lambda_i \)–strict pseudo-contractive mappings \( \{T_i\}_{i=1}^{\infty} \). The results presented here extend and improve some recent results.
2. Preliminaries

Let $E$ be a Banach space with dimension $E \geq 2$ and $E^*$ be its dual. The modulus of convexity of $E$ is the function $\delta_E : (0, 2] \rightarrow [0, 1]$, defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \right\}.$$

A Banach space $E$ is uniformly convex if and only if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$. A Banach space $E$ is said to be strictly convex if $\|x\| = \|y\| = 1, x \neq y$ implies $\|x + y\|^2 < 1$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. The space $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists, for all $x, y \in S(E)$. For any $x, y \in E(x \neq 0)$, we denote this limit by $(x, y)$. The norm is said to be uniformly Gâteaux differentiable if for $y \in S(E)$, the limit is attained uniformly for $x \in S(E)$. The norm $\|\cdot\|$ of $E$ is said to be Fréchet differentiable if for all $x \in S(E)$, the limit $(x, y)$ exists uniformly, for all $y \in S(E)$. It is known that $E$ is smooth if and only if each normalized duality mapping $J$ is single-valued.

Let $\rho_E : [0, \infty) \rightarrow [0, \infty)$ be the modulus of smoothness of $E$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}.$$

A Banach space $E$ is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \rightarrow 0$, as $t \to 0$. A Banach space $E$ is said to be $q$–uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well-known that $E$ is uniformly smooth if and only if the norm of $E$ is uniformly Fréchet differentiable, a typical example of both uniformly convex and uniformly smooth Banach space is $L^p$ ($p > 1$).

Let $K$ be a nonempty subset of a Banach space $E$. For $x \in K$, the inward set of $x$, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \rightarrow E$ is said to be weakly inward if $Tx \in cl[I_K(x)]$, for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. It is obvious that every self-map is trivially weakly inward.
Let $C$ and $D$ be nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$. Then a mapping $P : C \to D$ is said to be retraction if $P \mathbf{x} = \mathbf{x}$, for all $\mathbf{x} \in C$. A retraction $P : C \to D$ is said to be sunny \cite{25} if $P(\mathbf{P} \mathbf{x} + t(\mathbf{x} - \mathbf{P} \mathbf{x})) = \mathbf{P} \mathbf{x}$, for all $\mathbf{x} \in C$ and $t \geq 0$, with $\mathbf{P} \mathbf{x} + t(\mathbf{x} - \mathbf{P} \mathbf{x}) \in C$.

Suppose that $\{ \mathbf{x}_n \}$ is a sequence in $E$. In the sequel, $\mathbf{x}_n \to \mathbf{x}$ (respectively, $\mathbf{x}_n \xrightarrow{w} \mathbf{x}$) will denote strong (respectively, weak) convergence of the sequence $\{ \mathbf{x}_n \}$ to $\mathbf{x}$.

Concerning $W_n$, the following two lemmas play crucial roles in proving our main results.

**Lemma 2.1.** \textit{(see \cite{14})} Let $K$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $K$ into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 < \gamma_n \leq b < 1$, for any $n \geq 1$. Then, for any $\mathbf{x} \in K$ and $k \in \mathbb{N}$,

$$\lim_{n \to \infty} U_{n,k} \mathbf{x}$$

exists.

Using Lemma 2.1, we can define the mapping $W$ of $K$ into itself as follows:

$$W \mathbf{x} = \lim_{n \to \infty} W_n \mathbf{x} = \lim_{n \to \infty} U_{n,1} \mathbf{x}, \quad \forall \mathbf{x} \in K.$$ 

Such a mapping $W$ is said to be the $W$-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots, t_1, t_2, \ldots, t_n$. Throughout this paper, we will assume that $0 < \gamma_n \leq b < 1$, for all $n \geq N$, $t_i \in (0, \mu)$ and $\mu = \min\{1, (\frac{\lambda}{\mu})^{\frac{1}{\lambda - 1}}\}$, where $\lambda = \inf \lambda_i > 0$, $\forall i \in \mathbb{N}$.

**Lemma 2.2.** \textit{(see \cite{14})} Let $K$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $S_1, S_2, \ldots$ be nonexpansive mappings of $K$ into itself such that $\bigcap_{n=1}^{\infty} F(S_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 < \gamma_n \leq b < 1$, for any $n \geq 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.

It follows from Lemma 1.1 and Lemma 2.2 that $F(W) = \bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n)$.

We also need the following lemmas for the proof of our main results.

**Lemma 2.3.** Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $\mathbf{x}, \mathbf{y} \in E$, the following inequality holds:

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2(\mathbf{y}, j(\mathbf{x} + \mathbf{y})), \quad j(\mathbf{x} + \mathbf{y}) \in J(\mathbf{x} + \mathbf{y}).$$
Lemma 2.4. (see [6, Lemma 2.5]) Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following relation:

\[
a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \sigma_n + \mu_n, \quad n \geq 0,
\]

where, (i) \( \{\lambda_n\} \subset [0, 1], \sum_{n=1}^{\infty} \lambda_n = \infty; \) (ii) \( \limsup_{n \to \infty} \sigma_n \leq 0; \) and (iii) \( \mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n < \infty. \) Then, \( \{a_n\} \) converges to zero, as \( n \to \infty. \)

Lemma 2.5. (see [26]) Let \( \{x_n\}, \{y_n\} \) be two bounded sequences in a Banach space \( E \) and \( \beta_n \in [0, 1] \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \) Suppose

\[
x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n, \quad \text{for all integers } n \geq 0,
\]

and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \) Then, \( \lim_{n \to \infty} \|x_n - y_n\| = 0. \)

Lemma 2.6. (see [27]) Let \( E \) be a uniformly smooth Banach space and \( K \) be a closed convex subset of \( E. \) Let \( T_i : K \to K \) be a nonexpansive mapping with \( F(T_i) \neq \emptyset \) and \( f \in \Pi_K. \) Then, the sequence \( \{x_t\}, \) defined by

\[
x_t = tf(x_t) + (1 - t)Tx_t,
\]

converges strongly to a point in \( F(T). \) If we define a mapping \( P : \Pi_K \to F(T) \) by

\[
P(f) := \lim_{t \to 0} x_t, \quad \forall f \in \Pi_K,
\]

then \( P(f) \) solves the following variational inequality:

\[
\langle (I - f)P(f), J(P(f) - p) \rangle \leq 0, \quad \forall f \in \Pi_K, \quad p \in F(T).
\]

3. Main Results

Theorem 3.1. Let \( K \) be a closed convex subset of a real \( q-\)uniformly smooth and strictly convex Banach space \( E. \) Let \( T_i \) be a \( \lambda_i \)-strict pseudo-contractive mapping from \( K \) into itself, for \( i \in \mathbb{N}. \) Assume that \( F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \) and \( f \in \Pi_K. \) Suppose that the sequences \( \{\alpha_n\}, \{\beta_n\}, \{b_n\}, \{c_n\} \) and \( \{\alpha_n + \beta_n\} \) in \( (0, 1) \) satisfy the following conditions:

(1) \( \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0; \)
(2) \( \lim_{n \to \infty} b_n = 0, \lim_{n \to \infty} c_n = 0; \)
(3) \( \limsup_{n \to \infty} \beta_n < 1. \)

Then, the modified Noor iterative scheme defined by (1.8) converges
strongly to $P(f) \in F$, where $P(f)$ is the unique solution of the following variational inequality:

$$\langle (I - f)P(f), J(P(f) - p) \rangle \leq 0, \quad \forall f \in \Pi_K, \quad p \in F.$$  

**Proof.** We proceed with the following steps.

*Step 1.* We should prove that

\[ \|x_n - p\| \leq \max\{\|x_0 - p\|, (1/(1 - \alpha))\|f(p) - p\|\}, \quad \text{for all } n \geq 0 \text{ and all } p \in F. \]

So, \( \{y_n\}, \{z_n\}, \{f(x_n)\}, \{W_n x_n\}, \{W_n y_n\} \) and \( \{W_n z_n\} \) are bounded. Indeed, take a point \( p \in F \). It follows from (1.8) that

\[
\begin{align*}
\|z_n - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) \\
&\quad + (1 - \alpha_n - \beta_n)(W_n x_n - p)\| \\
&\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)\|W_n x_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \|f(p) - p\| + \beta_n\|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= (1 - (1 - \alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{1}{1 - \alpha}\|f(p) - p\|\}, \\
\end{align*}
\]

(3.1)

\[
\begin{align*}
\|y_n - p\| &= \|(1 - b_n)(z_n - p) + b_n(W_n z_n - p)\| \\
&\leq (1 - b_n)\|z_n - p\| + b_n\|W_n z_n - p\| \\
&\leq (1 - b_n)\|z_n - p\| + b_n\|z_n - p\| = \|z_n - p\|. \\
\end{align*}
\]

(3.2)

It follows from (1.8), (3.1) and (3.2) that

\[
\begin{align*}
\|x_{n+1} - p\| &= \|(1 - c_n)(y_n - p) + c_n(W_n y_n - p)\| \\
&\leq (1 - c_n)\|y_n - p\| + c_n\|y_n - p\| = \|y_n - p\| \\
&\leq \|z_n - p\| \leq \max\{\|x_n - p\|, \frac{1}{1 - \alpha}\|f(p) - p\|\}. \\
\end{align*}
\]

(3.3)

Using mathematical induction, we obtain:

\[
\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - \alpha}\|f(p) - p\|\};
\]

for all \( n \geq 0 \). Hence, \( \{x_n\} \) is bounded, and so are \( \{y_n\}, \{z_n\}, \{f(x_n)\}, \{W_n x_n\}, \{W_n y_n\} \) and \( \{W_n z_n\} \).
Step 2. We prove:

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}
\]

Indeed, putting \(l_n = \frac{x_{n+1} - \rho_n x_n}{1 - \rho_n}\), we have

\[
x_{n+1} = \rho_n x_n + (1 - \rho_n) l_n, \quad \forall n \geq 0. \tag{3.6}
\]

It follows from (1.5) and (1.8) that

\[
z_n = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \gamma_1 S_1 U_{n,2} x_n
\]

\[
= \alpha_n f(x_n) + (1 - \gamma_1) \alpha_n - (1 - \beta_n) \gamma_1 f(x_n)
\]

\[
+ (1 - \alpha_n - \beta_n) \gamma_1 S_1 U_{n,2} x_n
\]

\[
= \alpha_n f(x_n) + \rho_n x_n + (1 - \alpha_n - \rho_n) S_1 U_{n,2} x_n, \tag{3.7}
\]

where \(\rho_n = 1 - (1 - \gamma_1) \alpha_n - (1 - \beta_n) \gamma_1\). It follows from conditions (1) and (3) that \(0 < \lim \inf_{n \to \infty} \rho_n \leq \lim \sup_{n \to \infty} \rho_n < 1\). It follows from (3.6), (3.7) and (1.8) that

\[
l_{n+1} - l_n = \frac{(1 - c_{n+1}) y_{n+1} + c_{n+1} W_{n+1} y_{n+1} - \rho_{n+1} x_{n+1}}{1 - \rho_{n+1}}
\]

\[
- \frac{(1 - c_n) y_n + c_n W_n y_n - \rho_n x_n}{1 - \rho_n}
\]

\[
= \frac{c_{n+1} (W_{n+1} y_{n+1} - y_{n+1}) + y_{n+1} - \rho_{n+1} x_{n+1}}{1 - \rho_{n+1}}
\]

\[
- \frac{c_n (W_n y_n - y_n) + y_n - \rho_n x_n}{1 - \rho_n}
\]

\[
= \frac{c_{n+1}}{1 - \rho_{n+1}} (W_{n+1} y_{n+1} - y_{n+1}) - \frac{c_n}{1 - \rho_n} (W_n y_n - y_n)
\]

\[
+ \frac{b_{n+1}}{1 - \rho_{n+1}} (W_{n+1} z_{n+1} - z_{n+1}) - \frac{b_n}{1 - \rho_n} (W_n z_n - z_n)
\]

\[
+ \frac{\alpha_{n+1}}{1 - \rho_{n+1}} (f(x_{n+1}) - S_1 U_{n+1,2} x_{n+1})
\]

\[
- \frac{\alpha_n}{1 - \rho_n} (f(x_n) - S_1 U_{n,2} x_n) + (S_1 U_{n+1,2} x_{n+1})
\]

\[
- S_1 U_{n+1,2} x_n + (S_1 U_{n+1,2} x_n - S_1 U_{n,2} x_n).
\]
We have

\[
\left\| l_{n+1} - l_n \right\| \leq \frac{c_{n+1}}{1 - \rho_{n+1}} \left\| W_{n+1}y_{n+1} - y_{n+1} \right\| + \frac{c_n}{1 - \rho_n} \left\| W_n y_n - y_n \right\|
\]

\[
+ \frac{b_{n+1}}{1 - \rho_{n+1}} \left\| W_{n+1}z_{n+1} - z_{n+1} \right\| + \frac{b_n}{1 - \rho_n} \left\| W_n z_n - z_n \right\|
\]

\[
+ \frac{\alpha_{n+1}}{1 - \rho_{n+1}} \left\| f(x_{n+1}) - S_1 U_{n+1,2} x_{n+1} \right\|
\]

\[
+ \frac{\alpha_n}{1 - \rho_n} \left\| f(x_n) - S_1 U_{n,2} x_n \right\|
\]

(3.8)

\[
+ \left\| x_{n+1} - x_n \right\| + \left\| S_1 U_{n+1,2} x_n - S_1 U_{n,2} x_n \right\|
\]

Since $S_i$ and $U_{n,i}$ are nonexpansive, from (1.5), we obtain:

\[
\left\| S_1 U_{n+1,2} x_n - S_1 U_{n,2} x_n \right\| \leq \left\| U_{n+1,2} x_n - U_{n,2} x_n \right\|
\]

\[
= \left\| \gamma_2 S_2 U_{n+1,3} x_n - \gamma_2 S_2 U_{n,3} x_n \right\|
\]

\[
\leq \gamma_2 \left\| U_{n+1,3} x_n - U_{n,3} x_n \right\|
\]

\[
= \gamma_2 \gamma_3 \left\| U_{n+1,4} x_n - U_{n,4} x_n \right\|
\]

\[
\leq \ldots
\]

\[
\leq \gamma_2 \gamma_3 \cdots \gamma_n \left\| U_{n+1,n+1} x_n - U_{n,n+1} x_n \right\|
\]

(3.9)

\[
\leq M \prod_{i=2}^{n} \gamma_i,
\]

where $M \geq 0$ is a constant such that $\left\| U_{n+1,n+1} x_n - U_{n,n+1} x_n \right\| \leq M$, for all $n \geq 0$. Substituting (3.9) into (3.8), we have

\[
\left\| l_{n+1} - l_n \right\| - \left\| x_{n+1} - x_n \right\| \leq \frac{c_{n+1}}{1 - \rho_{n+1}} \left\| W_{n+1}y_{n+1} - y_{n+1} \right\|
\]

\[
+ \frac{c_n}{1 - \rho_n} \left\| W_n y_n - y_n \right\|
\]

\[
+ \frac{b_{n+1}}{1 - \rho_{n+1}} \left\| W_{n+1}z_{n+1} - z_{n+1} \right\|
\]

\[
+ \frac{b_n}{1 - \rho_n} \left\| W_n z_n - z_n \right\|
\]

\[
+ \frac{\alpha_{n+1}}{1 - \rho_{n+1}} \left\| f(x_{n+1}) - S_1 U_{n+1,2} x_{n+1} \right\|
\]

\[
+ \frac{\alpha_n}{1 - \rho_n} \left\| f(x_n) - S_1 U_{n,2} x_n \right\| + M \prod_{i=2}^{n} \gamma_i.
\]
From the conditions (1), (2), (3) and $0 < \gamma_n \leq b < 1$, we get
\[ \lim_{n \to \infty} \sup(||l_{n+1} - l_n|| - ||x_{n+1} - x_n||) \leq 0. \]

It follows from Lemma 2.5 that $\lim_{n \to \infty} ||l_n - x_n|| = 0$. Notice that using (3.6), we obtain:
\[ x_{n+1} - x_n = (1 - \rho_n)(l_n - x_n). \]

Thus, we get $\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0$.

Step 3. We will show $\lim_{n \to \infty} ||Wz_n - z_n|| = 0$. Observing that $x_{n+1} - y_n = c_n(W_ny_n - y_n)$, $y_n - z_n = b_n(W_nz_n - z_n)$ and condition (2), we have
\[ \lim_{n \to \infty} ||x_{n+1} - y_n|| = 0, \quad \lim_{n \to \infty} ||y_n - z_n|| = 0. \]

On the other hand, we have
\[ ||y_n - x_n|| \leq ||y_n - x_{n+1}|| + ||x_{n+1} - x_n||. \]

This, together with (3.5) and (3.10), imply:
\[ \lim_{n \to \infty} ||y_n - x_n|| = 0. \]

It follows from (1.8) that
\[
\begin{align*}
||x_n - W_nx_n|| &\leq ||x_n - y_n|| + ||y_n - z_n|| + ||z_n - W_nx_n|| \\
&\leq ||x_n - y_n|| + ||y_n - z_n|| + \beta_n||x_n - W_nx_n|| \\
&\quad + \alpha_n||f(x_n) - W_nx_n||.
\end{align*}
\]

This implies:
\[
(1 - \beta_n)||x_n - W_nx_n|| \leq ||x_n - y_n|| + ||y_n - z_n|| + \alpha_n||f(x_n) - W_nx_n||.
\]

From condition (2), (3), (3.10) and (3.11), we have
\[ \lim_{n \to \infty} ||x_n - W_nx_n|| = 0. \]

It follows from (1.8) that $z_n - x_n = (1 - \beta_n)(W_nx_n - x_n) + \alpha_n||f(x_n) - W_nx_n||$. Therefore, we have
\[
\begin{align*}
||z_n - x_n|| &\leq (1 - \beta_n)||W_nx_n - x_n|| + \alpha_n||f(x_n) - W_nx_n|| \\
&\leq ||W_nx_n - x_n|| + \alpha_n(||f(x_n)|| + ||W_nx_n||).
\end{align*}
\]

This, together with (3.12) and $\lim_{n \to \infty} \alpha_n = 0$, imply $\lim_{n \to \infty} ||z_n - x_n|| = 0$. Noticing that
\[
\begin{align*}
||z_n - W_nz_n|| &\leq ||z_n - x_n|| + ||x_n - W_nx_n|| + ||W_nx_n - W_nz_n|| \\
&\leq 2||z_n - x_n|| + ||x_n - W_nx_n||,
\end{align*}
\]
we have \( \lim_{n \to \infty} \| z_n - W_n z_n \| = 0 \). On the other hand, we have
\[
\| W z_n - z_n \| \leq \| W z_n - W_n z_n \| + \| W_n z_n - z_n \|. \tag{3.13}
\]
From [28, Remark 2.2], we have
\[
\| W z_n - W_n z_n \| \to 0 \quad (n \to \infty).
\]
This together with (3.13) imply:
\[
\lim_{n \to \infty} \| W z_n - z_n \| = 0. \tag{3.14}
\]

**Step 4.** We show that \( \limsup_{n \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_n) \rangle \leq 0 \), where \( P(f) = \lim_{t \to 0^+} x_t \), with \( x_t \) being the fixed point of the contraction,
\[
x \mapsto tf(x) + (1 - t)W x.
\]
Then, we can write
\[
x_t - z_{n_j} = t(f(x_t) - z_{n_j}) + (1 - t)(W x_t - z_{n_j}). \tag{3.15}
\]
Suppose that a subsequence \( \{ z_{n_j} \} \subset \{ z_n \} \) is such that
\[
\limsup_{n \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_n) \rangle = \lim_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle \tag{3.16}
\]
and \( z_{n_j} \rightharpoonup p \), for some \( p \in E \). It follows from (3.14) that \( \lim_{j \to \infty} \| z_{n_j} - W z_{n_j} \| = 0 \). Putting
\[
f_j(t) = (1 - t)^2 \| z_{n_j} - W z_{n_j} \| (2 \| x_t - z_{n_j} \|)
\]
(3.17)
\[
+ \| z_{n_j} - W z_{n_j} \| \to 0 \quad (j \to \infty),
\]
it follows from (3.15), Lemma 2.3 and Step 3 that
\[
\| x_t - z_{n_j} \|^2 \leq (1 - t)^2 \| W x_t - z_{n_j} \|^2 + 2t \langle f(x_t) - z_{n_j}, J(x_t - z_{n_j}) \rangle
\]
\[
\leq (1 - t)^2 (\| W x_t - W z_{n_j} \| + \| W z_{n_j} - z_{n_j} \|)^2
\]
\[
+ 2t \langle f(x_t) - x_t, J(x_t - z_{n_j}) \rangle
\]
\[
+ 2t \langle x_t - z_{n_j}, J(x_t - z_{n_j}) \rangle
\]
\[
= (1 - t)^2 \| x_t - z_{n_j} \|^2 + f_j(t) + 2t \langle f(x_t) - x_t, J(x_t - z_{n_j}) \rangle
\]
\[
+ 2t \| x_t - z_{n_j} \|^2. \tag{3.18}
\]
The last inequality implies:
\[
\langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq \frac{t}{2} \| x_t - z_{n_j} \|^2 + \frac{1}{2t} f_j(t).
\]
Letting $j \to \infty$ and noting (3.17) yield:

\begin{equation}
\limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq \frac{t}{2} M_1,
\end{equation}

where $M_1 > 0$ is a constant such that $M_1 \geq \|x_t - z_{n_j}\|^2$, for all $n \geq 0$ and $t \in (0, 1)$. Taking $t \to 0$ in (3.19) and noticing the fact that the two limits are interchangeable due to the fact that $J$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the weak* topology of $E^*$, we have

\begin{equation}
\limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle \leq 0.
\end{equation}

Indeed, letting $t \to 0$, from (3.19) we have

\begin{equation}
\limsup_{t \to 0} \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq 0.
\end{equation}

Thus, for arbitrary $\epsilon > 0$, there exists a positive number $\delta_1$ such that for any $t \in (0, \delta_1)$, we have

\begin{equation}
\limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle \leq \frac{\epsilon}{2}.
\end{equation}

Since $x_t \to P(f)$, as $t \to 0$, the set $\{x_t - z_{n_j}\}$ is bounded and the duality mapping $J$ is norm-to-norm uniformly continuous on bounded subset of $E$, then there exists $\delta_2 > 0$ such that, for any $t \in (0, \delta_2)$,

\begin{align*}
&\|\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle - \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle\|
\leq \|\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) - J(x_t - z_{n_j}) \rangle + \langle P(f) - f(P(f)) - (x_t - f(x_t)), J(x_t - z_{n_j}) \rangle\|
\leq \|P(f) - f(P(f)) - (x_t - f(x_t))\||x_t - z_{n_j}| < \epsilon/2.
\end{align*}

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, for all $t \in (0, \delta)$ and $j \in N$, we have

\begin{equation}
\langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle < \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle + \frac{\epsilon}{2},
\end{equation}

which implies:

\begin{align*}
\limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle &\leq \limsup_{j \to \infty} \langle x_t - f(x_t), J(x_t - z_{n_j}) \rangle + \frac{\epsilon}{2},
\end{align*}

This together with (3.22) imply:

\begin{equation}
\limsup_{j \to \infty} \langle P(f) - f(P(f)), J(P(f) - z_{n_j}) \rangle \leq \epsilon.
\end{equation}
Since $\epsilon$ is arbitrary, we have $\limsup_{j \to \infty} (P(f) - f(P(f)), J(P(f) - z_{n_j})) \leq 0$.

**Step 5.** We claim that $\lim_{n \to \infty} \|x_n - P(f)\| = 0$. Indeed, it follows from (3.1) and (1.8) that

$$\|x_{n+1} - P(f)\| \leq \|z_n - P(f)\| = \|(1 - \alpha_n - \beta_n)(W_n x_n - P(f)) + \beta_n(x_n - P(f)) + \alpha_n(f(x_n) - P(f))\|.$$ 

Thus, it follows from Lemma 2.3 and (3.1) that

$$\|x_{n+1} - P(f)\|^2 \leq \|z_n - P(f)\|^2 + 2\alpha_n\langle f(x_n) - P(f), J(z_n - P(f)) \rangle + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f)) \rangle$$

Therefore, we obtain:

$$\|x_{n+1} - P(f)\|^2 \leq (1 - 2(1 - \alpha)\alpha_n + \alpha_n^2)\|x_n - P(f)\|^2 + 2\alpha_n\langle f(P(f)) - P(f), J(z_n - P(f)) \rangle$$

(3.23)

$$\|x_{n+1} - P(f)\|^2 \leq \frac{\lambda_n}{\sigma_n} M_2^2 + \alpha_n\langle f(P(f)) - P(f), J(z_n - P(f)) \rangle,$$

where $M_2 = \sup_{n \geq 0} \|x_n - P(f)\|$. Set

$$\lambda_n = 2(1 - \alpha)\alpha_n, \quad \sigma_n = \frac{\alpha_n}{2(1 - \alpha)} M_2^2 + \frac{1 - \alpha}{2(1 - \alpha)} \langle f(P(f)) - P(f), J(z_n - P(f)) \rangle.$$

It follows from condition (1) and Step 4 that $\lambda_n \to 0$, $\sum_{n=1}^\infty \lambda_n = \infty$, and $\limsup_{n \to \infty} \sigma_n \leq 0$. Then, (3.23) reduces to

$$\|x_{n+1} - P(f)\|^2 \leq (1 - \lambda_n)\|x_n - P(f)\|^2 + \alpha_n\sigma_n.$$ 

From Lemma 2.4 with $\mu_n = 0$, we see that $\lim_{n \to \infty} \|x_n - P(f)\| = 0$. This completes the proof. \qed
Remark 3.2. If \( \{T_i\}_{i=1}^{\infty} \) is composed of nonexpansive mappings, then the \( S_i = t_i T_i + (1 - t_i) I \) are also nonexpansive mappings. Therefore, \( q \)-uniformly smoothness corresponding to \( E \) in Theorem 3.1 can be extended to uniformly smooth. If we take \( b_n = c_n = 0 \) in Theorem 3.1, then Theorem 3.1 becomes Theorem 2.1 of Cho et al. [12] and Theorem 3.1 of Yao [13].

Remark 3.3. Theorem 3.1 partially improves main results of [11] from a finite family of \( \lambda_i \)-strict pseudo-contractions to an infinite family of \( \lambda_i \)-strict pseudo-contractions.

If \( f(x) = u \) for all \( x \in K \), in Theorem 3.1, then we have the following result.

Theorem 3.4. Let \( K \) be a closed convex subset of a real \( q \)-uniformly smooth and strictly convex Banach space \( E \). Let \( T_i \) be a \( \lambda_i \)-strict pseudo-contractive mapping from \( K \) into itself, for \( i \in \mathbb{N} \). Assume that \( F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Suppose that the sequences \( \{\alpha_n\}, \{\beta_n\}, \{b_n\}, \{c_n\} \) and \( \{\alpha_n + \beta_n\} \) in \( (0,1) \) satisfy the following conditions:

1. \( \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0 \);
2. \( \lim_{n \to \infty} b_n = 0, \lim_{n \to \infty} c_n = 0 \);
3. \( \limsup_{n \to \infty} \beta_n < 1 \).

Let \( \{x_n\} \) be the three-step iterative scheme defined by

\[
\begin{cases}
  x_0 = x \in K, \\
  z_n = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n x_n, \\
  y_n = (1 - b_n) z_n + b_n W_n z_n, \\
  x_{n+1} = (1 - c_n) y_n + c_n W_n y_n, \quad \forall n \geq 0,
\end{cases}
\]

(3.24)

where \( \{\alpha_n\}, \{b_n\}, \{\beta_n\}, \{c_n\}, \{\alpha_n + \beta_n\} \subset (0,1) \), and \( W_n \) is a mapping defined by (1.5). Then, \( \{x_n\} \) converges strongly to \( z \in F \), where \( z = P_F(u) \), and \( P_F : K \to F \) is the unique sunny nonexpansive retraction from \( K \) onto \( F \).

Remark 3.5. Theorem 3.4 mainly improves Theorem 2.3 of Zhou [29] from a single \( \lambda \)-strict pseudo-contractive mapping to an infinite family of \( \lambda_i \)-strict pseudo-contractive mappings and from one-step iteration scheme to three-step iteration scheme if \( K \) is a closed convex subset of a \( 2 \)-uniformly smooth and strictly convex Banach space \( E \).
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References

Noor iterations for strict pseudo-contraction mappings


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